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On the existence of undominated elements of acyclic relations ☆

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ABSTRACT

We study the existence of undominated elements of acyclic relations. A sufficient condition for the existence is given without any topological assumptions when the dominance relation is finite valued. The condition says that there is a point such that all dominance sequences starting from this point are reducible. A dominance sequence is reducible, if it is possible to remove some elements from it so that the resulting subsequence is still a dominance sequence. Necessary and sufficient conditions are formulated for closed acyclic relations on compact Hausdorff spaces. Reducibility is the key concept also in this case. A representation theorem for such relations is given.

1. Introduction

The existence of undominated (or maximal) members of acyclic relations has interested economists for a long time. These kind of relations appear e.g. in preference theory and in social choice theory (see Alcantud, 2002; Bergstrom, 1975; Campbell and Walker, 1990; Walker, 1977). The interpretation of being undominated in such applications means that there exists a choice for which there exist no strictly better choices. Since acyclicity seems quite natural in consumer choice theory for example, it is clear why such relations interest economists.

Acyclic relations have applications also in equilibrium theory and in the analysis of dynamic systems. Equilibrium existence results are always some kind of fixed point theorems. Existence of fixed points and existence of undominated members of a relation in turn are closely related problems. If a relation describes a dynamical process, then undominated elements could be the instances when the process stops. A game tree is a representative example.

A sufficient condition for the existence of undominated members is given in the general case without any topological assumptions (Theorem 1). We assume in this case that the dominance relation is acyclic and finite valued (dominance relation is the asymmetric part of a relation). The key assumption is that there is a point x_0 such that every dominance sequence starting from this point is *reducible*. Reducibility means that if we have a *dominance sequence* $\{x_m\}_{m=0}^{\infty}$, *i.e.*, x_{m+1} dominates x_m, then for some m the element x_m is dominated also by another member x_k, $k > m+1$. Transitivity would imply that x_m is dominated by x_k whenever $m < k$, so reducibility is a much weaker condition.

Necessary and sufficient conditions for the existence of undominated elements are given also for closed relations on a compact Hausdorff space. It turns out (Example 2) that the simple reducibility condition is too weak in this case since relations need not be finite valued. However, reducibility condition can be strengthened to *proper reducibility* that guarantees the existence of undominated elements (Theorem 2). Proper reducibility means that a dominance sequence has a subsequence that is smaller in a topological sense than the original sequence. More precisely, the closure of the subsequence is a proper subset of the closure of the original sequence. The (simple) reducibility says only that a dominance sequence can be made shorter, but the subsequence could still be as large (in a topological sense) than the original sequence.

The results of Bergstrom (1975), Walker (1977), Campbell and Walker (1990), Peris and Subiza (1994) and Alcantud (2002) are based on different assumptions than our results. In these papers it is usually assumed that relations are "lower continuous", which means that the lower contour sets are open (see Campbell and Walker, 1990; Alcantud, 2002, for weaker continuity assumptions).

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We analyze closed relations, and the lower contour sets are not open in general. Bergstrom (1975) and Walker (1977) prove that a lower continuous, acyclic and asymmetric relation has undominated elements. Alcantud (2002) gives sufficient *and* necessary conditions for the existence of undominated elements.

We prove a representation theorem for closed acyclic relations on compact Hausdorff spaces. Functions that represent relations in this paper are "one-way" in the sense that if x dominates y then the value of the utility function is strictly higher at x, but a higher utility value does not necessarily imply dominance. We show that if a closed asymmetric and acyclic relation has undominated elements in every uncountable closed subset space, then this relation can be represented by a lower semicontinuous function (Theorem 3). This function has a maximizer in *every* nonempty subset, and maximizers are undominated in that subset.

Peleg (1970) shows that an irreflexive, transitive, separable, lower continuous, and spacious relation on a topological space has a continuous one-way representation. A relation is spacious if, when a lower contour set A is a subset of another lower contour set B, then the closure of A is also a subset of B. The maximizers of this function are undominated elements. Peris and Subiza (1995) replace transitivity by acyclicity and show that *to each subset* of the space there exists a function that represents the relation restricted to this subset. Moreover, the maximizers of that function are precisely the elements that are undominated in this subset. We assume neither separability nor spaciousness in Theorem 3.

The paper is organized in the following way. In Section 2 some notation is introduced. In Section 3 we give examples of situations when undominated members do not exist. Examples are simple but reveal something essential about what goes wrong when undominated members do not exist. The results are presented in Section 4.

2. Preliminaries

Let X be a nonempty set, and R a binary relation on X, so R is a subset of $X \times X$. The set X is called the *field of* R. We may denote $(x, y) \in R$ by xRy as usual. We will now define quickly the most important properties of relations we need, see Aleskerov and Monjardet (2002) for a more detailed analysis and comparison of these properties.

A relation R is *reflexive*, if xRx holds for every $x \in X$. R is *transitive*, if xRy and yRz imply xRz, for all $x, y, z \in X$. R is *antisymmetric*, if xRy and yRx implies $x = y$, for every $x, y \in X$. A relation satisfying reflexivity, transitivity and antisymmetry is a *partial order*. A relation R is *irreflexive*, if xRx does not hold for any x. R is *asymmetric*, if xRy implies that yRx does not hold for any $x, y \in X$. A relation satisfying transitivity and asymmetry is a *strict partial order*. The asymmetric part of any partial order is a strict partial order.

The *inverse* R^{-1} of a relation is a relation defined by $yR^{-1}x$ iff xRy, for all $x, y \in X$. We define formally the asymmetric or *strict* part P of any relation R by $P = R \setminus (R \cap R^{-1})$. We say that *$y$ dominates x* if xRy but not yRx. That is, the asymmetric part P of R represents dominance. An element is *undominated*, if there is no element that dominates it.

A finite subset $\{x_0, \dots, x_n\} \subset X$ such that x_iRx_{i+1} for $i = 0, \dots, n-1$, is called a *path in* R. A path $\{x_0, \dots, x_n\}$ is a *dominance path* if x_{i+1} dominates x_i for all $i = 0, \dots, n-1$. If it is clear what relation R is in question, we may simply say that $\{x_0, \dots, x_n\}$ is a (dominance) path. R is *acyclic*, if $x_0 \neq x_n$ for every dominance path $\{x_0, \dots, x_n\}$, $n > 0$.

A sequence $\{x_m\}_{m=0}^{\infty}$ is a *dominance sequence* starting from x_0, if all of its initial segments $\{x_0, \dots, x_k\}$, $k > 0$ are dominance paths. We say that a dominance sequence $\{x_m\}_{m=0}^{\infty}$ is *reducible*, if x_iRx_k holds for some i and k such that $k > i + 1$. That is, some member of the dominance sequence is dominated by at least two members of the sequence. If this does not hold for any x_i, then the dominance sequence is *irreducible*. A subsequence $\{x_{m_k}\}_{k=0}^{\infty}$ of a dominance sequence $\{x_m\}_{m=0}^{\infty}$ is called a *reduction* of $\{x_m\}_{m=0}^{\infty}$, if $\{x_{m_k}\}_{k=0}^{\infty}$ is a also a dominance sequence starting from x_0, and $\{x_{m_k}\}_{k=0}^{\infty} \neq \{x_m\}_{m=0}^{\infty}$.

If X is a topological space, we say that a dominance sequence $\{x_m\}_{m=0}^{\infty}$ is *properly reducible* if it has reduction $\{x_{m_k}\}_{k=0}^{\infty}$ such that $\mathrm{cl}\,\{x_{m_k}\}_{k=0}^{\infty}$ is a proper subset of $\mathrm{cl}\,\{x_m\}_{m=0}^{\infty}$, where $\mathrm{cl}\,Z$ denotes the closure of any $Z \subset X$. In this case we may say that $\{x_{m_k}\}_{k=0}^{\infty}$ is a *proper reduction of* $\{x_{m_k}\}_{k=0}^{\infty}$. (We abuse the notation slightly by denoting the set of members of a sequence $\{x_m\}_{m=0}^{\infty}$ with the same symbol as the sequence itself.)

If R is a relation on a nonempty set X, and Y is a nonempty subset of X, define the restriction of R to Y by $R_{|Y} = R \cap (Y \times Y)$. The field of the relation $R_{|Y}$ is Y. Then $R_{|Y}$ has no undominated members, if and only if for each $y \in Y$ there is $y' \in Y$ such that yRy'. We say that $y \in Y$ is undominated in Y, if yRy' does not hold for any $y' \in Y$, *i.e.*, if $R_{|Y}$ has undominated elements.

Given a nonempty $Y \subset X$, let $YR = \{x \in X \mid yRx \text{ for some } y \in Y\}$, and $RY = \{x \in X \mid xRy \text{ for some } y \in Y\}$. YR is called *the image of* Y, and RY is called the *inverse image of* Y. RX is the *domain of* R, and XR is the *range of* R. In case of singleton sets $\{x\}$, we may simply denote its image and inverse image by xR and Rx.

If R is asymmetric, it has undominated members if and only if the domain is not the whole X, and in this case $X \setminus RX$ is the set of undominated members. So undominated members of an asymmetric R exist precisely when the domain of R is a proper subset of the field of R.

Let X be a topological space. A relation R on X is *closed* if R is a closed subset of the product space $X \times X$ which is equipped with the product topology. If X is compact Hausdorff and $Z \subset X$ is closed, then both ZR and RZ are closed. A relation R on X is *lower continuous*, if Px is open for any $x \in X$. A relation is *spacious*, if xPy implies $\mathrm{cl}\,(Px) \subset Py$ for all $x, y \in X$ (Peleg, 1970).

3. Examples

Let us give next two examples of acyclic relations in which undominated members do not exist.

Example 1. Let $X \subset \mathbb{R}^2$ be the boundary of the closed unit ball with center at the origin. Define a relation R on X such that xRy, if the distance along X from x to y is 1, when we move from x to y clockwise. Since for all x there is a y such that xRy, no undominated members exist. Further R is acyclic: there are no points $x_1, \dots, x_n$ such that x_iRx_{i+1}, $i = 1, \dots, n-1$, and x_nRx_1. This follows from the fact that X has length 2π. Clearly, the relation R is also asymmetric and closed. In fact, there is a homeomorphism $f : X \longrightarrow X$ such that R is the graph of f: $y = f(x)$ iff xRy.

So acyclicity and closedness alone do not guarantee the existence of undominated members, even when the relation is "single valued". The relation in Example 1 has the feature that all dominance sequences are irreducible.

The existence of irreducible sequences from any initial value x in fact precludes the possibility that there are undominated members. We show a partial converse in Theorem 1. Suppose there is some initial value x_0 such that all dominance sequences are reducible. Then there are undominated members, if R is acyclic and xP is finite for every x. In this result, no topological assumptions are needed.

However, if we drop the assumption that xP is finite, then there need not exist undominated elements even when there are initial points from which all dominance sequences are reducible.

Example 2. Let X be as in Example 1, and let relation $\bar{R}$ be as the relation R in Example 1, except that $x_0 = (1, 0)$ is dominated by infinitely many elements as follows. Let x_1 be the second element of the dominance sequence $\{x_m\}_{m=0}^{\infty}$. Let $z_1 \neq x_1$ be the first member of this dominance sequence lying on the short segment between x_0 and x_1 such that the distance to x_1 is shorter than the distance to x_0.

Given that $z_1, \ldots, z_{k-1}$ have been chosen, $k > 1$, let $z_k \neq x_1$ be the first element of the dominance sequence lying on the short segment between z_{k-1} and x_1 and being closer to x_1 than to z_{k-1}.

In this way we get a sequence $\{z_k\}_{k=1}^{\infty}$ converging to x_1. Define relation $\bar{R}$ so that $x_0\bar{R} = \{x_1\} \cup \{z_k \mid k = 1, \ldots\}$, and elsewhere $\bar{R}$ is like R. By construction, $\bar{R}$ is acyclic, asymmetric and closed. All the dominance sequences of $\bar{R}$ starting from x_0 are reducible. But even this relation has no undominated elements.

Note that in Example 2, $\mathrm{cl}\,\{x_m\} = X$ for *all* dominance sequences in X. In particular, none of the dominance sequences has a proper reduction. In fact, to guarantee the existence of undominated elements in X, it would suffice that there is at least one point x_0 such that some dominance sequence starting from x_0 has a proper reduction (Theorem 2).

4. Undominated elements

Theorem 1. *Suppose that R is acyclic and xP is finite for every $x \in X$. There are undominated elements, iff there exists $x_0 \in X$ such that every dominance sequence starting from x_0 is reducible.*

Proof. If x_0 is undominated, then there are no dominance sequences starting from x_0, and therefore all dominance sequence starting from x_0 are reducible.

Let $x_0 \in X$ be such that every dominance sequence starting from x_0 is reducible. Assume that there are no undominated elements in X. Then there exists at least one dominance sequence $\{x_n\}_{n\geq 0}$ starting from x_0. Construct a reduction $\{x_{n_k}\}_{k\geq 0}$ of $\{x_n\}_{n\geq 0}$ in the following way.

Let $x_{n_0} = x_0$. Let n_1 be the highest index $m > 0$ such that x_0Px_m. A highest index exists since xP is finite for all x. Hence x_{n_1} has been defined. Suppose x_{n_k} is defined for $k \geq 1$. Let n_{k+1} be the highest index $m > n_k$ such that $x_{n_k}Px_m$. A highest index exists since xP is finite for all x.

By construction $\{x_{n_k}\}_{k\geq 0}$ is a dominance sequence starting from x_0. By assumption also this dominance sequence is reducible. So there exists n_k and $m \in \{n_{k+2}, n_{k+3}, \ldots\}$ such that $x_{n_k}Px_m$. But n_{k+1} is the highest index m such that $x_{n_k}Px_m$, a contradiction.

Hence there cannot exist dominance sequences starting from x_0. Let Y be the set of all elements belonging to some dominance path $\{x_0, \ldots, x_m\}$. If Y is finite then the acyclicity of R implies that undominated elements exist. Suppose then that Y is infinite. Now the set of all dominance paths starting from x_0 is a finite branching tree Γ having infinitely many nodes. The tree Γ can be constructed in the following way.

Take x_0 and all its successors $x_0P = \{x_{01}, \ldots, x_{0k}\}$ of x_0 as nodes of the tree. Given any $x_{0i} \in x_0P$, take all its successors $x_{0i}P = \{x_{0i1}, \ldots, x_{0in}\}$ as nodes of the tree, where n may depend on the string $0i$. Proceeding recursively, suppose a node $x_{0\cdots j}$ has been constructed. Take the set of all its successors $x_{0\cdots j}P = \{x_{0\cdots j1}, \ldots, x_{0\cdots jm}\}$ as nodes, where m may depend on the string $0\cdots j$. Some of the subsets $x_{0\cdots j}P$ may be empty, but there is a node $x_{0\cdots t}$ such that $x_{0\cdots t}P$ is nonempty and the strings $0\cdots j$ and $0\cdots t$ are equally long. This follows since Y is infinite. Note that a given $y \in Y$ may belong to several dominance paths, but given a dominance path $\{x_0, \ldots, z\}$, y appears in this path at most once. To each node z of Γ there is a unique dominance path $\{x_0, \ldots, z\}$ starting from x_0, so Γ is a finite branching tree with infinitely many nodes.

Then by König's lemma Γ has an infinite branch, *i.e.*, a dominance sequence $\{x_n\}_{n=0}^{\infty}$, a contradiction. Therefore Y is finite and undominated elements exist. □

The proof of Theorem 1 is non-constructive, since no constructive way of finding the root x_0 of the tree Γ was provided.

In the following, X is a compact Hausdorff space and R is a closed relation on it.

Lemma 1. *If R is a closed asymmetric and acyclic relation on a compact Hausdorff space X with no undominated members, then there is a minimal nonempty closed $Y \subset X$ such that every $y \in Y$ is dominated in Y.*

Proof. See the Appendix. □

Recall that a nonempty closed subset Y of a metric space is called *perfect*, if Y contains no points that are isolated in Y, *i.e.*, there is no $y \in Y$ such that for some open neighbourhood $V(y)$ of y, $Y \cap V(y) = \{y\}$. Perfect subsets are uncountable.

Lemma 2. *Let R be a closed asymmetric and acyclic relation on a compact Hausdorff space X with no undominated members, and let $Y \subset X$ be a minimal closed subset in which all elements are dominated as in Lemma 1. Then Y is homeomorphic to a compact perfect metric space, and $Y = R_{|Y}Y = YR_{|Y}$.*

Proof. See the Appendix. □

Now we are ready to prove our second main result.

Theorem 2. *Suppose R is a closed asymmetric and acyclic relation on a compact Hausdorff space X. Every closed subset C of X has elements that are undominated in C, iff in every uncountable closed $Z \subset X$ there is x_0 such that a dominance sequence $\{x_m\}_{m=0}^{\infty} \subset Z$ starting from x_0 has a proper reduction.*

Proof. If a closed subset C has an undominated element x, then there are no dominance sequences starting from x. Hence any dominance sequence starting from x has a proper reduction. We show the implication to the other direction by proving the cases when C is countable and when C is uncountable separately.

Case 1: *$C \subset X$ is closed and countable.* If there were no undominated members in C, then by Lemmas 1 and 2 there would exist a nonempty subset $Y \subset C$ such that Y is homeomorphic to a compact perfect metric space. But the cardinality of such a set is that of the continuum, a contradiction.

Case 2: *$C \subset X$ is closed and uncountable.* If there were no undominated members in C, then by Lemma 1 there would exist a minimal closed $Y \subset C$ such that all elements of Y are dominated in Y. By Lemma 2, Y is homeomorphic to a compact perfect metric space. We showed in the proof of Lemma 2 that $Y = \mathrm{cl}\,\{x_m\}_{m=0}^{\infty}$ where $\{x_m\}_{m=0}^{\infty}$ is an arbitrary dominance sequence in Y. By assumption, there is $x_0 \in Y$ such that at least one dominance sequence $\{x_m\}_{m=0}^{\infty}$ has a proper reduction, say $\{x_{m_k}\}_{k=0}^{\infty}$. By definition, this means that $\mathrm{cl}\,\{x_{m_k}\}_{k=0}^{\infty}$ is a proper subset of $\mathrm{cl}\,\{x_m\}_{m=0}^{\infty} = Y$, a contradiction. □

We say that a function $u : X \longrightarrow \mathbb{R}$ is a *one-way representation* of the strict part P of relation R, if xPy implies $u(x) < u(y)$. That is, if y dominates x then $u(x) < u(y)$.

The idea of the proof of the following result is simple. We eliminate iteratively *undominated elements* from the set X. It turns out that this process converges after finitely many steps. The process induces a finite partition of X, and members of this partition are taken as indifference classes of a function u that represents R.

Theorem 3. *Suppose R is a closed and acyclic relation on a compact Hausdorff space X such that the strict part P of R is closed as well. Each uncountable closed $Z \subset X$ contains members that are undominated in Z, iff there is a lower semicontinuous function $u : X \longrightarrow \mathbb{R}$ with finite range such that xPy implies $u(x) < u(y)$ for all $x, y \in X$.*

Proof (*Necessity*)**.** Suppose each uncountable closed $Z \subset X$ contains members that are undominated in Z. Then this holds for every nonempty closed Z by Lemma 2. Let $X_0 = X$, and define $X_{n+1} = PX_n$, $n > 0$. Then $X_{n+1} \subset X_n$ and each X_n is closed. Let us show first that there is $K > 0$ such that $X_n = \emptyset$ for all $n \geq K$.

If $X_n \neq \emptyset$ for all n, then $Z = \cap_n X_n$ is a nonempty closed set. If Z is uncountable then it has undominated elements, and if Z is countable it also has undominated elements (see *Case* 1 in the proof of Theorem 2). Let $z \in Z$ be undominated. Since $z \in X_{n+1}$ for every n, it follows that $yR \cap X_n \neq \emptyset$ for every n. Hence $yR \cap Z \neq \emptyset$, and so y is dominated in Z, a contradiction. Hence there is $K > 0$ such that $X_n = \emptyset$ for all $n > K$. Assume w.l.o.g. that K is the least such integer.

Let $S_n = X_n \setminus X_{n+1}$, $n < K$, and $S_K = X_K$. Then S_n contains all members of X_n that are undominated in X_n. Note that since each X_{n+1} is closed for $n < K$, the union $S_0 \cup \cdots \cup S_n = X \setminus X_n$ is open, and that subsets S_n and S_m are disjoint when $n \neq m$. The union of all subset S_n is X, so $\{S_n\}_{n=0}^{K}$ is a partition of X.

Define a function u on X by $u(x) = (K - n)/K$, where n is the unique number such that $x \in S_n$. Then xPz implies $u(x) < u(z)$. To see this, note that $u(x) = k/K$ iff $x \in S_{K-k}$. Since x is undominated in X_{K-k}, $z \in S_n$ implies $n < K - k$. Hence $u(z) = (K - n)/K > k = u(x)$. Fix $a \in \mathbb{R}$, and note that $\{x \in X \mid a < u(x)\}$ is open, and so u is lower semicontinuous.

Sufficiency. Suppose there is a lower semicontinuous function $u : X \longrightarrow \mathbb{R}$ with finite range such that xPz implies $u(x) < u(z)$ for all $x, z \in X$. We may assume w.l.o.g. that $u[X] = \{0, 1/K, \ldots, 1\}$, for some natural number $K > 0$. Let $T_n = \{x \in X \mid n - 1 < u(x) \leq n\}$, and note that the unions $T_k \cup \cdots \cup T_K$ are open for $k \leq K$, and that $\{T_n \mid 0 \leq n \leq K\}$ is a partition of X.

Let Z be a nonempty closed subset of X, and let k be the largest number n such that $Z \cap T_n$ is nonempty. For each $x \in Z \cap T_k$, $u(x) = k/K$, and xPy implies $u(x) < u(y)$. But then $y \notin Z$. So members of $Z \cap T_k$ are undominated in Z. □

As demonstrated in the next example, Theorem 3 does not hold if we drop the assumption that the strict part P of the relation R is closed.

Example 3. Let λ be the cardinality of the set $2^{[0,1]}$. Consider the interval $X = [0, \lambda]$ of ordinal numbers with its usual well-ordering $\preceq$. When X is equipped with the order topology, it becomes a compact Hausdorff space (see Givant and Halmos, 2009, p. 309). (The sub-basis of the order topology consists of intervals $[0, \alpha)$, $(\beta, \lambda]$, $\alpha, \beta \in X$.) Define a relation R on X by xRy iff $y \preceq x$, *i.e.*, R is the inverse of $\preceq$. The order topology makes R a closed relation. Since $\preceq$ is a well-ordering, every nonempty subset $Z \subset X$ has a least element $\inf Z \in Z$. So $\inf Z$ is the greatest element in Z with respect to R, and hence $\inf Z$ is undominated (w.r.t. R) in Z. Next we show that he strict part P of R is not closed. Take any $\alpha \in X, \alpha \neq \lambda$, and note that $\lambda P \alpha$. Since $\lambda = \inf[0, \lambda)$ (w.r.t. R), P is not closed since $\lambda P \lambda$ does not hold. Note that all the other assumptions of Theorem 3 are satisfied. But Theorem 3 does not hold since the totally ordered set X has a strictly greater cardinality than the set of real numbers $\mathbb{R}$, and hence there cannot be any function $u : X \longrightarrow \mathbb{R}$ representing P.

Note that since the range of the function u in Theorem 3 is finite, u has a maximum in every nonempty subset. The maximizers are clearly undominated in this subset. Here is a simple example.

Example 4. Let $X = [0, 3]$, and define an asymmetric relation P by

$$Px = \begin{cases} x - 1 & 1 \leq x \leq 3 \\ \{x - 1, x - 2\} & 2 \leq x \leq 3 \\ \emptyset & 0 \leq x < 1. \end{cases}$$

Using the same notation as in the proof of Theorem 3, $S_0 = (2, 3]$, $S_1 = (1, 2]$, $S_2 = (0, 1]$, $S_3 = \{0\}$. Then S_0 is the set of elements that are undominated in X, S_1 is the set of elements that are undominated in $X \setminus S_0 = [0, 2]$, and so on. The function u that represents P is given by $u(x) = (3 - n)/3$ for $x \in S_n$, $n = 0, 1, 2, 3$.

Appendix

Proof of Lemma 1. Since R is asymmetric, xRy implies that y dominates x. Partially order by set inclusion the set $\mathcal{C}$ of all nonempty closed subsets Z of X, such that all members of Z are dominated in Z. $\mathcal{C}$ is nonempty, since by assumption $X \subset \mathcal{C}$. Let $\mathcal{T}$ be a maximal totally ordered subset of $\mathcal{C}$. Then $\mathcal{T}$ exists by the Hausdorff Maximality Principle. Let Y be the intersection of the members of $\mathcal{T}$. Then Y is nonempty and closed, since every $Z \in \mathcal{T}$ is nonempty and closed and X is a compact Hausdorff space. Since every $z \in Z$ is dominated in Z, we have $Z \subset RZ$ for all $Z \in \mathcal{T}$.

Choose $y \in Y$. Since y is dominated in Z, $yR \cap Z$ is nonempty and closed for any $Z \in \mathcal{T}$. Therefore $yR \cap Y$ is nonempty and closed, and y is dominated in Y. Since y was chosen arbitrarily, we are done. □

Proof of Lemma 2. By Lemma 1, $R_{|Y}$ has no undominated members, and therefore $Y = R_{|Y}Y$. Since $R_{|Y}$ is viewed as a relation on Y, we have $YR_{|Y} \subset Y$. Let $Z = YR_{|Y}$, and note that Z consists of all those members of Y that dominate some member of Y. Since every $z \in Z$ is dominated by some $y \in Y$, we must have $y \in Z$. Since Y is minimal and Z is closed, we have $Z = YR_{|Y} = Y$ by Lemma 1.

Let $\{x_m\}_{m=0}^{\infty} \subset Y$ be any dominance sequence. Such a sequence exists, since Y contains no undominated members. Since Y is closed, $\mathrm{cl}\,\{x_m\}_{m=0}^{\infty} \subset Y$. Now $\{x_m\}_{m=0}^{\infty} = \mathrm{cl}\,\{x_m\}_{m=0}^{\infty}$ is impossible. To see this, suppose $\{x_m\}_{m=0}^{\infty} = \mathrm{cl}\,\{x_m\}_{m=0}^{\infty}$, that is, $\{x_m\}_{m=0}^{\infty}$ is closed. Then $\{x_m\}_{m=1}^{\infty} = \{x_m\}_{m=0}^{\infty}R \cap \{x_m\}_{m=0}^{\infty}$ would be closed as well, since CR is closed for any nonempty closed $C \subset Y$. By induction, for all natural numbers k, the dominance sequence $\{x_m\}_{m=k}^{\infty}$ starting from x_k would be closed. Since $\{x_m\}_{m=n}^{\infty} \subset \{x_m\}_{m=k}^{\infty}$ when $n > k$, the intersection $\cap_k \{x_m\}_{m=k}^{\infty}$ must be nonempty, because Y is a compact Hausdorff space. Then for some n, $x_n \in \{x_m\}_{m=k}^{\infty}$ for all k, a contradiction with acyclicity. Therefore $\{x_m\}_{m=0}^{\infty}$ is a proper subset of $\mathrm{cl}\,\{x_m\}_{m=0}^{\infty}$.

The set $\{x_m\}_{m=0}^{\infty}$ is a compact Hausdorff space having a countable dense subset $\{x_m\}_{m=0}^{\infty}$. Compact Hausdorff spaces are *normal*: disjoint closed subsets have disjoint open neighbourhoods. Urysohn's metrizability theorem says that every normal space with a countable dense subset is metrizable. Hence we may view $\mathrm{cl}\,\{x_m\}_{m=0}^{\infty}$ as a compact metric space.

Take any $y \in \mathrm{cl}\,\{x_m\}_{m=0}^{\infty} \setminus \{x_m\}_{m=0}^{\infty}$. There must be a subsequence $\{x_{m_k}\}_{k=0}^{\infty}$ of $\{x_m\}_{m=0}^{\infty}$ converging to y, since we may view $\mathrm{cl}\,\{x_m\}_{m=0}^{\infty}$ as a compact metric space. For the same reason the sequence $\{x_{m_k+1}\}_{k=0}^{\infty}$ has a subsequence converging to $z \in \mathrm{cl}\,\{x_m\}_{m=0}^{\infty}$. Assume *w.l.o.g.* that $\{x_{m_k+1}\}_{k=0}^{\infty}$ converges. Since R is a closed relation, and $x_{m_k}Rx_{m_k+1}$, it follows that yRz. Hence every member of $\mathrm{cl}\,\{x_m\}_{m=0}^{\infty}$ is dominated by some member of $\mathrm{cl}\,\{x_m\}_{m=0}^{\infty}$. Since $\mathrm{cl}\,\{x_m\}_{m=0}^{\infty}$ is a subset of Y, we must have $Y = \mathrm{cl}\,\{x_m\}_{m=0}^{\infty}$ by minimality of Y.

Recall that $\{x_m\}_{m=0}^{\infty} \subset Y$ was an arbitrarily chosen dominance sequence. If $y \in Y$ were isolated, we could take a dominance sequence $\{z_m\}_{m=0}^{\infty} \subset Y$ such that z_0 dominates y. Then by acyclicity, y cannot be a member of $\{z_m\}_{m=0}^{\infty}$. But then y cannot be a member of $\mathrm{cl}\,\{z_m\}_{m=0}^{\infty}$ either since y is isolated. Therefore $y \notin Y$ since $Y = \mathrm{cl}\,\{z_m\}_{m=0}^{\infty}$, a contradiction. Hence Y is perfect as desired. □

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