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Asymptotic Dirichlet problem on negatively curved spaces

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Abstract. We study the Dirichlet problem at infinity for the p-Laplacian and p-regularity of points at infinity on Cartan-Hadamard manifolds. We also survey the recent result of the authors and Lang on the asymptotic Dirichlet problem for p-harmonic functions on Gromov hyperbolic metric measure spaces.

Keywords. Cartan-Hadamard manifold, Gromov hyperbolic metric space, p-harmonic function, Dirichlet problem.

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1. Introduction

In [17] Greene and Wu conjectured that a Cartan-Hadamard manifold M admits a wealth of non-constant bounded harmonic functions if the sectional curvatures of M have an upper bound

$$K_M(P) \leq -C\,\rho^{-2}(x)$$

outside a compact set for some constant $C > 0$, where $\rho = d(\cdot, o)$ is the distance function to a fixed point $o \in M$ and P is any 2-dimensional subspace of T_xM. Recall that a Cartan-Hadamard manifold is a complete, connected and simply connected Riemannian n-manifold, $n \geq 2$, of non-positive sectional curvature. By the Cartan-Hadamard theorem, the exponential map $\exp_o\colon T_oM \to M$ is a diffeomorphism for every point $o \in M$. In particular, M is diffeomorphic to $\mathbb{R}^n$. It is well-known that M can be compactified by adding a *sphere at infinity* (or a *boundary at infinity*), denoted by $M(\infty)$, so that the resulting space $\bar{M} = M \cup M(\infty)$ equipped with the *cone topology* will be homeomorphic to a closed Euclidean ball.

The conjecture of Greene and Wu can be approached by studying the so-called *Dirichlet problem at infinity* (or the *asymptotic Dirichlet problem*) on a Cartan-Hadamard manifold. Thus one asks whether every continuous function on $M(\infty)$ has a (unique) harmonic extension to M. In general, the answer is no

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since the simplest Cartan-Hadamard manifold $\mathbb{R}^n$ admits no positive harmonic functions other than constants.

The Dirichlet problem at infinity was solved by Choi [13] under the condition that sectional curvatures have a negative upper bound $K \leq -a^2 < 0$ and any two points of the sphere at infinity can be separated by convex neighborhoods. Such appropriate convex sets were constructed by Anderson [6] for manifolds of pinched sectional curvature $-b^2 \leq K \leq -a^2 < 0$. At the same time, the Dirichlet problem at infinity was independently solved by Sullivan [32] under the same pinched curvature assumption by using probabilistic arguments. In [7], Anderson and Schoen presented a simple and direct solution to the Dirichlet problem again in the case of pinched negative curvature. Major contributions to the Dirichlet problem were given by Ancona in a series of papers [2], [3], [4], and [5]. In [2] he was able to replace the lower curvature bound by a bounded geometry assumption that each ball up to a fixed radius is L-bi-Lipschitz equivalent to an open set in $\mathbb{R}^n$ for some fixed $L \geq 1$. He also considered a more general class of linear equations than merely the Laplace equation. On the other hand, in [5] he showed that the Dirichlet problem is not solvable, in general, if there are neither curvature lower bounds nor the bounded geometry assumption; see also [9]. Furthermore, in [3] Ancona studied the asymptotic Dirichlet problem on Gromov hyperbolic graphs and in [4] on Gromov hyperbolic Riemannian manifolds with bounded geometry and a positive lower bound $\lambda_1(M) > 0$ for the Dirichlet eigenvalues. Cheng [12] solved the Dirichlet problem at infinity under a pointwise pinching condition on the sectional curvature for Cartan-Hadamard manifolds M with $\lambda_1(M) > 0$. There are several papers where assumptions on curvature have been weakened by allowing curvature decay (or growth) at a certain rate; see e.g. [8], [24], [28], and [23].

In the general case of the p-Laplacian, $1 < p < \infty$, Pansu [29] showed the existence of nonconstant bounded p-harmonic functions with finite p-energy on Cartan-Hadamard manifolds of pinched curvature

$$-b^2 \leq K \leq -a^2$$

for $p > (n-1)b/a$. The Dirichlet problem at infinity for the p-Laplacian was solved in [21] on Cartan-Hadamard manifolds of pinched negative sectional curvature by modifying the direct approach of Anderson and Schoen [7]. In [33] the asymptotic Dirichlet problem was considered for $\mathcal{A}$-harmonic functions and it was shown that $x_0 \in M(\infty)$ is p-regular if it has a neighborhood V in the cone topology such that the sectional curvatures satisfy a pointwise pinching condition in $V \cap M$ and

$$K_M \leq -\frac{\phi(\phi-1)}{\rho^2}$$

in $V \cap M$ for some $\phi > 1$ with $p < 1+\phi(n-1)$. Recall that a *p-harmonic function* in an open subset U of a Riemannian manifold M is a continuous (weak) solution u of the p-Laplace equation

$$-\operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big) = 0.$$

More precisely, u belongs to the local Sobolev space $W^{1,p}_{\mathrm{loc}}(U)$ and

$$\int_U \langle |\nabla u|^{p-2}\nabla u, \nabla\varphi\rangle \, dm = 0 \tag{1.1}$$

for every $\varphi \in C_0^\infty(U)$. Note that (1.1) is the Euler-Lagrange equation for the (p-energy) variational integral

$$\int_U |\nabla u|^p \, dm. \tag{1.2}$$

For the nonlinear potential theory associated with the p-Laplacian and more general quasilinear elliptic operators, we refer to the book [20] by Heinonen, Kilpeläinen, and Martio. In [22] the authors, together with Lang, considered the Dirichlet problem at infinity for p-harmonic functions in a very general setting of Gromov hyperbolic metric measure spaces. Suppose that X is a connected and locally compact Gromov hyperbolic metric measure space equipped with a Borel regular measure μ. Assume that X has a (local) bounded geometry in a sense that the measure μ is locally doubling, the measures of balls of a fixed sufficiently small radius have a uniform positive lower bound, and that X supports a local Poincaré-type inequality. Furthermore, suppose that X has at most exponential volume growth and that a global Sobolev-type inequality holds for compactly supported functions. The main result in [22] then states that given a bounded continuous function $f\colon \partial_G X \to \mathbb{R}$, there exists a continuous function $u\colon X^* \to \mathbb{R}$ which is p-harmonic in X and equal to f in $\partial_G X$. We refer to Chapter 4 for exact assumptions and the notation employed in the theorem. It is worth pointing out that the metric spaces above do not have, in general, a manifold structure not to mention a smooth structure. Therefore, p-harmonic functions can not, in general, be defined as solutions of an equation like (1.1) but rather as minimizers of a variational integral such as (1.2).

This paper can be divided roughly into two parts. In the first part of the paper (Section 3) we study in detail the Dirichlet problem at infinity for the p-Laplacian and the p-regularity of a point x_0 at infinity on a Cartan-Hadamard manifold M under a curvature assumption

$$-(b\circ\rho)^2 \le K_M \le -(a\circ\rho)^2$$

in $V \cap M$, where V is a neighborhood of x_0. Here $a, b\colon [0,\infty) \to [0,\infty),\ b \ge a$, are functions that will be specified later. In the second part of the paper we present basics on Gromov hyperbolic metric spaces and introduce upper gradients, Sobolev spaces, and p-harmonic functions on metric measure spaces. Then we describe briefly how to solve the asymptotic Dirichlet problem on Gromov hyperbolic metric measure spaces following [22].

2. Preliminaries

2.1. Notation. Let M be a connected Riemannian manifold. We denote the Hessian of a smooth function f by $D^2 f$. If $x \in M$, define

$$S_x M = \{v \in T_x M : |v| = 1\}$$

and $SM = \bigcup_{x \in M} S_x M$. If $x \in M$, let $\rho^x : M \to \mathbb{R}$ be the distance function $\rho^x(y) = d(x, y)$. The sectional curvature of a 2-plane $P \subset T_x M$ is denoted by $K_M(P)$. If $v \in TM$, γ^v is the unique maximal geodesic with $\dot{\gamma}_0^v = v$.

Suppose then that M is a Cartan-Hadamard manifold. We let $M(\infty)$ stand for its sphere at infinity and denote $\bar{M} = M \cup M(\infty)$. The sphere at infinity is defined as the set of all equivalence classes of geodesic unit speed rays in M; two such rays γ_1 and γ_2 are equivalent if

$$\sup_{t \geq 0} d(\gamma_1(t), \gamma_2(t)) < \infty.$$

The equivalence class of γ is denoted by $\gamma(\infty)$. For each $x \in M$ and $y \in \bar{M} \setminus \{x\}$ there exists a (unique) unit speed geodesic $\gamma^{x,y} : \mathbb{R} \to M$ such that $\gamma^{x,y}(0) = x$ and $\gamma^{x,y}(t) = y$ for some $t \in (0, \infty]$. If $x \in M$ and $y, z \in \bar{M} \setminus \{x\}$, then we denote by

$$\sphericalangle_x(y, z) = \sphericalangle(\dot{\gamma}_0^{x,y}, \dot{\gamma}_0^{x,z})$$

the angle between vectors $\dot{\gamma}_0^{x,y}$ and $\dot{\gamma}_0^{x,z}$ in $T_x M$. If $v \in T_x M \setminus \{0\}$, $\delta > 0$, and $r > 0$, then we define a cone

$$C(v, \delta) = \{y \in \bar{M} \setminus \{x\} : \sphericalangle(v, \dot{\gamma}_0^{x,y}) < \delta\}$$

and a truncated cone

$$T(v, \delta, r) = C(v, \delta) \setminus \bar{B}(x, r).$$

All cones and open balls in M form a basis for a topology on $\bar{M}$. This topology is called the cone topology and we always equip $\bar{M}$ with it. Then $\bar{M}$ is homeomorphic to the closed unit ball $\bar{B}^n \subset \mathbb{R}^n$ and $M(\infty)$ to the sphere $S^{n-1} = \partial B^n$; see [15] for a detailed study on the sphere at infinity and the cone topology.

Throughout the paper c is a positive constant, and $c(x, y, \ldots)$ denotes a positive constant depending on $x, y, \ldots$. The actual value of c may vary, even within a line.

2.2. Solutions of the differential equation $f'' = a^2 f$. In this subsection we consider the Jacobi equation $f'' = a^2 f$. We will later prove Proposition 2.5, which explains why we are interested in these solutions.

If $a : [0,\infty) \to [0,\infty)$ is a smooth function, let $f_a : [0,\infty) \to \mathbb{R}$ be the function determined by

$$\begin{cases} f_a(0) = 0, \\ f_a'(0) = 1, \\ \quad f_a'' = a^2 f_a. \end{cases}$$

Then $f_a \geq 0$ is a smooth function.

Example 2.1. Suppose that $\alpha > 0$ and $t_0 > 0$ are constants and that

$$a(t) = \frac{\alpha}{t}$$

for all $t \geq t_0$. It is easy to verify that then

$$f_a(t) = c_1 t^{\phi} + c_2 t^{1-\phi}$$

for all $t \geq t_0$, where

$$\phi = \frac{1 + \sqrt{1 + 4\alpha^2}}{2} > 1,$$

$$c_1 = t_0^{-\phi} \frac{f_a(t_0)(\phi - 1) + t_0 f_a'(t_0)}{2\phi - 1} > 0,$$

and

$$c_2 = t_0^{\phi-1} \frac{f_a(t_0)\phi - t_0 f_a'(t_0)}{2\phi - 1}.$$

In particular,

$$\lim_{t\to\infty} \frac{t f_a'(t)}{f_a(t)} = \phi. \tag{2.1}$$

Lemma 2.2. *Suppose that $a, b : [0,\infty) \to [0,\infty)$ are smooth and $a \leq b$. Then*

$$\frac{f_a'}{f_a} \leq \frac{f_b'}{f_b}.$$

In particular, $f_a \leq f_b$.

Proof. Since

$$(f_b' f_a - f_a' f_b)' = f_b'' f_a - f_a'' f_b = f_a f_b \Big(\frac{f_b''}{f_b} - \frac{f_a''}{f_a} \Big) = f_a f_b (b^2 - a^2) \geq 0,$$

we have $f_b' f_a - f_a' f_b \geq 0$. Therefore $f_b'/f_b \geq f_a'/f_a$. On the other hand,

$$\Big(\frac{f_b}{f_a} \Big)' = \frac{f_b' f_a - f_a' f_b}{f_a^2} \geq 0.$$

Now $\lim_{t\to 0} f_b(t)/f_a(t) = 1$ so that $f_b/f_a \geq 1$. ■

Lemma 2.3. *Suppose that $t_0 > 0$ is a constant, $b : [0,\infty) \to [0,\infty)$ is a smooth function such that $b(t) > 0$ for every $t \geq t_0$, and that*

$$\lim_{t\to\infty} \frac{b'(t)}{b(t)^2} = 0.$$

Then

$$\lim_{t\to\infty} \frac{f_b'(t)/f_b(t)}{b(t)} = 1.$$

Proof. Let $\varepsilon \in (0, 1/2)$. Choose $t_1 \geq t_0$ such that

$$|b'(t)| < \varepsilon\, b(t)^2$$

for all $t \geq t_1$. It follows that

$$b(t) \geq \frac{c}{t}$$

for all $t \geq t_1$. In particular,

$$\int_0^\infty b(t)\, dt = \infty. \tag{2.2}$$

If $k > \varepsilon$, let

$$g_k(t) = \exp\Big(k \int_0^t b(s)\, ds\Big).$$

Then

$$g_k'(t) = kb(t)g_k(t)$$

and

$$g_k''(t) = k\big(b'(t) + kb(t)^2\big)g_k(t),$$

so that

$$\frac{g_k''(t)}{g_k(t)} = kb'(t) + k^2 b(t)^2.$$

Therefore

$$(k^2 - \varepsilon k)b(t)^2 \leq \frac{g_k''(t)}{g_k(t)} \leq (k^2 + \varepsilon k)b(t)^2$$

for all $t \geq t_1$. In particular,

$$\frac{g_{1-\varepsilon}''(t)}{g_{1-\varepsilon}(t)} \leq b(t)^2 \leq \frac{g_{1+\varepsilon}''(t)}{g_{1+\varepsilon}(t)}$$

for every $t \geq t_1$. From this we see that

$$(f_b g_{1-\varepsilon}' - g_{1-\varepsilon} f_b')'(t) = f_b(t) g_{1-\varepsilon}''(t) - g_{1-\varepsilon}(t) f_b''(t) \leq 0$$

for all $t \geq t_1$. Hence

$$f_b(t) g_{1-\varepsilon}'(t) \leq c + g_{1-\varepsilon}(t) f_b'(t)$$

for all $t \geq t_1$. Therefore

$$\frac{g_{1-\varepsilon}'(t)/g_{1-\varepsilon}(t)}{f_b'(t)/f_b(t)} \leq 1 + \frac{c}{f_b'(t)g_{1-\varepsilon}(t)}$$

for all $t \geq t_1$. Since $g_{1-\varepsilon}(t) \to \infty$ as $t \to \infty$ by (2.2), we get

$$\limsup_{t\to\infty} \frac{b(t)}{f_b'(t)/f_b(t)} = \frac{1}{1-\varepsilon} \limsup_{t\to\infty} \frac{g_{1-\varepsilon}'(t)/g_{1-\varepsilon}(t)}{f_b'(t)/f_b(t)} \leq \frac{1}{1-\varepsilon}.$$

Similarly one gets

$$\liminf_{t\to\infty} \frac{b(t)}{f_b'(t)/f_b(t)} = \frac{1}{1+\varepsilon} \liminf_{t\to\infty} \frac{g_{1+\varepsilon}'(t)/g_{1+\varepsilon}(t)}{f_b'(t)/f_b(t)} \geq \frac{1}{1+\varepsilon}.$$

The claim follows by letting $\varepsilon \to 0$. ■

Lemma 2.4. *Let $t_0 > 0$ be a constant. Suppose that $a, b : [0,\infty) \to [0,\infty)$ are smooth functions that satisfy $a(t) = b(t)$ for every $t \geq t_0$. Then*

$$\lim_{t\to\infty} \frac{f_a'(t)/f_a(t)}{f_b'(t)/f_b(t)} = 1 \tag{2.3}$$

and there exists a constant $C > 0$ such that

$$\frac{1}{C} \leq \frac{f_a'(t)}{f_b'(t)} \leq C \tag{2.4}$$

for every $t \geq 0$.

Proof. If $t \geq t_0$, then

$$(f_a' f_b - f_b' f_a)'(t) = f_a''(t)f_b(t) - f_b''(t)f_a(t) = \big(a(t)^2 - b(t)^2\big)f_a(t)f_b(t) = 0.$$

Hence $f_a'(t)f_b(t) = c + f_b'(t)f_a(t)$ for every $t \geq t_0$. In particular,

$$\frac{f_a'(t)/f_a(t)}{f_b'(t)/f_b(t)} = 1 + \frac{c}{f_b'(t)f_a(t)} \to 1$$

when $t \to \infty$ as we claimed. On the other hand,

$$\Big(\frac{f_a}{f_b}\Big)'(t) = \frac{f_a'(t)f_b(t) - f_b'(t)f_a(t)}{f_b(t)^2} = \frac{c}{f_b(t)^2}$$

for all $t \geq t_0$. Therefore

$$\frac{f_a(t)}{f_b(t)} = \frac{f_a(t_0)}{f_b(t_0)} + \int_{t_0}^{t} \frac{c}{f_b(s)^2} ds \leq \frac{f_a(t_0)}{f_b(t_0)} + \int_{t_0}^{\infty} \frac{c}{s^2} ds = c < \infty$$

for every $t \geq t_0$. Since $\lim_{t\to 0} f_a(t)/f_b(t) = 1$, it follows that $f_a \leq c f_b$ everywhere. Similarly one shows that $f_b \leq c f_a$ everywhere. Hence $1/c \leq f_a/f_b \leq c$ everywhere. To get (2.4) we apply this to (2.3) that we already proved and use the fact that $f_a'(0) = 1 = f_b'(0)$. ■

2.3. Comparison results. Let $M \ni o$ be a Cartan-Hadamard n-manifold and denote $\rho = \rho^o$. Let $a, b : [0, \infty) \to [0, \infty)$ be smooth functions that are constant in some neighborhood of 0.

Proposition 2.5. *Suppose that $v \in S_oM$ is a unit vector and let $\gamma = \gamma^v$. Suppose that*

$$-b(t)^2 \le K_M(P) \le -a(t)^2$$

for every $t > 0$ and for every 2-dimensional subspace $P \subset T_{\gamma(t)}M$ that contains the radial vector $\dot\gamma_t$.

(a) *Let W be a Jacobi field along γ with $W_0 = 0$, $|W_0'| = 1$, and $W_0' \perp v$. Denote $f = |W|$. Then*

$$\frac{f_a'(t)}{f_a(t)} \le \frac{f'(t)}{f(t)} \le \frac{f_b'(t)}{f_b(t)}$$

for all $t > 0$. In particular, $f_a \le f \le f_b$.

(b) *Let $t > 0$ and $X \in T_{\gamma(t)}M$. Then*

$$\Big(\frac{f_a' \circ \rho}{f_a \circ \rho}(g - d\rho \otimes d\rho)\Big)(X, X) \le D^2\rho(X, X) \le \Big(\frac{f_b' \circ \rho}{f_b \circ \rho}(g - d\rho \otimes d\rho)\Big)(X, X).$$

Proof. We only prove the first inequality in (a) and (b) since the second ones are similar.

Let M_{-a^2} be $\mathbb{R}^n$ equipped with the Riemannian metric $dr^2 + f_a(r)^2 d\theta^2$, where r is the distance function from 0 and $d\theta^2$ is the standard metric on $\mathbb{S}^{n-1}$. Note that since a is constant in a neighborhood of 0, the metric $dr^2 + f_a(r)^2 d\theta^2$ extends smoothly over 0. Hence M_{-a^2} is a rotationally symmetric manifold with radial curvature function $-a^2$.

(a) Let $\tilde v \in T_0M_{-a^2}$, $|\tilde v| = 1$, and let $\tilde W$ be a Jacobi field along the unit speed geodesic $\gamma^{\tilde v}$ with $\tilde W_0 = 0$, $\tilde W_0' \perp \tilde v$, and $|\tilde W_0'| = 1$. Then

$$|\tilde W(t)| = f_a(t)$$

for every $t \ge 0$. Applying the Rauch comparison theorem to W and $\tilde W$ shows that $f_a'/f_a \le f'/f$.

(b) This follows from the Hessian comparison theorem [17, Theorem A] since

$$D^2 r = \frac{f_a' \circ r}{f_a \circ r}(\bar g - dr \otimes dr)$$

on $M_{-a^2} \setminus \{0\}$, where $\bar g$ is the Riemannian metric on M_{-a^2}. ■

Let $\psi : (0, \infty) \times S_oM \to M \setminus \{o\}$,

$$\psi(t, \xi) = \exp_o(t\xi).$$

Denote $\lambda = |J_\psi|$, the absolute value of the Jacobian of ψ. Then

$$m_M\big(B(o,t)\big) = \int_0^t \int_{S_oM} \lambda(s,\xi)\, d\xi\, ds$$

for every $t \geq 0$. For all $\xi \in S_oM$ we have

$$\lim_{t\to 0} \frac{\lambda(t,\xi)}{t^{n-1}} = 1. \tag{2.5}$$

Also,

$$\frac{\big(\lambda(\cdot,\xi)\big)'(t)}{\lambda(t,\xi)} = \Delta\rho\big(\psi(t,\xi)\big)$$

for all $t > 0$ and $\xi \in S_oM$. Therefore, in the situation of Proposition 2.5 we have

$$(n-1)\frac{f_a'(t)}{f_a(t)} \leq \frac{\big(\lambda(\cdot,v)\big)'(t)}{\lambda(t,v)} \leq (n-1)\frac{f_b'(t)}{f_b(t)}$$

for all $t > 0$ and in particular,

$$f_a(t)^{n-1} \leq \lambda(t,v) \leq f_b(t)^{n-1} \tag{2.6}$$

for all $t > 0$.

Lemma 2.6. *Let $x_0 \in M \setminus \{o\}$, $U = M \setminus \gamma^{o,x_0}(\mathbb{R})$, and define $\theta : U \to [0,\pi]$, $\theta(x) = \sphericalangle_o(x_0,x)$. Let $x \in U$ and $\gamma = \gamma^{o,x}$. Suppose that*

$$K_M(P) \leq -a(t)^2$$

for every $t > 0$ and for every 2-dimensional subspace $P \subset T_{\gamma(t)}M$ that contains the radial vector $\dot\gamma_t$. Then

$$|\nabla\theta(x)| \leq \frac{1}{(f_a\circ\rho)(x)}.$$

Proof. Let $\varphi : M \to \mathbb{R}^n$ be a normal coordinate chart at o and let $X \in S_xM$ be a unit vector. We want to prove that $|X\theta| \leq 1/f_a(\rho(x))$. If $X = \dot\gamma_{\rho(x)}$, then $X\theta = 0$ so without loss of generality we can assume that $X \perp \dot\gamma_{\rho(x)}$. Now

$$\theta = \tilde\theta \circ \varphi|U,$$

where $\tilde\theta : \varphi U \to [0,\pi]$,

$$\tilde\theta(z) = \sphericalangle_0\big(\varphi(x_0),z\big) = \arccos\Big(\frac{v\cdot z}{|z|}\Big),$$

and $v = \frac{\varphi(x_0)}{|\varphi(x_0)|}$. It is clear that $|\nabla\tilde\theta(z)| = |z|^{-1}$.

Let $w = \frac{\varphi(x)}{|\varphi(x)|}$ and define

$$W_t = (\varphi^{-1})_{*tw}\big(t\tfrac{\varphi_* X}{|\varphi_* X|}\big).$$

Then W is a Jacobi field along the geodesic γ and satisfies $W_0 = 0$, $|W_0'| = 1$, and $W_{\rho(x)} = \rho(x)\frac{X}{|\varphi_* X|}$. Since $W_0 = 0$ and $W_{\rho(x)} \perp \dot\gamma_{\rho(x)}$, $W \perp \dot\gamma$ everywhere. Proposition 2.5(a) now implies that

$$|X\theta| = |X(\tilde\theta \circ \varphi)| = \left|\nabla\tilde\theta\big(\varphi(x)\big)\cdot \varphi_* X\right| \le \frac{|\varphi_* X|}{\rho(x)} = \frac{1}{|W_{\rho(x)}|} \le \frac{1}{f_a\big(\rho(x)\big)}$$

as we wanted. ■

3. Dirichlet problem at infinity on Cartan-Hadamard manifolds

In this section we consider the Dirichlet problem at infinity on a Cartan-Hadamard manifold M. We use the simple approach taken by Anderson and Schoen in [7]. The same approach was used by Holopainen in [21] for p-harmonic functions. The idea is that given a continuous function on the boundary at infinity, we extend it radially to the whole $\bar M$. This extended function is then smoothened and it is shown that a slight perturbation of the smoothened function gives a p-superharmonic function. Starting with a suitable boundary function the resulting p-superharmonic function behaves like a barrier function and a point at infinity can be shown to be p-regular.

We apply Perron's method to solve the Dirichlet problem at infinity. Our definitions of the upper and lower Perron solutions follow [20]. Fix an exponent $p \in (1,\infty)$.

Definition 3.1. A lower semicontinuous function $u : U \to (-\infty,\infty]$, where $U \subset M$ is an open set, is *p-superharmonic* if $u \not\equiv \infty$ in each component of U, and for each open $D \subset\subset U$ and each $h \in C(\bar D)$, p-harmonic in D, $h \le u$ on ∂D implies $h \le u$ in D.

Definition 3.2. A function $u : M \to (-\infty,\infty]$ belongs to the *upper class* $\mathcal{U}_f$ of $f : M(\infty) \to [-\infty,\infty]$ if

(i) u is p-superharmonic in M,
(ii) u is bounded below, and
(iii) $\liminf_{x\to x_0} u(x) \ge f(x_0)$ for all $x_0 \in M(\infty)$.

We call

$$\overline{H}_f = \inf\{u : u \in \mathcal{U}_f\}$$

the *upper Perron solution.*

Theorem 3.3. *One of the following holds:*

(i) *$\overline{H}_f$ is p-harmonic in M,*

(ii) $\overline{H}_f \equiv \infty$ *in* M,
(iii) $\overline{H}_f \equiv -\infty$ *in* M.

Proof. As in [20, Theorem 9.2]. ∎

The upper Perron solution $\overline{H}_f$ is a good candidate to the solution of the Dirichlet problem at infinity.

Definition 3.4. A point $x_0 \in M(\infty)$ is *p-regular* if

$$\lim_{x \to x_0} \overline{H}_f(x) = f(x_0)$$

for every continuous $f : M(\infty) \to \mathbb{R}$.

Define the *lower class* $\mathcal{L}_f = -\mathcal{U}_{-f}$ and the *lower Perron solution* $\underline{H}_f = -\overline{H}_{-f}$. Then $\overline{H}_f \geq \underline{H}_f$.

Remark 3.5. If $x_1, x_2 \in M(\infty)$, $x_1 \neq x_2$, are both p-regular, then $\overline{H}_f$ is a non-constant bounded p-harmonic function on M whenever $f : M(\infty) \to \mathbb{R}$ is a continuous function such that $f(x_1) \neq f(x_2)$. The Dirichlet problem at infinity for p-harmonic functions is solvable if and only if all points at infinity are p-regular.

Assumptions. We will carry the following notation and assumptions with us until we reach Subsection 3.3. We fix a point $o \in M$ and write $\rho = \rho^o$ for the distance function $\rho(x) = d(x, o)$. Let

$$a, b \colon [0, \infty) \to [0, \infty), \ b \geq a,$$

be smooth functions that are constant in some neighborhood of 0. We suppose that $v_0 \in S_oM$ and $L \in (8/\pi, \infty)$ are given. For $k > 0$ we denote

$$\Omega = C(v_0, 1/L) \cap M$$

and

$$k\Omega = C(v_0, k/L) \cap M.$$

Suppose that

$$-(b \circ \rho)^2(x) \leq K_M(P) \leq -(a \circ \rho)^2(x)$$

for all $x \in 4\Omega$ and all 2-dimensional subspaces $P \subset T_xM$.

Suppose that b is monotonic, in other words it is increasing or decreasing. Suppose also that there exist constants $T_1 > 0$, $C_1, C_2, C_3 > 0$, and $Q \in (0, 1)$ such that

$$\text{(A1)} \qquad a(t) \begin{cases} = C_1 t^{-1} & \text{if } b \text{ is decreasing,} \\ \geq C_1 t^{-1} & \text{if } b \text{ is increasing} \end{cases}$$

for all $t \geq T_1$ and

$$a(t) \leq C_2, \tag{A2}$$

$$b(t+1) \leq C_2 b(t), \tag{A3}$$

$$b(t/2) \leq C_2 b(t), \tag{A4}$$

$$b(t) \geq C_3(1+t)^{-Q} \tag{A5}$$

for all $t \geq 0$. In addition, we assume that

$$\lim_{t\to\infty} \frac{b'(t)}{b(t)^2} = 0 \tag{A6}$$

and that there exists a constant $C_4 > 0$ such that

$$\lim_{t\to\infty} \frac{t^{1+C_4} b(t)}{f_a'(t)} = 0. \tag{A7}$$

We collect all these constants and functions together and denote

$$C = (a, b, T_1, C_1, C_2, C_3, C_4, Q, n, L).$$

Define $h : M(\infty) \to \mathbb{R}$,

$$h(x) = \min\Big(1, L\sphericalangle(v_0, \dot\gamma_0^{o,x})\Big). \tag{3.1}$$

Our aim is to extend h to a function $h \in C^\infty(M) \cap C(\bar M)$ with controlled first and second order derivatives. Let us first make a crude extension by defining $\tilde h : \bar M \to \mathbb{R}$,

$$\tilde h(x) = \min\Big(1, \max\big(2 - 2\rho(x), L\sphericalangle(v_0, \dot\gamma_0^{o,x})\big)\Big). \tag{3.2}$$

Then $\tilde h \in C(\bar M)$ and $\tilde h | M(\infty) = h$.

If V is an inner product space and $\alpha : V^2 \to \mathbb{R}$ is a symmetric 2-covariant tensor, denote

$$\|\alpha\| = \sup_{|X|\leq 1} |\alpha(X,X)|.$$

It is clear that $\|\cdot\|$ defines a norm in the set of all symmetric 2-covariant tensors of V. If α is a symmetric 2-covariant tensor field, we naturally define $\|\alpha\|_x = \|\alpha_x\|$.

Lemma 3.6. *(Cauchy-Schwarz inequality) If V is an inner product space, $\alpha : V^2 \to \mathbb{R}$ is a symmetric 2-covariant tensor, and $X, Y \in V$, then*

$$|\alpha(X,Y)| \leq \|\alpha\|\,|X||Y|.$$

Proof. By linearity we can assume that $|X| = 1 = |Y|$. Since α is symmetric, we have

$$\alpha(X,Y) = \frac{1}{4}\Big(\alpha(X+Y, X+Y) - \alpha(X-Y, X-Y)\Big).$$

Applying the triangle inequality gives

$$|\alpha(X,Y)| \le \frac{1}{4}\Big(\|\alpha\||X+Y|^2 + \|\alpha\||X-Y|^2\Big) = \frac{1}{2}\|\alpha\|\big(|X|^2+|Y|^2\big) = \|\alpha\|.$$

■

Lemma 3.7. *Suppose that N is a Riemannian manifold and $h_1, h_2 \in C^\infty(N)$. Then*

$$|\nabla\langle\nabla h_1, \nabla h_2\rangle| \le |\nabla h_1|\|D^2 h_2\| + |\nabla h_2|\|D^2 h_1\|.$$

Proof. Let $x \in N$. If $X \in T_x N$, then

$$\begin{aligned} X\big((\nabla h_1)h_2\big) &= D^2h_2(X,\nabla h_1) + \big(\nabla_X(\nabla h_1)\big)h_2 \\ &= D^2h_2(X,\nabla h_1) + \big\langle\nabla_X(\nabla h_1),\nabla h_2\big\rangle \\ &= D^2h_2(X,\nabla h_1) + D^2h_1(X,\nabla h_2). \end{aligned}$$

By applying the Cauchy-Schwarz inequality we get

$$\begin{aligned} |\nabla\langle\nabla h_1,\nabla h_2\rangle|_x &= \big|\nabla\big((\nabla h_1)h_2\big)\big|_x \\ &= \sup_{X\in S_xN}\big|X\big((\nabla h_1)h_2\big)\big| \\ &= \sup_{X\in S_xN}\Big|D^2h_2(X,\nabla h_1) + D^2h_1(X,\nabla h_2)\Big| \\ &\le |\nabla h_1|_x\|D^2h_2\|_x + |\nabla h_2|_x\|D^2h_1\|_x. \end{aligned}$$

Here $x \in N$ was arbitrary so the claim follows. ■

3.1. Extending the function h. In this subsection we extend h, defined in (3.1), to a function $h \in C(\bar{M}) \cap C^\infty(M)$ with controlled first and second order derivatives.

Lemma 3.8. *Let N be a Cartan-Hadamard manifold and $f : N \times N \to \mathbb{R}$ a function. Suppose that $f(\cdot, y) \in C^\infty(N)$ for all $y \in N$ and that*

$$(x,y) \mapsto X_m(X_{m-1}(\cdots(X_1(f(\cdot,y)))\cdots))(x)$$

is continuous for all smooth vector fields $X_i \in \mathcal{T}(N)$ and all $m \ge 0$. Suppose also that each $x_0 \in N$ has a neighborhood $V \ni x_0$ such that the set

$$\bigcup_{x\in V} \operatorname{supp} f(x,\cdot)$$

is bounded. Define $u : N \to \mathbb{R}$,

$$u(x) = \int_N f(x,y)\, dm_N(y).$$

Then $u \in C^\infty(N)$ *and*

$$Xu = \int_N X\big(f(\cdot, y)\big)\, dm_N(y) \tag{3.3}$$

for all $X \in TN$, *and*

$$D^2u(X,Y) = \int_N D^2\big(f(\cdot, y)\big)(X,Y)\, dm_N(y). \tag{3.4}$$

for all $X, Y \in T_xN$, $x \in N$.

Proof. Let us first consider the special case $N = \mathbb{R}^n$. Fix $x_0 \in \mathbb{R}^n$. The assumptions imply that there exists a compact set $K \subset \mathbb{R}^n$ such that $f(z, y) = 0$ whenever $z \in \bar{B}(x_0, 2)$ and $y \in \mathbb{R}^n \setminus K$. If $x \in B(x_0, 1)$, $X \in S_x\mathbb{R}^n = \mathbb{S}^{n-1}$, and $\sigma \in [-1, 1] \setminus \{0\}$, then

$$\begin{aligned}\frac{u(x+\sigma X) - u(x)}{\sigma} &= \int_{\mathbb{R}^n} \frac{f(x+\sigma X, y) - f(x,y)}{\sigma}\, dm(y) \\ &= \int_K \frac{f(x+\sigma X, y) - f(x,y)}{\sigma}\, dm(y).\end{aligned}$$

The integrand is uniformly bounded since

$$\sup_{\substack{x\in B(x_0,1),\ X\in\mathbb{S}^{n-1},\\ 0<|\sigma|\le 1,\ y\in K}} \left|\frac{f(x+\sigma X, y) - f(x,y)}{\sigma}\right| \le \sup_{y\in K,\, z\in \bar{B}(x,2)} \big|\nabla\big(f(\cdot, y)\big)(z)\big| < \infty.$$

Therefore we can use the dominated convergence theorem to conclude that the equation (3.3) holds if $|X| = 1$ and thus holds for all X. We also see that u is locally Lipschitz, therefore continuous. Now note that if $1 \le i \le n$, then the function $\bar{f} : (x, y) \mapsto \partial_i(f(\cdot, y))(x)$ satisfies the assumptions of the lemma. The above reasoning then shows that $\bar{u} = \partial_i u$ is continuous and satisfies (3.3). By repeating this argument we see that $u \in C^\infty(\mathbb{R}^n)$.

We return to the case of a general Cartan-Hadamard manifold N. Let $\varphi : \mathbb{R}^n \to N$ be a diffeomorphism. Then

$$u \circ \varphi = \int_N f(\varphi(\cdot), y)\, dm_N(y) = \int_{\mathbb{R}^n} |J_\varphi(y)| f(\varphi(\cdot), \varphi(y))\, dm(y).$$

The function $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $(x, y) \mapsto |J_\varphi(y)| f(\varphi(x), \varphi(y))$ satisfies the assumptions of the lemma. By what we already have shown in $\mathbb{R}^n$ we get that $u \circ \varphi$ is a smooth function. Therefore $u \in C^\infty(N)$. Fix $x \in N$, $X \in T_xN$, and denote

$Z = (\varphi^{-1})_* X$. Then

$$\begin{aligned} Xu &= Z(u\circ\varphi) \\ &= Z\int_{\mathbb{R}^n} |J_\varphi(y)| f(\varphi(\cdot),\varphi(y))\, dm(y) \\ &= \int_{\mathbb{R}^n} |J_\varphi(y)| Z\big(f(\varphi(\cdot),\varphi(y))\big)\, dm(y) \\ &= \int_N Z\big(f(\varphi(\cdot),y)\big)\, dm_N(y) \\ &= \int_N X\big(f(\cdot,y)\big)\, dm_N(y). \end{aligned}$$

Therefore (3.3) is valid.

We still have to show (3.4). Fix $x\in N$ and $X,Y\in T_xN$. Define $W\in\mathcal{T}(N)$ by $W=\sum_i\langle Y,(\partial_i)_x\rangle\partial_i$, where ∂_i is the i-th coordinate vector field of the normal coordinate chart at x. Then $W_x=Y$ and $\nabla_X W=0$ so that

$$D^2\eta(X,Y) = D^2\eta(X,W_x) = X(W\eta)-(\nabla_X W)\eta = X(W\eta) \tag{3.5}$$

for all $\eta\in C^\infty(N)$. Define $\bar f : N\times N\to\mathbb{R}$, $(z,y)\mapsto W_z\big(f(\cdot,y)\big)$. Then $\bar f$ satisfies the assumptions of the lemma. Using (3.5) and (3.3) we get

$$\begin{aligned} D^2u(X,Y) = X(Wu) &= X\int_N \bar f(\cdot,y)\, dm_N(y) \\ &= \int_N X\Big(W\big(f(\cdot,y)\big)\Big)\, dm_N(y) \\ &= \int_N D^2\big(f(\cdot,y)\big)(X,Y)\, dm_N(y). \end{aligned}$$

■

Lemma 3.9. *Let N be a Riemannian manifold, $h_1,h_2\in C^\infty(N)$, and $h_2(x)\neq 0$ for every $x\in N$. Then*

$$\begin{aligned} D^2\Big(\frac{h_1}{h_2}\Big) \;&=\; \frac{1}{h_2}D^2h_1-\frac{h_1}{h_2^2}D^2h_2+2\frac{h_1}{h_2^3}dh_2\otimes dh_2 \\ &\quad -\frac{1}{h_2^2}dh_2\otimes dh_1-\frac{1}{h_2^2}dh_1\otimes dh_2. \end{aligned}$$

Proof. This is a direct computation. Fix $X, Y \in \mathcal{T}(N)$. Then

$$\begin{aligned}
\nabla_X\big(\nabla(h_1/h_2)\big) &= \nabla_X\big((1/h_2)\nabla h_1\big) - \nabla_X\big((h_1/h_2^2)\nabla h_2\big) \\
&= X\Big(\frac{1}{h_2}\Big)\nabla h_1 + \frac{1}{h_2}\nabla_X(\nabla h_1) - X\Big(\frac{h_1}{h_2^2}\Big)\nabla h_2 - \frac{h_1}{h_2^2}\nabla_X(\nabla h_2) \\
&= -\frac{Xh_2}{h_2^2}\nabla h_1 + \frac{1}{h_2}\nabla_X(\nabla h_1) - \frac{h_2^2 Xh_1 - 2h_1h_2\,Xh_2}{h_2^4}\nabla h_2 \\
&\quad -\frac{h_1}{h_2^2}\nabla_X(\nabla h_2).
\end{aligned}$$

It follows that

$$\begin{aligned}
D^2\Big(\frac{h_1}{h_2}\Big)(X,Y) &= \big\langle \nabla_X\big(\nabla(h_1/h_2)\big), Y\big\rangle \\
&= -\frac{Xh_2}{h_2^2}Yh_1 + \frac{1}{h_2}D^2h_1(X,Y) - \frac{Xh_1}{h_2^2}Yh_2 \\
&\quad +\frac{2h_1\,Xh_2}{h_2^3}Yh_2 - \frac{h_1}{h_2^2}D^2h_2(X,Y).
\end{aligned}$$

■

The following lemma tells us that for given $k > 0$ and $x \in M$, the function $b \circ \rho$ does not change much in the set $\{y \in M : b(\rho(y))d(x,y) \le k\}$. Since $-(b\circ\rho)^2$ represents the curvature lower bound in 4Ω, this enables the use of comparison theorems.

Lemma 3.10. *Let $k > 0$. There exists a constant $c_{1,k} = c_{1,k}(C,k) > 1$ such that if $x, y \in M$ and $b(\rho(y))d(x,y) \le k$, then*

$$\frac{1}{c_{1,k}} b\big(\rho(x)\big) \le b\big(\rho(y)\big) \le c_{1,k}\, b\big(\rho(x)\big).$$

Proof. *Case b increasing:* Let $x, y \in M$ be such that $b(\rho(y))d(x,y) \le k$. Since b is increasing, we have $d(x,y) \le k/b(0)$. Hence (A3) implies

$$b(\rho(y)) \ge b\big(\max(0, \rho(x) - k/b(0))\big) \ge \frac{1}{c_{1,k}} b(\rho(x)),$$

where $c_{1,k} = C_2^{k/b(0)+1}$. Similarly, (A3) implies

$$b(\rho(y)) \le b\big(\rho(x) + k/b(0)\big) \le c_{1,k}\, b(\rho(x)).$$

Case b decreasing: Let $x, y \in M$ be such that $b(\rho(y))d(x,y) \le k$.

Since $Q \in (0,1)$, there exists a constant $c' = c'(C_3, k, Q)$ such that

$$t^Q \le \frac{C_3}{2k}(c' + t) \tag{3.6}$$

for all $t \geq 0$. Now (A5) and (3.6) imply

$$\begin{aligned} d(x,y) &\leq \frac{k}{b(\rho(y))} \leq \frac{k}{C_3}\big(1+\rho(y)\big)^Q \\ &\leq \frac{k}{C_3}\big(1+\rho(x)+d(x,y)\big)^Q \\ &\leq \frac{k}{C_3}\Big(\big(1+\rho(x)\big)^Q + d(x,y)^Q\Big) \\ &\leq \frac{k}{C_3}\Big(\big(1+\rho(x)\big)^Q + \frac{C_3}{2k}\big(c'+d(x,y)\big)\Big) \end{aligned}$$

so that

$$d(x,y) \leq c' + \frac{2k}{C_3}\big(1+\rho(x)\big)^Q. \tag{3.7}$$

Again since $Q \in (0,1)$, there exists a constant $R = R(Q, C_3, k)$ such that

$$c' + \frac{2k}{C_3}\big(1+t\big)^Q \leq \frac{t}{2} \tag{3.8}$$

for all $t \geq R$. So, if $\rho(x) \geq R$, then $d(x,y) \leq \rho(x)/2$ by (3.7) and (3.8) and therefore

$$b\big(\rho(y)\big) \geq b\big(\rho(x)+d(x,y)\big) \geq b\big(2\rho(x)\big) \geq \frac{1}{C_2} b\big(\rho(x)\big),$$

where the last inequality follows from (A4). Similarly, if $\rho(x) \geq R$, then

$$b\big(\rho(y)\big) \leq b\big(\rho(x)-d(x,y)\big) \leq b\big(\rho(x)/2\big) \leq C_2\, b\big(\rho(x)\big).$$

This is what we wanted but with the extra assumption $\rho(x) \geq R$. Suppose now that $\rho(x) \leq R$. Then (3.7) and (3.8) imply

$$d(x,y) \leq c' + \frac{2k}{C_3}\big(1+\rho(x)\big)^Q \leq c' + \frac{2k}{C_3}(1+R)^Q \leq \frac{R}{2}.$$

In particular, $\rho(y) \leq 2R$. Since $b > 0$ is continuous, there exists a constant $\beta = \beta(b, R)$ such that

$$\frac{1}{\beta} \leq b(t) \leq \beta$$

for all $t \in [0, 2R]$. Then

$$\frac{1}{\beta^2} \leq \frac{b\big(\rho(y)\big)}{b\big(\rho(x)\big)} \leq \beta^2.$$

By combining the cases $\rho(x) \geq R$ and $\rho(x) \leq R$ we see that the claim holds with $c_{1,k} = \max\{C_2, \beta^2\}$. ∎

Lemma 3.11. *For each $k > 0$ there exists a constant $c_2 = c_2(C, k)$ such that*

$$m_M\Big(B\Big(x, \frac{k}{b(\rho(x))}\Big)\Big) \le \frac{c_2}{b(\rho(x))^n}$$

holds for all $x \in M$ that satisfy $B\big(x, k/b(\rho(x))\big) \subset 4\Omega$.

Proof. Fix $k > 0$ and $x \in M$ that satisfies $B_x \subset 4\Omega$, where $B_x = B\big(x, k/b(\rho(x))\big)$. Let $y \in B_x$. Then $b(\rho(x))d(x, y) < k$. Lemma 3.10 (applied with x and y interchanged) then implies that

$$b(\rho(y)) \le c_{1,k}\, b(\rho(x)) =: \beta.$$

Since $B_x \subset 4\Omega$ and the sectional curvature of M is bounded from below by $-(b \circ \rho)^2$ in 4Ω, we see that the sectional curvature of M is bounded from below by $-\beta^2$ in the ball B_x. Denote $N = M^n_{-\beta^2}$, the model space with constant sectional curvature $-\beta^2$, and fix any $x_N \in N$. We use the Bishop-Gromov volume comparison theorem and (2.6) to obtain

$$\begin{aligned}
m_M(B_x) &\le m_N\Big(B\Big(x_N, \frac{k}{b(\rho(x))}\Big)\Big) \\
&= \int_0^{k/b(\rho(x))} \int_{S_{x_N}N} \big(\beta^{-1}\sinh(\beta t)\big)^{n-1}\, d\xi\, dt \\
&= \omega_{n-1}\beta^{1-n} \int_0^{k/b(\rho(x))} \sinh^{n-1}(\beta t)\, dt \\
&= \omega_{n-1}\beta^{-n} \int_0^{\beta k/b(\rho(x))} \sinh^{n-1}(s)\, ds \\
&= \omega_{n-1} \frac{1}{c_{1,k}^n b(\rho(x))^n} \int_0^{kc_{1,k}} \sinh^{n-1}(s)\, ds.
\end{aligned}$$

From this we see that the claim holds with the constant

$$c_2 = \omega_{n-1} c_{1,k}^{-n} \int_0^{kc_{1,k}} \sinh^{n-1}(s)\, ds.$$

■

Fix $\chi \in C^\infty(\mathbb{R})$ such that $0 \le \chi \le 1$, $\operatorname{supp}\chi \subset [-2, 2]$, and $\chi|[-1, 1] \equiv 1$.

Lemma 3.12. *If $\varphi \in C(M)$, then the function $f : M \times M \to \mathbb{R}$,*

$$f(x, y) = \chi\big(b(\rho(y))d(x, y)\big)\varphi(y),$$

satisfies the assumptions of Lemma 3.8.

Proof. Let $X_1, X_2, \ldots$ be smooth vector fields on M. We prove by induction with respect to m that the function $(x,y) \mapsto X_m(X_{m-1}(\cdots(X_1(f(\cdot,y)))\cdots))(x)$ can be written as a finite sum of functions of the form

$$(x,y) \mapsto \chi^{(k)}\big(b(\rho(y))d(x,y)\big)b(\rho(y))^k\varphi(y)u(x,y), \tag{3.9}$$

where $k \in \mathbb{N}$ and $u \in C^\infty(M \times M)$.

The basic case $m = 0$ is trivial.

Suppose that the claim holds for $m = m_0 - 1$. We show that it holds for $m = m_0$. Let η,

$$\eta(x,y) = \chi^{(k)}\big(b(\rho(y))d(x,y)\big)b(\rho(y))^k\varphi(y)u(x,y),$$

be one of the functions in the finite sum corresponding to the function

$$(x,y) \mapsto X_{m_0-1}(X_{m_0-2}(\cdots(X_1(f(\cdot,y)))\cdots))(x).$$

We have to show that $(x,y) \mapsto X_{m_0}(\eta(\cdot,y))(x)$ can be written as a finite sum of functions of the form (3.9). If $x \neq y$, then

$$\begin{aligned} X_{m_0}(\eta(\cdot,y))(x) &= \overbrace{\chi^{(k)}\big(b(\rho(y))d(x,y)\big)b(\rho(y))^k\varphi(y)\,X_{m_0}\big(u(\cdot,y)\big)(x)}^{\eta_1(x,y):=} \\ &\quad + \underbrace{\chi^{(k+1)}\big(b(\rho(y))d(x,y)\big)b(\rho(y))^{k+1}\varphi(y)u(x,y)\,X_{m_0}\big(d(\cdot,y)\big)(x)}_{=:\eta_2(x,y)}. \end{aligned}$$

The function η_1 is of the form (3.9). The problem with η_2 is that the metric $d : M \times M \to \mathbb{R}$ is not smooth on the diagonal. This is no real problem since if $b(\rho(y))d(x,y) \leq 1$, then $\chi^{(k+1)}\big(b(\rho(y))d(x,y)\big) = 0$ and thus

$$\eta_2(x,y) = \chi^{(k+1)}\big(b(\rho(y))d(x,y)\big)b(\rho(y))^{k+1}\varphi(y)\big(u\,\bar{u}\big)(x,y),$$

where $\bar{u} : M \times M \to \mathbb{R}$,

$$\bar{u}(x,y) = \begin{cases} \Big(1 - \chi\big(2\,b(\rho(y))^2 d(x,y)^2\big)\Big)\,X_{m_0}\big(d(\cdot,y)\big)(x) & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

It is easy to see that $\bar{u} \in C^\infty(M \times M)$, so that η_2 is of the form (3.9). This ends the induction proof. Every function of the form (3.9) is continuous, so the first assumption in Lemma 3.8 is verified.

Let $x_0 \in M$ be arbitrary. Then $V = B(o, \rho(x_0) + 1)$ is a neighborhood of x_0. Let $x \in V$ and $y \in M$ be such that $f(x,y) \neq 0$. Then $b(\rho(y))d(x,y) < 2$. Denote $\beta = \inf_{0 \leq t \leq 1+\rho(x_0)} b(t) > 0$. By Lemma 3.10 we have

$$\beta \leq b(\rho(x)) \leq c_{1,2}b(\rho(y))$$

so that

$$d(x,y) < \frac{2}{b(\rho(y))} \le \frac{2c_{1,2}}{\beta}.$$

Therefore

$$\bigcup_{x\in V} \operatorname{supp} f(x,\cdot) \subset \bar{B}\Big(o, \rho(x_0) + 1 + 2c_{1,2}\beta^{-1}\Big)$$

is a bounded set. ■

Let $\varphi \in C(M)$ and f be as in Lemma 3.12. Then we can define

$$\mathcal{R}(\varphi)(x) = \int_M f(x,y)\, dm_M(y)$$

by Lemma 3.8 and Lemma 3.12. Since $\mathcal{R}(1) > 0$, we can also define $\mathcal{P}(\varphi) : M \to \mathbb{R}$ by

$$\mathcal{P}(\varphi) = \frac{\mathcal{R}(\varphi)}{\mathcal{R}(1)}.$$

Then $\mathcal{P}$ is linear: if $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\varphi_1, \varphi_2 \in C(M)$, then

$$\mathcal{P}(\lambda_1\varphi_1 + \lambda_2\varphi_2) = \lambda_1\mathcal{P}(\varphi_1) + \lambda_2\mathcal{P}(\varphi_2).$$

Also, if $k : M \to \mathbb{R}$ is a constant function, then

$$\mathcal{P}(k) = k.$$

Lemma 3.13. *Suppose that $\varphi \in C(\bar{M})$. Extend the function $\mathcal{P}(\varphi) : M \to \mathbb{R}$ to a function $\bar{M} \to \mathbb{R}$ by setting*

$$\mathcal{P}(\varphi)(\bar{x}) = \varphi(\bar{x})$$

whenever $\bar{x} \in M(\infty)$. Then the extended function satisfies

$$\mathcal{P}(\varphi) \in C^\infty(M) \cap C(\bar{M}).$$

Proof. We already know that $\mathcal{P}(\varphi) \in C^\infty(M)$ by Lemma 3.8. It is therefore enough to show continuity at infinity. Fix $\bar{x} \in M(\infty)$ and $\varepsilon > 0$. Since φ is continuous at $\bar{x}$, there exist $\delta \in (0,1)$ and $R > 0$ such that $|\varphi(x) - \varphi(\bar{x})| < \varepsilon$ for every $x \in T(\dot{\gamma}_0^{o,\bar{x}}, \delta, R)$.

Since $Q \in (0,1)$, we can choose $R' > 3R/2$ such that

$$\frac{2c_{1,2}}{C_3}(1+t)^Q \le \frac{\delta}{3}t$$

for all $t \ge R'$. Let $x \in M\setminus B(o,R')$ and let $y \in M$ be such that $b(\rho(y))d(x,y) \le 2$. Lemma 3.10 and (A5) imply

$$d(x,y) \le \frac{2}{b(\rho(y))} \le \frac{2c_{1,2}}{b(\rho(x))} \le \frac{2c_{1,2}}{C_3}\big(1+\rho(x)\big)^Q \le \frac{\delta}{3}\rho(x). \tag{3.10}$$

Lemma 2.6 (applied with $a \equiv 0$) together with (3.10) imply that

$$\sphericalangle_o(x,y) \le \frac{d(x,y)}{\rho(x)-d(x,y)} \le \delta/2.$$

Also by (3.10),

$$\rho(y) \ge \rho(x) - d(x,y) \ge \frac{2\rho(x)}{3} \ge \frac{2R'}{3} > R.$$

So, if $x, y \in M$ are such that $\rho(x) \ge R'$, $\sphericalangle_o(x,\bar{x}) < \delta/2$, and $b(\rho(y))d(x,y) \le 2$, then

$$\sphericalangle_o(y,\bar{x}) \le \sphericalangle_o(y,x) + \sphericalangle_o(x,\bar{x}) < \delta$$

and $\rho(y) > R$, therefore $y \in T(\dot{\gamma}_0^{o,\bar{x}}, \delta, R)$ and thus $|\varphi(y)-\varphi(\bar{x})| < \varepsilon$. This implies that if $x \in T(\dot{\gamma}_0^{o,\bar{x}}, \delta/2, R') \cap M$, then

$$\begin{aligned}
|\mathcal{P}(\varphi)(x) - \mathcal{P}(\varphi)(\bar{x})| &= |\mathcal{P}(\varphi)(x) - \varphi(\bar{x})| = \big|\mathcal{P}\big(\varphi - \varphi(\bar{x})\big)(x)\big| \\
&= \left| \frac{\int_M \chi\big(b(\rho(y))d(x,y)\big)\big(\varphi(y)-\varphi(\bar{x})\big)\, dm_M(y)}{\int_M \chi\big(b(\rho(y))d(x,y)\big)\, dm_M(y)} \right| \\
&\le \frac{\int_M \chi\big(b(\rho(y))d(x,y)\big)\big|\varphi(y)-\varphi(\bar{x})\big|\, dm_M(y)}{\int_M \chi\big(b(\rho(y))d(x,y)\big)\, dm_M(y)} \\
&\le \sup_{y\in M,\ b(\rho(y))d(x,y)<2} |\varphi(y)-\varphi(\bar{x})| \le \varepsilon.
\end{aligned}$$

This shows that $\mathcal{P}(\varphi)$ is continuous at $\bar{x}$. ■

Lemma 3.14. *Let $\varphi \in C(M)$. Let $x \in M$ be such that $B(x, 2c_{1,2}/b(\rho(x))) \subset 4\Omega$ and let $X \in S_xM$. Then*

$$|\mathcal{R}(\varphi)(x)| \le c_3 b(\rho(x))^{-n} \sup_{y\in B(x,2c_{1,2}/b(\rho(x)))} |\varphi(y)|, \tag{3.11}$$

$$\big|X\big(\mathcal{R}(\varphi)\big)\big| \le c_3 b(\rho(x))^{1-n} \sup_{y\in B(x,2c_{1,2}/b(\rho(x)))} |\varphi(y)|, \tag{3.12}$$

and

$$\big|D^2\big(\mathcal{R}(\varphi)\big)(X,X)\big| \le c_3 b(\rho(x))^{2-n} \sup_{y\in B(x,2c_{1,2}/b(\rho(x)))} |\varphi(y)|. \tag{3.13}$$

Also,

$$\mathcal{R}(1)(x) \ge c_3^{-1} b(\rho(x))^{-n}. \tag{3.14}$$

Here $c_3 = c_3(C)$ is a constant.

Proof. To shorten our notation we denote $B_x = B\big(x, 2c_{1,2}/b(\rho(x))\big)$. Then

$$m_M(B_x) \leq cb(\rho(x))^{-n} \tag{3.15}$$

by Lemma 3.11. By Lemma 3.10 and Lemma 3.8 we can write

$$\mathcal{R}(\varphi)(x) = \int_{B_x} \chi\big(b(\rho(y))d(x,y)\big)\varphi(y)\,dm_M(y),$$

$$X\big(\mathcal{R}(\varphi)\big) = \int_{B_x} X\big(f(\cdot,y)\big)\,dm_M(y),$$

and

$$D^2\big(\mathcal{R}(\varphi)\big)(X,X) = \int_{B_x} D^2\big(f(\cdot,y)\big)(X,X)\,dm_M(y).$$

Here $X\big(f(\cdot,y)\big)$ and $D^2\big(f(\cdot,y)\big)(X,X)$ can be computed using the formulas

$$\nabla\big(f(\cdot,y)\big)(x) = \varphi(y)b(\rho(y))\chi'\big(b(\rho(y))d(x,y)\big)\nabla\rho^y(x) \tag{3.16}$$

and

$$\begin{aligned} &D^2\big(f(\cdot,y)\big)(x) \\ &= \varphi(y)b(\rho(y))\Big(b(\rho(y))\chi''\big(b(\rho(y))d(x,y)\big)(d\rho^y)_x \otimes (d\rho^y)_x \\ &\quad + \chi'\big(b(\rho(y))d(x,y)\big)D^2\rho^y(x)\Big). \end{aligned} \tag{3.17}$$

We get (3.11) from (3.15) by estimating

$$\begin{aligned} |\mathcal{R}(\varphi)(x)| &= \left|\int_{B_x} \chi(b(\rho(y))d(x,y))\varphi(y)\,dm_M(y)\right| \\ &\leq m_M(B_x) \sup_{y\in B_x} |\varphi(y)| \\ &\leq c\,b(\rho(x))^{-n} \sup_{y\in B_x} |\varphi(y)|. \end{aligned}$$

Suppose that $y \in M$ is such that $b(\rho(x))d(x,y) \leq \min\{2, c_{1,2}^{-1}\}$. Then Lemma 3.10 implies that

$$\frac{1}{c_{1,2}} b(\rho(x)) \leq b(\rho(y)) \leq c_{1,2} b(\rho(x)).$$

Therefore $b(\rho(y))d(x,y) \le 1$. This and the Bishop-Gromov volume comparison theorem imply that

$$\begin{aligned}|\mathcal{R}(1)(x)| &= \left|\int_M \chi(b(\rho(y))d(x,y))\, dm_M(y)\right| \\ &\ge m_M\Big(B\Big(x, \frac{\min\{2, c_{1,2}^{-1}\}}{b(\rho(x))}\Big)\Big) \\ &\ge m_{\mathbb{R}^n}\Big(B\Big(0, \frac{\min\{2, c_{1,2}^{-1}\}}{b(\rho(x))}\Big)\Big) \\ &= c\, b(\rho(x))^{-n}.\end{aligned}$$

This shows that (3.14) holds.

Inequality (3.12) follows from (3.16), Lemma 3.10, and (3.15) since

$$\begin{aligned}\big|X\big(\mathcal{R}(\varphi)\big)\big| &= \Big|\int_{B_x} X\big(f(\cdot,y)\big)\, dm_M(y)\Big| \\ &= \Big|\int_{B_x} \varphi(y)b(\rho(y))\chi'(b(\rho(y))d(x,y))(X\rho^y)\, dm_M(y)\Big| \\ &\le c\, b(\rho(x))\, m_M(B_x) \sup_{y\in B_x} |\varphi(y)| \\ &\le c\, b(\rho(x))^{1-n} \sup_{y\in B_x} |\varphi(y)|.\end{aligned}$$

Suppose that $y \in M$ is such that $1 \le b(\rho(y))d(x,y) \le 2$. If z lies on the geodesic segment between the points x and y, then $b(\rho(y))d(z,y) \le 2$ and thus

$$b(\rho(z)) \le c_{1,2}b(\rho(y)) \le c_{1,2}^2 b(\rho(x))$$

by Lemma 3.10. The Hessian comparison theorem [17, Theorem A] and Lemma 3.10 then give us

$$\begin{aligned}0 \le D^2\rho^y(X,X) &\le c_{1,2}^2\, b(\rho(x)) \coth\big(c_{1,2}^2\, b(\rho(x))d(x,y)\big) \\ &\le c_{1,2}^2\, b(\rho(x)) \coth\big(c_{1,2}\, b(\rho(y))d(x,y)\big) \\ &\le c_{1,2}^2 \coth(c_{1,2})\, b(\rho(x)).\end{aligned}$$

This computation along with (3.17) and (3.15) are used to estimate

$$\begin{aligned}\big|D^2\big(\mathcal{R}(\varphi)\big)(X,X)\big| &= \Big|\int_{B_x} D^2\big(f(\cdot,y)\big)(X,X)\,dm_M(y)\Big| \\ &= \Big|\int_{B_x} \varphi(y)b(\rho(y))\Big(b(\rho(y))\chi''\big(b(\rho(y))d(x,y)\big)(X\rho^y)^2 \\ &\quad + \chi'\big(b(\rho(y))d(x,y)\big)D^2\rho^y(X,X)\Big)\,dm_M(y)\Big| \\ &\le c\,b(\rho(x))^2\,m_M(B_x)\sup_{y\in B_x}|\varphi(y)| \\ &\le c\,b(\rho(x))^{2-n}\sup_{y\in B_x}|\varphi(y)|.\end{aligned}$$

This gives us (3.13) and ends the proof. ∎

Lemma 3.15. *There exist constants $R_1 = R_1(C)$ and $c_4 = c_4(C)$ such that the following hold.*

(a) *If $x \in 3\Omega \setminus B(o, R_1)$ and $y \in B(x, 2c_{1,2}/b(\rho(x)))$, then*

$$\sphericalangle_o(x,y) \le \frac{c_4}{b(\rho(x))f_a(\rho(x))} \tag{3.18}$$

and

$$y \in 4\Omega \setminus B(o,1).$$

(b) *If $x \in M \setminus \big(2\Omega \cup B(o,R_1)\big)$, then $B(x, 2c_{1,2}/b(\rho(x))) \subset M \setminus \big(\Omega \cup B(o,1)\big)$.*

Proof. Suppose that $x \in M$ and $y \in B(x, 2c_{1,2}/b(\rho(x)))$. Since $Q \in (0,1)$, we can choose $R > 2$ such that

$$\frac{2c_{1,2}}{C_3}(1+t)^Q \le \frac{t}{2L}$$

for all $t \ge R$. Suppose that $\rho(x) \ge R$. Then (A5) implies

$$d(x,y) \le \frac{2c_{1,2}}{b(\rho(x))} \le \frac{2c_{1,2}}{C_3}\big(1+\rho(x)\big)^Q \le \frac{\rho(x)}{2L} \tag{3.19}$$

and thus $\rho(y) \ge \rho(x) - d(x,y) \ge \rho(x)/2 > 1$. Also, Lemma 2.6 (applied with $a \equiv 0$) gives

$$\sphericalangle_o(x,y) \le \frac{d(x,y)}{\rho(x) - d(x,y)} \le \frac{1/(2L)}{1 - 1/(2L)} < \frac{1}{L}.$$

From this we see that if $x \notin 2\Omega \cup B(o,R)$, then $y \notin \Omega \cup B(o,1)$. This proves (b). Also, if $x \in 3\Omega \setminus B(o,R)$, then $y \in 4\Omega \setminus B(o,1)$.

Suppose now that $x \in 3\Omega$ and $\rho(x) \ge R$. We know by above that then $B(x, 2c_{1,2}/b(\rho(x))) \subset 4\Omega \setminus B(o,1)$. We are left to verify the equation (3.18) in (a).

Suppose first that b is increasing and denote $r_1 = 2c_{1,2}/b(0) > 0$. Then

$$d(x,y) \le \frac{2c_{1,2}}{b(\rho(x))} \le r_1.$$

Since $a \le C_2$ by (A2), the function f_a/f_{C_2} is decreasing by Lemma 2.2. In particular,

$$\frac{f_a(\rho(x))}{f_{C_2}(\rho(x))} \le \frac{f_a(\rho(x)-r_1)}{f_{C_2}(\rho(x)-r_1)}.$$

Therefore

$$\begin{aligned} f_a(\rho(x)-d(x,y)) \ge f_a(\rho(x)-r_1) &\ge \frac{f_{C_2}(\rho(x)-r_1)}{f_{C_2}(\rho(x))} f_a(\rho(x)) \\ &= \frac{\sinh\big(C_2(\rho(x)-r_1)\big)}{\sinh\big(C_2\rho(x)\big)} f_a(\rho(x)). \end{aligned}$$

From this we see that if $\rho(x)$ is large enough, then

$$f_a(\rho(x)-d(x,y)) \ge c f_a(\rho(x)).$$

Suppose then that b is decreasing. Then $a(t) = C_1 t^{-1}$ for all $t \ge T_1$ by assumption (A1). By Example 2.1 we see that there exist constants $\phi > 1$, $c_1' > 0$, and $c_2' \in \mathbb{R}$ such that

$$f_a(t) = c_1' t^{\phi} + c_2' t^{1-\phi}$$

for all $t \ge T_1$. This implies that

$$f_a(t/2) \ge 2^{-1-\phi} f_a(t)$$

for all large t. Hence in this case (3.19) implies

$$f_a\big(\rho(x)-d(x,y)\big) \ge f_a(\rho(x)/2) \ge c f_a(\rho(x))$$

whenever $\rho(x)$ is large enough.

We have proved that regardless of whether b is increasing or decreasing, we get

$$f_a(\rho(x)-d(x,y)) \ge c f_a(\rho(x))$$

if $\rho(x)$ is large enough. Since we have proved that $B(x, 2c_{1,2}/b(\rho(x))) \subset 4\Omega$, Lemma 2.6 gives

$$\sphericalangle_o(x,y) \le \frac{d(x,y)}{f_a\big(\rho(x)-d(x,y)\big)} \le c\frac{d(x,y)}{f_a(\rho(x))} \le \frac{c}{b(\rho(x)) f_a(\rho(x))}.$$

This shows (3.18) and ends the proof. ∎

We extend $h : M(\infty) \to \mathbb{R}$, defined by (3.1), to a function $h : \bar{M} \to \mathbb{R}$ by setting

$$h(x) = \mathcal{P}(\tilde{h})(x), \quad x \in M,$$

where $\tilde{h}$ is the function given by (3.2). We already know by Lemma 3.13 that $h \in C^\infty(M) \cap C(\bar{M})$ and are now ready to estimate its derivatives.

Lemma 3.16. *(Main lemma) The extended function $h \in C^\infty(M) \cap C(\bar{M})$ satisfies*

$$\begin{aligned} |\nabla h(x)| &\le c_5 \frac{1}{(f_a \circ \rho)(x)}, \\ \|D^2 h(x)\| &\le c_5 \frac{(b \circ \rho)(x)}{(f_a \circ \rho)(x)}, \end{aligned} \tag{3.20}$$

for all $x \in 3\Omega \setminus B(o, R_1)$. In addition,

$$h(x) = 1$$

for every $x \in M \setminus \big(2\Omega \cup B(o, R_1)\big)$. Here $R_1 = R_1(C)$ is the constant in Lemma 3.15 and $c_5 = c_5(C)$ is a constant.

Proof. Suppose first that $x \in M \setminus \big(2\Omega \cup B(o, R_1)\big)$. Then

$$B(x, 2c_{1,2}b(\rho(x))) \subset M \setminus \big(\Omega \cup B(o, 1)\big)$$

by Lemma 3.15(b) and hence

$$\begin{aligned} \mathcal{P}(\tilde{h})(x) &= \frac{1}{\mathcal{R}(1)(x)} \int_{B(x, 2c_{1,2}/b(\rho(x)))} \chi\big(d(x,y)b(\rho(y))\big) \underbrace{\tilde{h}(y)}_{=1} \, dm_M(y) \\ &= \frac{\mathcal{R}(1)(x)}{\mathcal{R}(1)(x)} = 1. \end{aligned}$$

We are left to verify (3.20).

Fix $x \in 3\Omega \setminus B(o, R_1)$. Denote $\varphi = \tilde{h} - \tilde{h}(x)$. By Lemma 3.15(a)

$$B(x, 2c_{1,2}/b(\rho(x))) \subset 4\Omega \setminus B(o, 1)$$

and

$$\begin{aligned} \sup_{y \in B(x, 2c_{1,2}/b(\rho(x)))} |\varphi(y)| &\le L \sup_{y \in B(x, 2c_{1,2}/b(\rho(x)))} |\sphericalangle(v_0, \dot{\gamma}_0^{o,y}) - \sphericalangle(v_0, \dot{\gamma}_0^{o,x})| \\ &\le L \sup_{y \in B(x, 2c_{1,2}/b(\rho(x)))} |\sphericalangle_o(x, y)| \\ &\le \frac{c}{b(\rho(x)) f_a(\rho(x))}. \end{aligned} \tag{3.21}$$

Let $X \in S_xM$ be a unit vector. Then

$$X\big(\mathcal{P}(\varphi)\big) = \frac{1}{\mathcal{R}(1)} X\big(\mathcal{R}(\varphi)\big) - \frac{\mathcal{R}(\varphi)}{\big(\mathcal{R}(1)\big)^2} X\big(\mathcal{R}(1)\big) \tag{3.22}$$

and

(3.23)

$$\begin{aligned} D^2\big(\mathcal{P}(\varphi)\big)(X,X) = {} & \frac{1}{\mathcal{R}(1)} D^2\big(\mathcal{R}(\varphi)\big)(X,X) - \frac{\mathcal{R}(\varphi)}{\big(\mathcal{R}(1)\big)^2} D^2\big(\mathcal{R}(1)\big)(X,X) \\ & + 2\frac{\mathcal{R}(\varphi)}{\big(\mathcal{R}(1)\big)^3}\Big(X\big(\mathcal{R}(1)\big)\Big)^2 - 2\frac{1}{\big(\mathcal{R}(1)\big)^2} X\big(\mathcal{R}(1)\big)\, X\big(\mathcal{R}(\varphi)\big) \end{aligned}$$

by Lemma 3.9. We use (3.22), Lemma 3.14, and (3.21) to estimate

$$\begin{aligned} \big|X\big(\mathcal{P}(\tilde h)\big)\big| &= \big|X\big(\mathcal{P}(\tilde h) - \tilde h(x)\big)\big| = \big|X\big(\mathcal{P}(\varphi)\big)\big| \\ &\le \Big|\frac{1}{\mathcal{R}(1)} X\big(\mathcal{R}(\varphi)\big)\Big| + \Big|\frac{\mathcal{R}(\varphi)}{\big(\mathcal{R}(1)\big)^2} X\big(\mathcal{R}(1)\big)\Big| \\ &\le c\, b(\rho(x)) \sup_{y\in B(x,2c_{1,2}/b(\rho(x)))} |\varphi(y)| \\ &\le \frac{c}{f_a\big(\rho(x)\big)}. \end{aligned}$$

Similarly, using (3.23), Lemma 3.14, and (3.21) we get

$$\begin{aligned} \big|D^2\big(\mathcal{P}(\tilde h)\big)(X,X)\big| &= \big|D^2\big(\mathcal{P}(\tilde h) - \tilde h(x)\big)(X,X)\big| \\ &= \big|D^2\big(\mathcal{P}(\varphi)\big)(X,X)\big| \\ &\le \Big|\frac{1}{\mathcal{R}(1)} D^2\big(\mathcal{R}(\varphi)\big)(X,X)\Big| + \Big|\frac{\mathcal{R}(\varphi)}{\big(\mathcal{R}(1)\big)^2} D^2\big(\mathcal{R}(1)\big)(X,X)\Big| \\ &\quad + 2\Big|\frac{\mathcal{R}(\varphi)}{\big(\mathcal{R}(1)\big)^3}\Big(X\big(\mathcal{R}(1)\big)\Big)^2\Big| + 2\Big|\frac{1}{\big(\mathcal{R}(1)\big)^2} X\big(\mathcal{R}(1)\big)\, X\big(\mathcal{R}(\varphi)\big)\Big| \\ &\le c\, b(\rho(x))^2 \sup_{y\in B(x,2c_{1,2}/b(\rho(x)))} |\varphi(y)| \\ &\le c\frac{b(\rho(x))}{f_a\big(\rho(x)\big)}. \end{aligned}$$

■

3.2. Constructing a p-superharmonic function. The function $\rho^{-\delta}$, $\delta \in \mathbb{R}$, satisfies

$$\nabla(\rho^{-\delta}) = -\delta\rho^{-\delta-1}\nabla\rho$$

and

$$D^2(\rho^{-\delta}) = \delta(\delta+1)\rho^{-\delta-2}\big(d\rho \otimes d\rho\big) - \delta\rho^{-\delta-1}D^2\rho$$

in the set $M \setminus \{o\}$. In particular,

$$\Delta(\rho^{-\delta}) = \delta(\delta+1)\rho^{-\delta-2} - \delta\rho^{-\delta-1}\Delta\rho.$$

Lemma 3.17. *There exist constants $R_2 = R_2(C)$ and $c_6 = c_6(C)$ with the following property. If $\delta \in (0,1)$, then*

$$\begin{aligned} |\nabla h| &\le c_6/(f_a \circ \rho), \\ \|D^2 h\| &\le c_6\rho^{-C_4-1}(f_a' \circ \rho)/(f_a \circ \rho), \\ |\nabla\langle\nabla h, \nabla h\rangle| &\le c_6\rho^{-C_4-2}(f_a' \circ \rho)/(f_a \circ \rho), \\ |\nabla\langle\nabla h, \nabla(\rho^{-\delta})\rangle| &\le c_6\rho^{-C_4-2}(f_a' \circ \rho)/(f_a \circ \rho), \\ \nabla\big\langle\nabla(\rho^{-\delta}), \nabla(\rho^{-\delta})\big\rangle &= -2\delta^2(\delta+1)\rho^{-2\delta-3}\nabla\rho \end{aligned}$$

in the set $3\Omega \setminus B(o, R_2)$.

Proof. Choose $R_2 \ge \max\{1, R_1\}$ so large that

$$\begin{aligned} t^{1+C_4}b(t) &\le f_a'(t), \\ \frac{f_b'(t)}{f_b(t)} &\le 2b(t), \\ t^{-1} &\le b(t). \end{aligned} \tag{3.24}$$

for all $t \ge R_2$. Such R_2 exists by (A7), (A6), Lemma 2.3, and (A5).

We already know the first inequality by (3.20). The second inequality follows from (3.20) and (3.24) since

$$\|D^2 h\| \le c_5 \frac{b \circ \rho}{f_a \circ \rho} \le c_5 \rho^{-C_4-1}\frac{f_a' \circ \rho}{f_a \circ \rho}$$

in the set $3\Omega \setminus B(o, R_2)$. The third inequality follows from Lemma 3.7 and the first two inequalities by estimating

$$|\nabla\langle\nabla h, \nabla h\rangle| \le 2|\nabla h|\,\|D^2 h\| \le 2c_5^2\rho^{-C_4-1}\frac{f_a' \circ \rho}{(f_a \circ \rho)^2} \le 2c_5^2\rho^{-C_4-2}\frac{f_a' \circ \rho}{f_a \circ \rho}$$

in $3\Omega \setminus B(o, R_2)$.

We proceed to the fourth inequality. Using Lemma 3.7, (3.20), and Proposition 2.5(b) we get

$$\begin{aligned}\big|\nabla\big\langle\nabla h, \nabla(\rho^{-\delta})\big\rangle\big| &\le |\nabla h|\big\|D^2(\rho^{-\delta})\big\| + \big|\nabla(\rho^{-\delta})\big|\|D^2 h\| \\ &\le \frac{c_5}{f_a\circ\rho}\Big(\delta(\delta+1)\rho^{-\delta-2} + \delta\rho^{-\delta-1}\|D^2\rho\|\Big) + \delta\rho^{-\delta-1}\frac{c_5(b\circ\rho)}{f_a\circ\rho} \\ &\le \frac{c_5}{f_a\circ\rho}\Big(2\delta\rho^{-\delta-2} + \delta\rho^{-\delta-1}\frac{f_b'\circ\rho}{f_b\circ\rho}\Big) + \delta\rho^{-\delta-1}\frac{c_5(b\circ\rho)}{f_a\circ\rho} \\ &= \frac{c_5\delta\rho^{-\delta-1}}{f_a\circ\rho}\Big(2\rho^{-1} + \frac{f_b'\circ\rho}{f_b\circ\rho} + (b\circ\rho)\Big)\end{aligned}$$

in the set $3\Omega\setminus B(o,R_2)$. Hence (3.24) implies

$$\begin{aligned}\big|\nabla\big\langle\nabla h, \nabla(\rho^{-\delta})\big\rangle\big| &\le 5c_5\rho^{-\delta-1}\frac{b\circ\rho}{f_a\circ\rho} \\ &\le 5c_5\rho^{-\delta-1}\rho^{-C_4-1}\frac{f_a'\circ\rho}{f_a\circ\rho} \\ &\le 5c_5\rho^{-C_4-2}\frac{f_a'\circ\rho}{f_a\circ\rho}\end{aligned}$$

in $3\Omega\setminus B(o,R_2)$. This proves the fourth inequality.

The last equality holds since

$$\begin{aligned}\nabla\big\langle\nabla(\rho^{-\delta}), \nabla(\rho^{-\delta})\big\rangle &= \nabla\big\langle -\delta\rho^{-\delta-1}\nabla\rho, -\delta\rho^{-\delta-1}\nabla\rho\big\rangle \\ &= \delta^2\nabla(\rho^{-2\delta-2}) \\ &= -2\delta^2(\delta+1)\rho^{-2\delta-3}\nabla\rho\end{aligned}$$

in $M\setminus\{o\}$. ∎

Denote

$$\phi_1 = \frac{1+\sqrt{1+4C_1^2}}{2} > 1 \quad\text{and}\quad \delta_1 = \min\Big\{C_4, \frac{-1+(n-1)\phi_1}{1+(n-1)\phi_1}\Big\}.$$

Then $\delta_1\in(0,1)$.

Lemma 3.18. *Let $\delta\in(0,\delta_1)$. Denote*

$$\lambda = \frac{1+\delta}{(1-\delta)(n-1)\phi_1} \in (0,1).$$

There exists a constant $R_{3,\delta} = R_{3,\delta}(C,\delta) \ge R_2$ such that

$$-\Delta(\rho^{-\delta}) > 0, \qquad \frac{-\Delta(\rho^{-\delta})}{\delta\rho^{-\delta-1}\Delta\rho} \ge 1-\lambda, \qquad \frac{|\Delta h|}{-\Delta(\rho^{-\delta})} \le \delta,$$

in the set $3\Omega \setminus B(o, R_{3,\delta})$.

Proof. Since $0 < \delta < \delta_1$, we see that $0 < \lambda < 1$.

Define $a_1 : [0, \infty) \to [0, \infty)$ such that a_1 is a smooth function that is constant in some neighborhood of 0, $a_1 \leq a$ everywhere, and $a_1(t) = C_1 t^{-1}$ for every $t \geq T_1 + 1$. Such a function exists by the assumption (A1). Then

$$\frac{f_a'}{f_a} \geq \frac{f_{a_1}'}{f_{a_1}}$$

everywhere by Lemma 2.2. By (2.1) we have

$$\lim_{t\to\infty} \frac{t f_{a_1}'(t)}{f_{a_1}(t)} = \phi_1.$$

Therefore

$$\liminf_{t\to\infty} \frac{t f_a'(t)}{f_a(t)} \geq \phi_1.$$

Since $\delta < C_4$, we have

$$\lim_{t\to\infty} t^{-C_4+\delta} = 0.$$

Using these formulas we see that there exists $R_{3,\delta} \geq R_2$ such that

$$\frac{\rho(x)(f_a' \circ \rho)(x)}{(f_a \circ \rho)(x)} \geq (1-\delta)\phi_1 \tag{3.25}$$

and

$$\frac{2c_6}{\delta}\rho(x)^{-C_4+\delta} \leq \delta(1-\lambda) \tag{3.26}$$

whenever $x \in M \setminus B(o, R_{3,\delta})$.

By Proposition 2.5(b) we have

$$\Delta\rho \geq (n-1)\frac{f_a' \circ \rho}{f_a \circ \rho} > 0 \tag{3.27}$$

in 4Ω. Inequalities (3.27) and (3.25) imply

$$\frac{1+\delta}{\rho\Delta\rho} \leq \frac{1+\delta}{(n-1)\rho(f_a' \circ \rho)/(f_a \circ \rho)} \leq \frac{1+\delta}{(n-1)(1-\delta)\phi_1} = \lambda \tag{3.28}$$

in $4\Omega \setminus B(o, R_{3,\delta})$. By (3.28) we have

$$\frac{-\Delta(\rho^{-\delta})}{\delta\rho^{-\delta-1}\Delta\rho} = \frac{-\delta(\delta+1)\rho^{-\delta-2} + \delta\rho^{-\delta-1}\Delta\rho}{\delta\rho^{-\delta-1}\Delta\rho} = 1 - \frac{1+\delta}{\rho\Delta\rho} \geq 1-\lambda \tag{3.29}$$

in the set $4\Omega \setminus B(o, R_{3,\delta})$. In particular, $\Delta(\rho^{-\delta}) < 0$. By (3.29) and (3.27) we now have

$$\frac{|\Delta h|}{-\Delta(\rho^{-\delta})} \le (1-\lambda)^{-1} \frac{|\Delta h|}{\delta \rho^{-\delta-1}\Delta\rho} \le (1-\lambda)^{-1} \frac{|\Delta h|}{\delta(n-1)\rho^{-\delta-1}(f_a' \circ \rho)/(f_a \circ \rho)}$$

in $4\Omega \setminus B(o, R_{3,\delta})$. Here we can estimate $|\Delta h|$ with Lemma 3.17 and get

$$\begin{aligned}\frac{|\Delta h|}{-\Delta(\rho^{-\delta})} &\le (1-\lambda)^{-1} \frac{nc_6\rho^{-C_4-1}(f_a' \circ \rho)/(f_a \circ \rho)}{\delta(n-1)\rho^{-\delta-1}(f_a' \circ \rho)/(f_a \circ \rho)} \\ &\le (1-\lambda)^{-1} \frac{2c_6}{\delta} \rho^{-C_4+\delta}\end{aligned}$$

in the set $3\Omega \setminus B(o, R_{3,\delta})$. Using (3.26) we see that

$$-|\Delta h|/\Delta(\rho^{-\delta}) \le \delta$$

in $3\Omega \setminus B(o, R_{3,\delta})$. ∎

Lemma 3.19. *Suppose that*

$$p \in \big(1, 1 + (n-1)\phi_1\big).$$

Then there exist constants $\delta = \delta(p, C) \in (0, \delta_1)$ and $R_4 = R_4(p, C)$ such that the function $\rho^{-\delta} + \sigma h$ is a p-superharmonic in the set $3\Omega \setminus \bar{B}(o, R_4)$ for every $\sigma \in [-1, 1]$.

Proof. Fix $\sigma \in [-1, 1]$. Choose $\delta \in (0, \delta_1)$ so that $(1-\phi_1)+\delta < 0$, $-C_4+2\delta < 0$, and

$$\delta + \Big(\max\{p-2, 0\} + \delta|p-2|\Big)(1-\delta)^{-3} \frac{(1+\delta)}{(1-\delta)(n-1)\phi_1 - (1+\delta)} < 1.$$

Such δ exists since $p - 2 < (n-1)\phi_1 - 1$. As in Lemma 3.18, we denote

$$\lambda = \frac{1+\delta}{(1-\delta)(n-1)\phi_1}.$$

Then $0 < \lambda < 1$.

Let a_1 be as in the proof of Lemma 3.18. Since $a \ge a_1$, $f_a \ge f_{a_1}$ by Lemma 2.2. Denote

$$c' = \liminf_{t\to\infty} \frac{f_a(t)}{t^{\phi_1}}.$$

By Example 2.1 we have

$$c' \ge \liminf_{t\to\infty} \frac{f_{a_1}(t)}{t^{\phi_1}} > 0.$$

Lemma 3.17 now implies that

$$\frac{|\nabla h|}{|\nabla(\rho^{-\delta})|} \le \frac{c_6/(f_a \circ \rho)}{\delta\rho^{-\delta-1}} \le \frac{2c_6}{\delta c'} \rho^{(1-\phi_1)+\delta}$$

in $3\Omega \setminus B(o,r)$ if r is large. Note that we chose δ such that $(1-\phi_1)+\delta<0$. By this, Lemma 3.18, and (3.25) we can choose $R_4 \geq R_{3,\delta}$ such that

$$\frac{|\nabla h|}{|\nabla(\rho^{-\delta})|} \leq \delta, \tag{3.30}$$

$$\frac{|\Delta h|}{-\Delta(\rho^{-\delta})} \leq \delta, \tag{3.31}$$

$$\frac{\rho(f_a'\circ\rho)}{f_a\circ\rho} \geq (1-\delta)\phi_1, \tag{3.32}$$

and

$$\frac{2c_6|p-2|(1-\delta)^{-2}}{(1-\lambda)(n-1)}\,\frac{\rho^{-C_4+2\delta}}{\delta^2} \leq \delta \tag{3.33}$$

in $3\Omega \setminus B(o,R_4)$.

Denote $u=\rho^{-\delta}+\sigma h$. Then $|\nabla u|>0$ and $\Delta u<0$ in $3\Omega\setminus B(o,R_4)$ by (3.30), (3.31), and Lemma 3.18. We now have

$$\begin{aligned}
\frac{\frac{p-2}{2}|\nabla u|^{p-4}\big\langle\nabla\langle\nabla u,\nabla u\rangle,\nabla u\big\rangle}{-|\nabla u|^{p-2}\Delta u} &= \frac{p-2}{2}\frac{\big\langle\nabla\langle\nabla u,\nabla u\rangle,\nabla u\big\rangle}{-|\nabla u|^2\Delta u}\\
&= \frac{p-2}{2}\Big(\frac{\big\langle\nabla\langle\nabla(\rho^{-\delta}),\nabla(\rho^{-\delta})\rangle,\nabla(\rho^{-\delta})\big\rangle}{-|\nabla u|^2\Delta u}\\
&\quad+\frac{\Big\langle\sigma^2\nabla\langle\nabla h,\nabla h\rangle+2\sigma\nabla\langle\nabla h,\nabla(\rho^{-\delta})\rangle,\nabla u\Big\rangle}{-|\nabla u|^2\Delta u}\\
&\quad+\frac{\sigma\big\langle\nabla\langle\nabla(\rho^{-\delta}),\nabla(\rho^{-\delta})\rangle,\nabla h\big\rangle}{-|\nabla u|^2\Delta u}\Big)\\
&\leq \underbrace{\frac{p-2}{2}\frac{\big\langle\nabla\langle\nabla(\rho^{-\delta}),\nabla(\rho^{-\delta})\rangle,\nabla(\rho^{-\delta})\big\rangle}{-|\nabla u|^2\Delta u}}_{=:T_1}\\
&\quad+\underbrace{\frac{|p-2|}{2}\frac{\big|\nabla\langle\nabla h,\nabla h\rangle\big|+2\big|\nabla\langle\nabla h,\nabla(\rho^{-\delta})\rangle\big|}{-|\nabla u|\Delta u}}_{=:T_2}\\
&\quad+\underbrace{\frac{|p-2|}{2}\frac{\big|\nabla\langle\nabla(\rho^{-\delta}),\nabla(\rho^{-\delta})\rangle\big||\nabla h|}{-|\nabla u|^2\Delta u}}_{=:T_3}
\end{aligned}$$

in $3\Omega \setminus B(o, R_4)$. We have to estimate the terms T_1, T_2, and T_3. To do this we first denote

$$T = \frac{\left|\nabla\langle\nabla(\rho^{-\delta}), \nabla(\rho^{-\delta})\rangle\right|}{-|\nabla u|\Delta u}$$

and estimate this. By (3.30), (3.31), Lemma 3.18, and Proposition 2.5(b) we have

$$\begin{aligned}\frac{1}{-|\nabla u|\Delta u} &\le (1-\delta)^{-2}\frac{1}{-|\nabla(\rho^{-\delta})|\Delta(\rho^{-\delta})}\\ &\le (1-\delta)^{-2}\frac{1}{\delta^2\rho^{-2\delta-2}(1-\lambda)\Delta\rho}\\ &\le (1-\delta)^{-2}\frac{1}{\delta^2\rho^{-2\delta-2}(1-\lambda)(n-1)}\frac{f_a\circ\rho}{f_a'\circ\rho}\end{aligned}\tag{3.34}$$

in $3\Omega \setminus B(o, R_4)$. By Lemma 3.17, (3.34), and (3.32) we have

$$\begin{aligned}T &= \frac{2\delta^2(1+\delta)\rho^{-2\delta-3}}{-|\nabla u|\Delta u}\\ &\le 2(1-\delta)^{-2}\frac{\delta^2(1+\delta)\rho^{-2\delta-3}}{\delta^2\rho^{-2\delta-2}(1-\lambda)(n-1)}\frac{f_a\circ\rho}{f_a'\circ\rho}\\ &= 2(1-\delta)^{-2}\frac{(1+\delta)}{(1-\lambda)(n-1)\rho}\frac{f_a\circ\rho}{f_a'\circ\rho}\\ &\le 2(1-\delta)^{-3}\frac{(1+\delta)}{(1-\lambda)(n-1)\phi_1}\\ &= 2(1-\delta)^{-2}\frac{(1+\delta)}{(1-\delta)(n-1)\phi_1-(1+\delta)}\end{aligned}\tag{3.35}$$

in $3\Omega \setminus B(o, R_4)$.

We start to estimate T_1. If $p \le 2$, then $T_1 \le 0$ in $3\Omega \setminus B(o, R_4)$ since

$$\begin{aligned}\left\langle\nabla\langle\nabla(\rho^{-\delta}), \nabla(\rho^{-\delta})\rangle, \nabla(\rho^{-\delta})\right\rangle &= \left\langle -2\delta^2(\delta+1)\rho^{-2\delta-3}\nabla\rho, -\delta\rho^{-\delta-1}\nabla\rho\right\rangle\\ &= 2\delta^3(\delta+1)\rho^{-3\delta-4} > 0.\end{aligned}$$

On the other hand, if $p > 2$, then by (3.35) and (3.30) we have

$$\begin{aligned}T_1 &\le \frac{p-2}{2}T\frac{|\nabla(\rho^{-\delta})|}{|\nabla u|}\\ &\le (p-2)(1-\delta)^{-3}\frac{(1+\delta)}{(1-\delta)(n-1)\phi_1-(1+\delta)}\end{aligned}$$

in $3\Omega \setminus B(o, R_4)$.

Next we estimate T_2. By (3.34), Lemma 3.17, and (3.33) we have

$$\begin{aligned} T_2 &= \frac{|p-2|}{2} \frac{\big|\nabla\langle\nabla h, \nabla h\rangle\big| + 2\big|\nabla\langle\nabla h, \nabla(\rho^{-\delta})\rangle\big|}{-|\nabla u|\Delta u} \\ &\le |p-2|(1-\delta)^{-2} \frac{\big|\nabla\langle\nabla h, \nabla h\rangle\big| + \big|\nabla\langle\nabla h, \nabla(\rho^{-\delta})\rangle\big|}{\delta^2\rho^{-2\delta-2}(1-\lambda)(n-1)} \frac{f_a\circ\rho}{f_a'\circ\rho} \\ &\le |p-2|(1-\delta)^{-2} \frac{2c_6\rho^{-C_4-2}(f_a'\circ\rho)/(f_a\circ\rho)}{\delta^2\rho^{-2\delta-2}(1-\lambda)(n-1)} \frac{f_a\circ\rho}{f_a'\circ\rho} \\ &= \frac{2c_6|p-2|(1-\delta)^{-2}}{(1-\lambda)(n-1)} \frac{\rho^{-C_4+2\delta}}{\delta^2} \\ &\le \delta \end{aligned}$$

in $3\Omega \setminus B(o, R_4)$.

We still have to estimate T_3. By (3.30) and (3.35) we have

$$\begin{aligned} T_3 &= \frac{|p-2|}{2} \frac{\big|\nabla\langle\nabla(\rho^{-\delta}), \nabla(\rho^{-\delta})\rangle\big||\nabla h|}{-|\nabla u|^2\Delta u} \\ &= \frac{|p-2|}{2} T \frac{|\nabla h|}{|\nabla u|} \\ &\le \frac{|p-2|}{2} T \frac{\delta}{1-\delta} \\ &\le \delta|p-2|(1-\delta)^{-3} \frac{(1+\delta)}{(1-\delta)(n-1)\phi_1 - (1+\delta)} \end{aligned}$$

in $3\Omega \setminus B(o, R_4)$. We combine our results and get

$$\begin{aligned} \frac{\frac{p-2}{2}|\nabla u|^{p-4}\big\langle\nabla\langle\nabla u, \nabla u\rangle, \nabla u\big\rangle}{-|\nabla u|^{p-2}\Delta u} &\le T_1 + T_2 + T_3 \\ &\le \delta + \Big(\max\{p-2, 0\} + \delta|p-2|\Big)(1-\delta)^{-3} \frac{(1+\delta)}{(1-\delta)(n-1)\phi_1 - (1+\delta)} \\ &< 1 \end{aligned}$$

in $3\Omega \setminus B(o, R_4)$ by our choice of δ. It follows that

$$\begin{aligned} \operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big) &= |\nabla u|^{p-2}\Delta u + \big\langle\nabla\big(|\nabla u|^{p-2}\big), \nabla u\big\rangle \\ &= |\nabla u|^{p-2}\Delta u + \tfrac{p-2}{2}|\nabla u|^{p-4}\big\langle\nabla\langle\nabla u, \nabla u\rangle, \nabla u\big\rangle \\ &< 0 \end{aligned}$$

in $3\Omega \setminus B(o, R_4)$. Therefore $u = \rho^{-\delta} + \sigma h$ is a continuous p-supersolution, hence p-superharmonic in $3\Omega \setminus \bar{B}(o, R_4)$. ■

Let δ and R_4 be as in Lemma 3.19.

Lemma 3.20. *Suppose that*

$$p \in \big(1, 1 + (n-1)\phi_1\big).$$

Then there exists a function $w : M \to [0,1]$ *such that* w *is continuous and* p*-superharmonic in* M*,* $w|\big(M \setminus (3\Omega \setminus \bar{B}(o, R_4))\big) \equiv 1$*, and*

$$\lim_{x \to \gamma^{v_0}(\infty),\ x \in M} w(x) = 0. \tag{3.36}$$

Proof. Denote $U = 3\Omega \setminus \bar{B}(o, R_4)$ and $U_k = U \cap B(o, k)$ if $k \in \mathbb{N} \cap (R_4 + 1, \infty)$. Let $\eta : M \to \mathbb{R}$,

$$\eta(x) = \min\Big(1, \max\big(R_4 + 1 - \rho(x), h(x)\big)\Big).$$

Then $\eta : M \to [0,1]$ is continuous, $\eta|\partial B(o, R_4) \equiv 1$, and $\eta(x) = h(x)$ for every $x \in M \setminus B(o, R_4 + 1)$. If $k \in \mathbb{N} \cap (R_4 + 1, \infty)$, let $u_k : U_k \to \mathbb{R}$ be the unique p-harmonic function with $u_k - \eta \in W_0^{1,p}(U_k)$. Let $k \in \mathbb{N} \cap (R_4 + 1, \infty)$ and $x_0 \in \partial U_k$. Then

$$\lim_{x \to x_0,\ x \in U_k} u_k(x) = \eta(x_0) = \begin{cases} 1 & \text{if } \rho(x_0) < k, \\ h(x_0) & \text{if } \rho(x_0) > R_4, \end{cases} \tag{3.37}$$

since U_k is p-regular and since $h|\big(M \setminus (2\Omega \cup B(o, R_1))\big) \equiv 1$ by Lemma 3.16.

Now (u_k) is a bounded sequence of p-harmonic functions. Hence there exists a subsequence, again denoted by (u_k), and a p-harmonic function $u : U \to [0,1]$ such that $u_k \to u$ locally uniformly in U. It follows from (3.37) that if $x_0 \in M \cap \partial U$, then

$$\lim_{x \to x_0,\ x \in U} u(x) = 1. \tag{3.38}$$

By (3.37) we have

$$\lim_{x \to x_0,\ x \in U_k} u_k(x) \le \rho(x_0)^{-\delta}/R_4^{-\delta} + h(x_0)$$

for every $k \in \mathbb{N} \cap (R_4 + 1, \infty)$ and every $x_0 \in \partial U_k$. The function $\rho^{-\delta}/R_4^{-\delta} + h$ is p-superharmonic in U_k by Lemma 3.19 and hence we get by the comparison principle that

$$u_k \le \rho^{-\delta}/R_4^{-\delta} + h$$

in U_k. It follows that

$$u \le \rho^{-\delta}/R_4^{-\delta} + h$$

in U. This implies that

$$\limsup_{x \to \gamma^{v_0}(\infty),\ x \in U} u(x) \le h\big(\gamma^{v_0}(\infty)\big) = 0.$$

Thus

$$\lim_{x\to\gamma^{v_0}(\infty),\ x\in U} u(x)=0. \tag{3.39}$$

Define $w\colon M\to[0,1]$,

$$w(x)=\begin{cases}\min(1,2u)(x) & \text{if } x\in U,\\ 1 & \text{if } x\in M\setminus U.\end{cases}$$

Because the minimum of two p-superharmonic functions is p-superharmonic and (3.38) holds for every $x_0\in M\cap\partial U$, we see that w is continuous and p-superharmonic in some neighborhood of each point in M. It follows that w is p-superharmonic since p-superharmonicity is a local property. The equation (3.36) follows from (3.39). ■

3.3. p-regular points at infinity. We are ready to prove our main result in this section.

Theorem 3.21. *Let $a,b\colon[0,\infty)\to[0,\infty)$ be smooth functions that are constant in some neighborhood of* 0. *Suppose that a and b satisfy the conditions* (A1)-(A7). *Suppose that $M\ni o$ is a Cartan-Hadamard n-manifold and $\rho=\rho^o$. Let $x_0\in M(\infty)$ and let U be a neighborhood of x_0 in the cone topology. Suppose that*

$$-(b\circ\rho)^2(x)\le K_M(P)\le-(a\circ\rho)^2(x)$$

for every $x\in U\cap M$ and every 2-dimensional subspace $P\subset T_xM$. Let

$$p\in\big(1,1+(n-1)\phi_1\big),$$

where

$$\phi_1=\frac{1+\sqrt{1+4C_1^2}}{2}.$$

Then x_0 is a p-regular point at infinity.

Proof. Fix an arbitrary continuous function $f\colon M(\infty)\to\mathbb{R}$. We have to show that the upper Perron solution $\overline{H}_f$ satisfies

$$\lim_{x\to x_0}\overline{H}_f(x)=f(x_0).$$

Fix $\varepsilon>0$. Denote $v_0=\dot\gamma_0^{o,x_0}$. Choose $R>0$ and $L\in(8/\pi,\infty)$ such that $T(v_0,4/L,R)\subset U$ and that $|f(x_1)-f(x_0)|<\varepsilon$ for all $x_1\in C(v_0,4/L)\cap M(\infty)$.

Let $k>1$ and define $\tilde b=kb$ and $\tilde a=\big(1-\chi(\cdot/k)\big)a$, where $\chi\in C^\infty(\mathbb{R})$ is such that $0\le\chi\le1$, $\operatorname{supp}\chi\subset[-2,2]$, and $\chi|[-1,1]\equiv1$. Then $\tilde a$ and $\tilde b$ satisfy the conditions (A1)-(A7) with constants $\tilde T_1=\max\{T_1,2k\}$, $\tilde C_1=C_1$, $\tilde C_2=C_2$,

$\tilde{C}_3 = C_3$, $\tilde{Q} = Q$, and $\tilde{C}_4 = C_4$. Conditions (A1)-(A6) are easy to verify and (A7) follows from Lemma 2.4. By choosing $k > 1$ large enough we have

$$-(\tilde{b} \circ \rho)^2(x) \leq K_M(P) \leq -(\tilde{a} \circ \rho)^2(x)$$

for every $x \in C(v_0, 4/L) \cap M$ and every 2-dimensional subspace $P \subset T_xM$. By considering $\tilde{a}$ and $\tilde{b}$ instead of a and b we are now in a situation where we can apply Lemma 3.20. Let w be the p-superharmonic function described there. Then

$$\overline{H}_f \leq f(x_0) + \varepsilon + 2(\sup |f|)w$$

by the definition of $\overline{H}_f$ and the properties of w. Since

$$\lim_{x \to x_0} w(x) = 0,$$

we get $\limsup_{x \to x_0} \overline{H}_f(x) \leq f(x_0) + \varepsilon$. Letting $\varepsilon \to 0$ gives

$$\limsup_{x \to x_0} \overline{H}_f(x) \leq f(x_0).$$

Similarly one proves that the lower Perron solution $\underline{H}_f$ satisfies

$$\liminf_{x \to x_0} \underline{H}_f(x) \geq f(x_0).$$

Taking into account $\overline{H}_f \geq \underline{H}_f$ we get $\lim_{x \to x_0} \overline{H}_f(x) = f(x_0)$. ■

The most interesting special cases of this result are given by the following two corollaries.

Corollary 3.22. *Let $\phi > 1$ and $\varepsilon > 0$. Suppose that $M \ni o$ is a Cartan-Hadamard n-manifold and $\rho = \rho^o$. Let $x_0 \in M(\infty)$ and let U be a neighborhood of x_0 in the cone topology. Suppose that*

$$-\rho(x)^{2\phi - 4 - \varepsilon} \leq K_M(P) \leq -\frac{\phi(\phi - 1)}{\rho(x)^2} \tag{3.40}$$

for every $x \in U \cap M$ and every 2-dimensional subspace $P \subset T_xM$. Then x_0 is a p-regular point at infinity for every $p \in \big(1, 1 + (n-1)\phi\big)$.

Proof. Without loss of generality we can assume that $U = T(\dot{\gamma}_0^{o,x_0}, \alpha, r)$ for some $\alpha > 0$ and $r > 1$. We can also assume that $\varepsilon < 2\phi - 2$.

Denote $C_1 = \sqrt{\phi(\phi - 1)}$. If $t \geq r + 1$, let

$$b(t) = t^{\phi - 2 - \varepsilon/2}.$$

Extend b to a smooth function $b : [0, \infty) \to (0, \infty)$ that is monotonic and constant in some neighborhood of 0. If $t \geq r + 1$, let

$$a(t) = C_1 t^{-1}$$

and extend a to a smooth function $a : [0,\infty) \to [0,\infty)$ that is constant in some neighborhood of 0 and that satisfies $a \le b$. This can be done since

$$C_1 t^{-1} \le t^{\phi-2-\varepsilon/2}$$

for every $t \ge r$ by (3.40). Now a and b satisfy (A1)-(A7) with constants $T_1 = r+1$, C_1, some $C_2 > 0$, some $C_3 > 0$, $Q = \max\{1/2, -\phi+2+\varepsilon/2\}$, and any $C_4 \in (0, \varepsilon/2)$. The sectional curvatures satisfy

$$-(b\circ\rho)^2(x) \le K_M(P) \le -(a\circ\rho)^2(x)$$

for every $x \in T(\dot\gamma^{o,x_0}, \alpha, r+1) \cap M$ and every 2-dimensional subspace $P \subset T_xM$ and the claim follows from Theorem 3.21. ∎

Corollary 3.23. *Let $k > 0$ and $\varepsilon > 0$. Suppose that $M \ni o$ is a Cartan-Hadamard n-manifold and $\rho = \rho^o$. Let $x_0 \in M(\infty)$ and let U be a neighborhood of x_0 in the cone topology. Suppose that*

$$-\rho(x)^{-2-\varepsilon} e^{2k\rho(x)} \le K_M(P) \le -k^2 \tag{3.41}$$

for every $x \in U \cap M$ and every 2-dimensional subspace $P \subset T_xM$. Then x_0 is a p-regular point at infinity for every $p \in (1,\infty)$.

Proof. Let $r_0 > 1$ be so large that $t \mapsto t^{-1-\varepsilon/2}e^{kt}$ is increasing on $[r_0,\infty)$. Without loss of generality we can assume that $U = T(\dot\gamma_0^{o,x_0}, \alpha, r)$ for some $\alpha > 0$ and $r > r_0$.

Fix $p \in (1,\infty)$. Let $a(t) = k$ for every $t \ge 0$. Let $b(t) = t^{-1-\varepsilon/2}e^{kt}$ for every $t \ge r+1$ and extend b to a smooth increasing function $b : [0,\infty) \to (0,\infty)$ such that b is constant in a neighborhood of 0 and $a \le b$ everywhere. This can be done since $k \le t^{-1-\varepsilon/2}e^{kt}$ for every $t \ge r$ by (3.41). Choose $C_1 > 0$ to be so large that ϕ, defined by

$$\phi = \frac{1+\sqrt{1+4C_1^2}}{2},$$

satisfies $p < 1 + (n-1)\phi$. Now a and b satisfy (A1)-(A7) with constants $T_1 = C_1/k$, C_1, some $C_2 > 0$, some $C_3 > 0$, $Q = 1/2$, and any $C_4 \in (0,\varepsilon/2)$. The sectional curvatures satisfy

$$-(b\circ\rho)^2(x) \le K_M(P) \le -(a\circ\rho)^2(x)$$

for every $x \in T(\dot\gamma_0^{o,x_0}, \alpha, r+1) \cap M$ and every 2-dimensional subspace $P \subset T_xM$ and it follows from Theorem 3.21 that x_0 is p-regular. ∎

Remark 3.24. 1) In the case of the Laplacian ($p = 2$), the results of the previous corollaries are similar to those presented by Hsu in [23]. Hsu used probabilistic methods whereas our proof is analytic.

2) The condition $p < 1+(n-1)\phi$, that appears in Theorem 3.21 and Corollary 3.22 is automatically satisfied if $p \le n$, in particular in the most important cases $p = 2$ and $p = n$. This condition appeared and was discussed in [33].
3) In the case $n = 2$ Theorem 3.21, Corollary 3.22, and Corollary 3.23 follow as special cases from [33]. In this case no curvature lower bound is needed.

4. Dirichlet problem at infinity on Gromov hyperbolic metric measure spaces

4.1. Hyperbolic metric spaces. In this subsection we recall the basic notions related to Gromov hyperbolic metric spaces. Our notation and terminology is similar to that in [34]. Let $X = (X, d)$ be a metric space. We usually write $|x-y|$ for the distance $d(x, y)$ between points $x, y \in X$. For points $x, y, o \in X$ in a metric space X, the *Gromov product* of x and y with respect to the basepoint o is defined by

$$(x\,|\,y)_o = \tfrac{1}{2}\big(|x-o| + |y-o| - |x-y|\big).$$

Note that $0 \le (x\,|\,y)_o \le \min\{|x-o|, |y-o|\}$. Moreover, if $o' \in X$ is another basepoint, then

$$|(x\,|\,y)_o - (x\,|\,y)_{o'}| \le |o-o'| \tag{4.1}$$

for all $x, y \in X$. The metric space X is called *(Gromov) δ-hyperbolic*, with $\delta \ge 0$, if

$$(x \mid z)_o \ge \min\{(x \mid y)_o, (y \mid z)_o\} - \delta \tag{4.2}$$

for all $x, y, z, o \in X$. The space X is called *(Gromov) hyperbolic* if it is δ-hyperbolic for some $\delta \ge 0$.

We assume from now on that X is δ-hyperbolic. We fix a basepoint $o \in X$ and abbreviate $(x\,|\,y) = (x\,|\,y)_o$. A sequence $\bar{x} = (x_i)$ of points in X is called a *Gromov sequence*, or a *sequence converging at infinity*, if

$$\lim_{i,j\to\infty} (x_i\,|\,x_j) = \infty. \tag{4.3}$$

The condition (4.3) is independent of the choice of the basepoint o by (4.1). It is worth observing that

$$|x_i - o| = (x_i\,|\,x_i) \to \infty$$

for a Gromov sequence (x_i). We say that two Gromov sequences $\bar{x} = (x_i)$ and $\bar{y} = (y_i)$ are *equivalent*, and write $\bar{x} \sim \bar{y}$, if $(x_i\,|\,y_i) \to \infty$ as $i \to \infty$. This defines an equivalence relation on the set of all Gromov sequences; the relation is transitive due to (4.2). Note that $\bar{x}$ is equivalent to all of its subsequences.

The *Gromov boundary* of X, also called the *boundary at infinity* of X, is the set of all equivalence classes

$$\partial_G X = \{[\bar{x}] \colon \bar{x} \text{ is a Gromov sequence in } X\}.$$

The set

$$X^* = X \cup \partial_G X$$

is called the *Gromov closure* of X.

To introduce an appropriate topology on X^* and a family of metrics in $\partial_G X$ we next define the Gromov product $(a \,|\, b)$ for all $a, b \in X^*$. Following [14, p. 18] we set

$$(a \,|\, b) = \inf \left\{ \liminf_{i,j\to\infty} (x_i \,|\, y_j) \colon \bar{x} \in a, \bar{y} \in b \right\}$$

for $a, b \in \partial_G X$, and

$$(a \,|\, y) = (y \,|\, a) = \inf \left\{ \liminf_{i\to\infty} (x_i \,|\, y) \colon \bar{x} \in a \right\}$$

for $a \in \partial_G X$ and $y \in X$. It then follows that

$$(a \,|\, c) \geq \min\{(a \,|\, b), (b \,|\, c)\} - \delta \tag{4.4}$$

for all $a, b, c \in X^*$. Note that $(a \,|\, b) = \infty$ if and only if $a = b \in \partial_G X$. Let $\varepsilon > 0$ and define

$$\varrho_\varepsilon(a, b) = \begin{cases} \exp\bigl(-\varepsilon (a \,|\, b)\bigr), & \text{if } a \neq b, \\ 0, & \text{if } a = b. \end{cases}$$

By (4.4), we then have

$$\varrho_\varepsilon(a, c) \leq e^{\varepsilon\delta} \max\{\varrho_\varepsilon(a, b), \varrho_\varepsilon(b, c)\}$$

for all $a, b, c \in X^*$. For $a, b \in X^*$ we define

$$d_\varepsilon(a, b) = \inf \sum_{j=1}^{k} \varrho_\varepsilon(a_j, a_{j-1}), \tag{4.5}$$

where the infimum is taken over all finite sequences $a = a_0, \ldots, a_k = b$ in X^*. If $\varepsilon > 0$ is so small that $e^{\varepsilon\delta} \leq 2$, then d_ε is a metric satisfying

$$\frac{1}{4} \varrho_\varepsilon(a, b) \leq d_\varepsilon(a, b) \leq \varrho_\varepsilon(a, b)$$

for all $a, b \in X^*$, cf. [16] and [34]. We say that the metric space (X^*, d_ε) is *obtained from* the δ-hyperbolic space X.

Let $\mathcal{T}_d$ denote the original topology of X induced by the metric d. Each metric d_ε induces a topology $\mathcal{T}_{d_\varepsilon}$ on X^*. For fixed $a \in X$, $\varrho_\varepsilon(a, b) \to \exp(-\varepsilon|a - o|)$ as $|a - b| \to 0$. Consequently, the open ball $B_{d_\varepsilon}(a, r)$, $a \in X$, with respect to d_ε is the singleton $\{a\}$ for all $r \leq \frac{1}{2}\exp(-\varepsilon|a - o|)$. Hence $\mathcal{T}_{d_\varepsilon}|X$ is discrete, i.e. all subsets of X are open with respect to $\mathcal{T}_{d_\varepsilon}$. However, we want to maintain the original topology of X, and therefore we choose

$$\mathcal{T}^* = \{U \in \mathcal{T}_{d_\varepsilon} : U \cap X \in \mathcal{T}_d\}$$

for the topology of X^* for the rest of the paper; see [1, 4.7] and [34, 5.29]. It is important to observe that $\mathcal{T}_{d_\varepsilon}$ and hence $\mathcal{T}^*$ is independent of the choices of o and ε.

Remark 4.1. If X is a Cartan-Hadamard whose sectional curvatures are bounded from above with a negative constant $-a^2 < 0$, then it is Gromov-hyperbolic; see [14, Théorème 5.1]. In this case $\bar{X} = X \cup X(\infty)$, equipped with cone topology, and the Gromov closure $X^* = X \cup \partial_G X$, equipped with the topology $\mathcal{T}^*$, are homeomorphic, cf. [10, Sect. III.H.3].

4.2. p-harmonic functions on a metric measure space. Suppose that $X = (X, d, \mu)$ is a connected, locally compact, and non-compact metric measure space with a metric d and a Borel regular measure μ. We assume that the measure μ is *locally doubling*, that is, there exist positive constants C_d and R_d such that

$$0 < \mu(B(x,2r)) \leq C_d\, \mu(B(x,r)) < \infty \tag{4.6}$$

for every ball $B(x,r) \subset X$, with $0 < r \leq R_d$.

Let Γ be a family of paths in X and let $1 \leq p < \infty$. The *p-modulus* of Γ is defined as

$$M_p(\Gamma) = \inf \int_X \rho^p \, d\mu,$$

where the infimum is taken over all Borel functions $\rho\colon X \to [0, +\infty]$ satisfying

$$\int_\gamma \rho \, ds \geq 1$$

for every locally rectifiable path $\gamma \in \Gamma$. We say that a property of paths hold for *p-almost all paths* if the family of paths for which the property fails is of zero p-modulus.

A Borel function $g\colon X \to [0, +\infty]$ is said to be an *upper gradient* of a function $u\colon X \to [-\infty, +\infty]$ if, for every rectifiable path $\gamma\colon [a,b] \to X$,

$$\big|u(\gamma(b)) - u(\gamma(a))\big| \leq \int_\gamma g \, ds \tag{4.7}$$

whenever both $u(\gamma(a))$ and $u(\gamma(b))$ are finite, and $\int_\gamma g\, ds = +\infty$ otherwise. We say that g is a *p-weak upper gradient* of u if (4.7) holds for p-almost all paths $\gamma\colon [a,b] \to X$.

If u has a p-weak upper gradient in $L^p(X)$, then it also has a *minimal p-weak upper gradient*, denoted by $|\nabla u|$, in the sense that $|\nabla u| \le g$ μ-a.e. for every p-weak upper gradient $g \in L^p(X)$ of u; see [18, 7.16].

We assume that the space (X, d, μ) supports a *local weak* $(1,p)$-*Poincaré inequality* which means that there exist constants $C_P > 0$, $R_P > 0$, and $\tau \ge 1$ such that for all balls $B = B(x,r) \subset X$, with $0 < r \le R_P$,

$$\fint_B |u - u_B|\, d\mu \le C_P r \left(\fint_{\tau B} g^p \, d\mu \right)^{1/p} \tag{4.8}$$

whenever u is an integrable function in $\tau B = B(x, \tau r)$ and g is a p-weak upper gradient of u. Here

$$u_B = \fint_B u\, d\mu = \frac{1}{\mu(B)} \int_B u \, d\mu.$$

Let $\tilde N^{1,p}(X)$ be the set of all functions $u \in L^p(X)$ that have a p-weak upper gradient $g \in L^p(X)$. We equip $\tilde N^{1,p}(X)$ with the seminorm

$$\|u\|_{\tilde N^{1,p}(X)} = \|u\|_{L^p(X)} + \inf \|g\|_{L^p(X)},$$

where the infimum is taken over all p-weak upper gradients g of u. As usual, we identify functions $u, v \in \tilde N^{1,p}(X)$, and write $u \sim v$, if

$$\|u - v\|_{\tilde N^{1,p}(X)} = 0.$$

The *Sobolev space* $N^{1,p}(X)$ is then the space $\tilde N^{1,p}(X)/\sim$ with the (well-defined) norm

$$\|u\|_{N^{1,p}(X)} = \|u\|_{\tilde N^{1,p}(X)}.$$

We say that u belongs to the *local Sobolev space* $N^{1,p}_{\mathrm{loc}}(X)$ if $u \in N^{1,p}(U)$ for every measurable $U \Subset X$. Here $U \Subset X$ means that $\bar U$ is compact. We refer to [30] for basic properties of the Sobolev spaces $N^{1,p}(X)$. In [11] Cheeger gives an alternative definition which leads to the same Banach space if $1 < p < \infty$; see [30]. Furthermore, Cheeger [11] proved the deep result that $N^{1,p}(X)$ is reflexive if $1 < p < \infty$.

The *(Sobolev) p-capacity* of a set $E \subset X$ is defined by

$$C_p(E) = \inf \|u\|_{N^{1,p}(X)},$$

where the infimum is taken over all functions $u \in N^{1,p}(X)$, with $u|E \ge 1$. For a subset $\Omega \subset X$ let $N^{1,p}_0(\Omega)$ be the space of all elements in $N^{1,p}(X)$ whose representatives u satisfy

$$C_p\big(\{x \in X \setminus \Omega \colon u(x) \ne 0\}\big) = 0.$$

The space $N_0^{1,p}(\Omega)$ equipped with the norm

$$\|u\|_{N_0^{1,p}(\Omega)} = \|u\|_{N^{1,p}(X)}$$

is called the *Sobolev space with zero boundary values*. It is worth observing that $N^{1,p}(X) = N_0^{1,p}(X)$, and therefore $N_0^{1,p}(\Omega)$, for a subset $\Omega \subset X$, depends on the ambient space X. In the literature the Sobolev spaces as above are usually called *Newtonian spaces*.

The local doubling condition (4.6) and the local weak $(1,p)$-Poincaré inequality (4.8) imply, by [19, Theorem 5.1], that there are constants $c > 0$, $R_S > 0$, and $\lambda > 1$ such that a *local Sobolev-Poincaré inequality*

$$\left(\fint_B |u - u_B|^{\lambda p} \, d\mu \right)^{1/\lambda p} \le cr \left(\fint_{5\tau B} g^p \, d\mu \right)^{1/p} \tag{4.9}$$

holds for all balls $B = B(x,r) \subset X$, with $0 < r \le R_S$, whenever u is an integrable function in $5\tau B = B(x, 5\tau r)$ and g is a p-weak upper gradient of u. Furthermore, if $u \in N_0^{1,p}\big(B(x,r)\big)$, with $0 < r \le R_S$, then a *local Sobolev inequality*

$$\left(\fint_{B(x,r)} |u|^{\lambda p} \, d\mu \right)^{1/\lambda p} \le C_S r \left(\fint_{B(x,r)} g^p \, d\mu \right)^{1/p} \tag{4.10}$$

holds.

Let $1 < p < \infty$. Suppose that $\Omega \subset X$ is open and $\vartheta \in N^{1,p}(\Omega)$. A function $u \in N^{1,p}(\Omega)$ is called a *p-minimizer in Ω with boundary values ϑ* if $u - \vartheta \in N_0^{1,p}(\Omega)$ and

$$\int_\Omega |\nabla u|^p \, d\mu \le \int_\Omega |\nabla v|^p \, d\mu \tag{4.11}$$

for every $v \in N^{1,p}(\Omega)$, with $v - \vartheta \in N_0^{1,p}(\Omega)$. Recall that $|\nabla u|$ and $|\nabla v|$ are the minimal p-weak upper gradients of u and v in Ω, respectively. Let then $U \subset X$ be an open set. A function $u \in N_{\mathrm{loc}}^{1,p}(U)$ is called a *p-minimizer in U* if (4.11) holds for every open set $\Omega \Subset U$ and for all functions $v \in N_{\mathrm{loc}}^{1,p}(U)$, with $u - v \in N_0^{1,p}(\Omega)$. Furthermore, a function u is called *p-harmonic in U* if it is a continuous p-minimizer in U. It is proved in [27] that every p-minimizer in U can be redefined in a set of measure zero so that it becomes locally Hölder continuous in U. We refer to [11, 7.12, 7.14], [25, 3.2], and [31, 5.6] for the existence and uniqueness of p-harmonic functions with prescribed boundary values in relatively compact open sets. More precisely, suppose that $\Omega \Subset X$ is open and that $\vartheta \in N^{1,p}(\Omega)$ is bounded. Then there exists a unique p-harmonic function u in Ω, with $u - \vartheta \in N_0^{1,p}(\Omega)$.

Remark 4.2. Let (X, d, μ) be a metric measure space, where X is a Riemannian manifold, d is the Riemannian distance, and μ is the Riemannian measure. If

u is a smooth real-valued function on X, then $|\nabla u|$, the norm of the gradient of u, is the minimal p-weak upper gradient of u for all $p \geq 1$. Thus the spaces $N^{1,p}(X)$, $N^{1,p}_{\mathrm{loc}}(X)$, and $N^{1,p}_0(\Omega)$, where $\Omega \Subset X$, coincide with the corresponding usual Sobolev spaces $W^{1,p}(X)$, $W^{1,p}_{\mathrm{loc}}(X)$, and $W^{1,p}_0(\Omega)$. Furthermore, p-harmonic functions in an open set $U \subset X$ defined as continuous p-minimizers in U are, equivalently, continuous (weak) solutions of the p-Laplace equation

$$-\operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big) = 0,$$

that is,

$$\int_U \langle |\nabla u|^{p-2}\nabla u, \nabla\varphi\rangle \, d\mu = 0$$

for every $\varphi \in C_0^\infty(U)$.

4.3. Solving the Dirichlet problem at infinity. Let X be a connected, locally compact, and non-compact metric space equipped with a Borel regular measure μ satisfying the local doubling condition (4.6). Furthermore, we assume that the local weak $(1,p)$-Poincaré inequality (4.8) holds on X, with fixed $1 < p < \infty$. The following two lemmata are crucial in solving the Dirichlet problem at infinity. Their proofs combine ideas from [12] and [26]; see [22] for the details.

Lemma 4.3. [22, Lemma 5.1] *Suppose that a global (p,p)-Sobolev inequality*

$$\|u\|_p \leq C\, \||\nabla u|\|_p \tag{4.12}$$

holds for all compactly supported functions $u \in N^{1,p}(X)$. Let $\Omega \Subset X$ be an open set and $f \in N^{1,p}(\Omega)$ a bounded continuous function. Then for every $q \geq p$ there exists a constant $c = c(p,q,C)$ such that

$$\|u - f\|_{L^q(\Omega)} \leq c\, \||\nabla f|\|_{L^q(\Omega)}, \tag{4.13}$$

where $u \in N^{1,p}(\Omega)$ is the unique p-harmonic function in Ω with $u - f \in N^{1,p}_0(\Omega)$.

Proof. The idea of the proof is to first obtain the inequality

$$\int_\Omega |u-f|^{q-p}|\nabla u|^p \, d\mu \leq \int_\Omega |u-f|^{q-p}|\nabla f|^p \, d\mu \tag{4.14}$$

by using the fact that u is a p-harmonic in Ω. Then the Sobolev inequality (4.12) applied to the function $|u-f|^{q/p}$ together with (4.14) and the Hölder inequality imply (4.13). ■

In the next lemma, C_d and R_d are the constants in the local doubling condition (4.6) and $C_S > 0$, $R_S > 0$, and $\lambda > 1$ are the constants in the local Sobolev inequality (4.10).

Lemma 4.4. [22, Lemma 5.2] *Let $1 < p < \infty$ and $Q \geq p$. Let $f \in N^{1,p}_{\mathrm{loc}}(X)$ be a bounded continuous function such that its minimal p-weak upper gradient $|\nabla f|$ is bounded. Suppose that $\Omega \Subset X$ and that u is a bounded p-harmonic function in Ω, with $u - f \in N^{1,p}_0(\Omega)$, and $u - f = 0$ in $X \setminus \Omega$. Then there exists a constant $d \in (0,1)$ such that for every $x \in X$ and $0 < R \leq \min\{R_d, R_S/2\}$ we have*

$$\sup_{B(x,R)} |u-f|^Q \leq C \left(\fint_{B(x,2R)} |u-f|^Q \, d\mu \right)^d, \tag{4.15}$$

where C depends on $p, Q, C_d, R_d, \lambda, C_S, R_S, \sup_X|u-f|$, and $\sup_X|\nabla f|$, but is independent of $x \in X$.

Proof. The idea of the proof is to first obtain a Caccioppoli type estimate

$$\int_X \eta^p \big|\nabla|u-f|^{q/p}\big|^p \, d\mu \leq cq^p \int_X \eta^p |u-f|^{q-p} |\nabla f|^p \, d\mu + c \int_X |u-f|^q |\nabla \eta|^p \, d\mu$$

for Lipschitz test functions η by using the fact that u is p-harmonic in Ω. This applied to suitable test functions η together with the local Sobolev inequality (4.10) then imply (4.15). Here the argument is similar to the Moser iteration. ■

We employ two additional assumptions on measures of balls. The first one is a global volume growth condition

$$\mu(B(o,R)) \leq Ce^{\beta R} \tag{4.16}$$

for all $R > 0$, where $\beta > 0$ and $C > 0$ are constants and $o \in X$ is fixed. The second new assumption is a uniform positive lower bound for measures of balls with fixed small radius. More precisely, we assume that there exist constants $C_v > 0$ and $0 < R_v \leq \min\{2R_d, R_S\}$ such that

$$\mu(B(x,R_v)) \geq C_v \tag{4.17}$$

for all $x \in X$.

Theorem 4.5. [22, Theorem 6.2] *Let X be a connected, locally compact, and non-compact δ-hyperbolic metric space equipped with a non-trivial Borel regular measure μ supported on all of X. Suppose that the local weak $(1,p)$-Poincaré inequality (4.8) and the global (p,p)-Sobolev inequality (4.12) hold on X, with a fixed $1 < p < \infty$. Furthermore, we assume that the measure μ satisfies the local doubling condition (4.6), the volume growth condition (4.16), and has the uniform positive lower bound (4.17). If $f\colon \partial_G X \to \mathbb{R}$ is a bounded continuous function, there exists a continuous function $u\colon X^* \to \mathbb{R}$ which is p-harmonic in X and equal to f in $\partial_G X$.*

The proof of Theorem 4.5 is carried out in several steps that we briefly describe next. Suppose first that the given bounded continuous function on $\partial_G X$ is Lipschitz. Applying the McShane-Whitney extension theorem to the metric space (X^*, d_ε) and to the L-Lipschitz function $f\colon \partial_G X \to \mathbb{R}$ we obtain a function $F\colon X^* \to \mathbb{R}$ which is L-Lipschitz with respect to the metric d_ε. Note that F need not be continuous in the topology $\mathcal{T}^*$ that we use for X^*. Next we define, by using a Lipschitz partition of unity, another extension of $f\colon \partial_G X \to \mathbb{R}$ which will be Lipschitz in the original metric d. We denote this extension by $f\colon X^* \to \mathbb{R}$. The Gromov hyperbolicity of X implies that $\operatorname{Lip} f(x)$ has an exponentially decreasing upper bound

$$\operatorname{Lip} f(x) \le cL \exp\bigl(-\varepsilon|x-o|\bigr) \tag{4.18}$$

for all $x \in X$ (cf. [22, Lemma 3.2]). Here

$$\operatorname{Lip} f(x) = \limsup_{r\to 0} \sup_{y\in B(x,r)} \frac{|f(x)-f(y)|}{r}$$

is the pointwise upper Lipschitz constant of f at $x \in X$. This exponential decay together with the volume growth condition (4.16) imply L^Q-integrability of $|\nabla f|$ for suitably large Q. Using an exhaustion of X by relatively compact domains and solving the Dirichlet problem with boundary values f in each of these domains, we obtain a sequence of p-harmonic functions converging, after passing to a subsequence, locally uniformly to a function that is p-harmonic on all of X. To show that the p-harmonic limit function has the right boundary values, we apply the crucial Lemmata 4.3 and 4.4. This solves the Dirichlet problem at infinity with Lipschitz-continuous boundary values. The general case follows by another limiting argument. We refer to [22, Section 6] for the details and to [22, Section 7] for the discussion on the uniqueness of the solution.

References

[1] J. M. Alonso, and et al., *Notes on word hyperbolic groups.* In *Group theory from a geometrical viewpoint (Trieste, 1990)*, World Sci. Publishing, River Edge, NJ, 1991, pp. 3–63. Edited by H. Short.

[2] A. Ancona, *Negatively curved manifolds, elliptic operators, and the Martin boundary.* Ann. of Math. **125**(1987), 495–536.

[3] ———, *Positive harmonic functions and hyperbolicity.* In *Potential theory—surveys and problems (Prague, 1987)*, vol. 1344 of *Lecture notes in Math.* Springer, Berlin, 1988, pp. 1–23.

[4] ———, *Théorie du potentiel sur les graphes et les variétés.* In *École d'été de Probabilités de Saint-Flour XVIII—1988*, vol. 1427 of *Lecture notes in Math.* Springer, Berlin, 1990, pp. 1–112.

[5] ———, *Convexity at infinity and Brownian motion on manifolds with unbounded negative curvature.* Rev. Mat. Iberoamericana **10**(1994), 189–220.

[6] M. T. Anderson, *The Dirichlet problem at infinity for manifolds of negative curvature.* J. Differential Geom. **18**(1983), 701–721.

[7] M. T. Anderson and R. Schoen, *Positive harmonic functions on complete manifolds of negative curvature.* Ann. Math. **121**(1985), 429–461.

[8] A. Borbély, *A note on the Dirichlet problem at infinity for manifolds of negative curvature.* Proc. Amer. Math. Soc. **114**(1992), 865–872.

[9] ———, *The nonsolvability of the Dirichlet problem on negatively curved manifolds.* Differential Geom. Appl. **8**(1998), 217–237.

[10] M. R. Bridson and A. Haefliger, *Metric spaces of non-positive curvature,* vol. 319 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences].* Springer-Verlag, Berlin, 1999.

[11] J. Cheeger, *Differentiability of Lipschitz functions on metric measure spaces.* Geom. Funct. Anal. **9**(1999), 428–517.

[12] S. Y. Cheng, *The Dirichlet problem at infinity for non-positively curved manifolds.* Comm. Anal. Geom. **1**(1993), 101–112.

[13] H. I. Choi, *Asymptotic Dirichlet problems for harmonic functions on Riemannian manifolds.* Trans. Amer. Math. Soc. **281**(1984), 691–716.

[14] M. Coornaert, T. Delzant, and A. Papadopoulos, *Géométrie et théorie des groupes,* vol. 1441 of *Lecture notes in Math.* Springer, Berlin, 1990.

[15] P. Eberlein and B. O'Neill, *Visibility manifolds.* Pacific J. Math. **46**(1973), 45–109.

[16] A. H. Frink, *Distance functions and the metrization problem.* Bull. Amer. Math. Soc. **43**(1937), 133–142.

[17] R. E. Greene and H. Wu, *Function theory on manifolds which possess a pole*, vol. 699 of *Lecture notes in Math.* Springer, Berlin, 1979.

[18] P. Hajłasz, *Sobolev spaces on metric-measure spaces.* In *Heat kernels and analysis on manifolds, graphs, and metric spaces (Paris, 2002)*, vol. 338 of *Contemp. Math.* Amer. Math. Soc., Providence, RI, 2003, pp. 173–218.

[19] P. Hajłasz and P. Koskela, *Sobolev met Poincaré.* Mem. Amer. Math. Soc. **145**(2000).

[20] J. Heinonen, T. Kilpeläinen, and O. Martio, *Nonlinear potential theory of degenerate elliptic equations.* The Clarendon Press, Oxford University Press, New York, 1993.

[21] I. Holopainen, *Asymptotic Dirichlet problem for the p-Laplacian on Cartan-Hadamard manifolds.* Proc. Amer. Math. Soc. **130**(2002), 3393–3400.

[22] I. Holopainen, U. Lang, and A. Vähäkangas, *Dirichlet problem at infinity on Gromov hyperbolic metric measure spaces.* Math. Ann. **339**(2007), 101-134.

[23] E. P. Hsu, *Brownian motion and Dirichlet problems at infinity.* Ann. Probab. **31**(2003), 1305–1319.

[24] P. Hsu and P. March, *The limiting angle of certain Riemannian Brownian motions.* Comm. Pure Appl. Math. **38**(1985), 755–768.

[25] J. Kinnunen and O. Martio, *Nonlinear potential theory on metric spaces.* Illinois J. Math. **46**(2002), 857–883.

[26] J. Kinnunen and O. Martio, *Sobolev space properties of superharmonic functions on metric spaces.* Results Math. **44**(2003), 114–129.

[27] J. Kinnunen and N. Shanmugalingam, *Regularity of quasi-minimizers on metric spaces.* Manuscripta Math. **105**(2001), 401–423.

[28] H. Kumura, *The Dirichlet problem at infinity on Hadamard manifolds.* Nagoya Math. J. **138**(1995), 1–18.

[29] P. Pansu, *Cohomologie L^p des variétés à courbure négative, cas du degré* 1. In *Conference on Partial Differential Equations and Geometry (Torino, 1988)*, *Rend. Sem. Mat. Univ. Politec. Torino*, Special Issue (1989), 95–120 (1990).

[30] N. Shanmugalingam, *Newtonian spaces: an extension of Sobolev spaces to metric measure spaces.* Rev. Mat. Iberoamericana **16**(2000), 243–279.

[31] ———, *Harmonic functions on metric spaces.* Illinois J. Math. **45**(2001), 1021–1050.

[32] D. Sullivan, *The Dirichlet problem at infinity for a negatively curved manifold.* J. Differential Geom. **18**(1983), 723–732.

[33] A. Vähäkangas, *Dirichlet problem at infinity for $\mathcal{A}$-harmonic functions.* Potential Anal. **27**(2007), 27–44.

[34] J. Väisälä, *Gromov hyperbolic spaces.* Expo. Math. **23**(2005), 187–231.

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