FDPE Decision Theory
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Problem set 1

1. Weak Axiom of Revealed Preference WARP is defined by: if $x, y \in A \cap B$, then $x \in c(A)$ and $y \in c(B)$ imply $x \in c(B)$

   (a) Prove that if choice function c satisfies WARP, then there is a preference relation $\succsim$ such that $c_{\succsim} = c$

   A: Take $\{(x,y) : x, y \in X\}$ and let $\succsim$ satisfy $x \succsim y$ iff $x \in c(\{x,y\})$. Since c is a choice function, $c(\{x,y\})$ is nonempty. Thus the derived $\succsim$ is complete. For transitivity, take any x, y, z, and let $x \in c(\{x,y,z\})$. By WARP, $x \in c(\{x,y\})$, and $x \in c(\{x,z\})$. Thus $x \succsim y$ and $x \succsim z$. This implies that there cannot be a cycle of the form $x \succ y \succ z \succ x$, and hence $\succsim$ is transitive.

   b. Prove that the combination of Sen's α and β are equivalent to WARP

   A: That WARP implies α, i.e. if $x \in c(A)$ and $x \in B \subseteq A$ then $x \in c(B)$, follows since by nonemptiness of c there is y such that $y \in c(B)$. Sen's β, i.e. if $x, y \in c(A)$ and $A \subseteq B$ and $y \in c(B)$, then $x \in c(B)$, follows directly from the statement of WARP. For the other direction, suppose that c satisfies α and β but violates WARP. Then there are $x, y \in A \cap B$, $x \in c(A)$ and $y \in c(B)$, but $x \notin c(B)$. By α, $x, y \in c(A \cap B)$. By β, $x \in c(B)$, a contradiction.

   c. Give an example of c that violates WARP

   A: A three cycle.

2. Let X be finite. Let complete and transitive preferences over lotteries satisfy the vNM independence axiom. Show that there is a maximal (and minimal) lottery L.

A: We show that 1_{x^*} such that $1_{x^*} \succsim 1_x$ for all $x \in X$ is the desired maximal lottery. Take any lottery p^0, and order the n elements in the support of p^0 by $1_{x_1} \succsim \cdots \succsim 1_{x_n}$. By transitivity and completeness of $\succsim$ it suffices to show that $1_{x_1} \succsim p^0$. Note that

$$p^0 = p^0(x_1) \cdot 1_{x_1} + (1 - p^0(x_1)) \cdot p^1$$

where

$$p^1(x_k) = \frac{p^0(x_k)}{1 - p^0(x_1)}, \text{ for all } k \geq 1$$

By independence, then,

$$1_{x_1} \succsim p^0 \text{ if and only if } \; 1_{x_1} \succsim p^1$$

By induction, for all $k = 1, ..., n$,

$$1_{x_k} \succsim p^{k-1} \text{ if and only if } \; 1_{x_k} \succsim p^k.$$

By transitivity,

$$1_{x_1} \succsim p^0 \text{ if and only if } 1_{x_n} \succsim p^n.$$

Since necessarily $p^n = 1_{x_n}$, it follows that $1_{x_1} \succsim p^0$.

3. Relax the assumption that X is finite but focus on *simple* lotteries that have a finite support. Prove the vNM theorem.

   A: Take any finite subset Y of X. Take any two outcomes $x_1, x_0 \in X$ such that $1_{x_1} \succ 1_{x_0}$. Let $u(x_1) = 1$ and $u(x_0) = 0$. For any other outcome x , take set $\{x_0, x_1, x\}$. There are three possible orderings

$$\begin{gathered} 1_{x_1} \succsim 1_{x_0} \succsim 1_x \\ 1_{x_1} \succsim 1_x \succsim 1_{x_0} \\ 1_x \succsim 1_{x_1} \succsim 1_{x_0} \end{gathered}$$

   Depending on which ordering materializes, scale the utility index from x such that

$$u(x) = \begin{cases} 1/\alpha, & \text{if } 1_{x_1} \sim \alpha \cdot 1_x + (1-\alpha) \cdot 1_{x_0} \\ \alpha, & \text{if } 1_x \sim \alpha \cdot 1_{x_1} + (1-\alpha) \cdot 1_{x_0} \\ \alpha/(\alpha - 1), & \text{if } 1_{x_0} \sim \alpha \cdot 1_{x_1} + (1-\alpha) \cdot 1_x \end{cases}$$

Extend the same utility indeces for outcomes in any finite sets $A, B \subseteq X$. Then the indeces can be interpreted as rescaled versions of the indeces obtained in the characterization theorem under $X = A \cup B$.

For any p, q with finite supports A and B respectively, $p \succsim q$ if and only if

$$\sum_{A \cup B} q(x)u(x) \geq \sum_{A \cup B} q(x)u(x)$$

as desired.

4. (Mixture Space Theorem) Let Δ be a set that is closed under a mixing operation (= Δ is a mixture space): for all $x, y \in \Delta$ and for all $\lambda \in [0,1]$ imply $x \cdot a + y \cdot (1-a) \in \Delta$. Let $\succsim$ be a preference order on Δ such that:

   (a) (Continuity) For all $x, y, z \in \Delta$, if $x \succ y \succ z$, then there are $\lambda, \mu \in (0,1)$ such that $\lambda \cdot x + (1-\lambda) \cdot z \succ y \succ \mu \cdot x + (1-\mu) \cdot z$

   (b) (Independence) For all $x, y, z \in \Delta$ and $\lambda \in (0,1)$, if $x \succsim y$ then $\lambda \cdot x + (1-\lambda) \cdot z \succsim \lambda \cdot y + (1-\lambda) \cdot z$

   Show that any utility function $U : L \to \mathbb{R}$ that represents $\succsim$ is additive: $U(x \cdot \lambda + z \cdot (1-\lambda)) = U(x)\lambda + U(z)(1-\lambda)$. Moreover, show that U is unique up to positive linear transformations.

   A: By continuity and independence there is, for any $x \succsim y \succsim z$, a unique λ_y such that ,

   $$\lambda_y \cdot x + (1-\lambda_y) \cdot z \sim y \tag{1}$$

   and, for any λ, λ',

   $$\lambda \cdot x + (1-\lambda) \cdot z \succsim \lambda' \cdot x + (1-\lambda') \cdot z \quad \text{if and only if} \quad \lambda \geq \lambda' \tag{2}$$

   (apply the lemmata in the lecture notes). Construct a Bernoulli utility function $U(\cdot)$ as in the previous exercise (use any $x_0, x_1 \in \Delta$ as the origin of scaling).

   Independence implies that $x \succsim y \succsim z$ for $y = x \cdot \lambda + z \cdot (1-\lambda)$. Hence, by (1)

   $$U(x \cdot \lambda + z \cdot (1-\lambda)) = U(x)\lambda_y + U(z)(1-\lambda_y).$$

By definition,

$$y \sim x \cdot \lambda + z \cdot (1-\lambda).$$

Thus

$$x \cdot \lambda_y + z \cdot (1-\lambda_y) \sim x \cdot \lambda + z \cdot (1-\lambda).$$

which implies, by (2), $\lambda = \lambda_y$ and, a fortiori,

$$U(x \cdot \lambda + z \cdot (1-\lambda)) = U(x)\lambda + U(z)(1-\lambda).$$

To see the uniqueness, take any $x, y, z \in \Delta$,

$$U(y) = \lambda U(x) + (1-\lambda)U(z).$$

Note that for any representation V, we can find unique $a \in \mathbb{R}_{++}$ and $b \in \mathbb{R}$ such that

$$V(x) = aU(x) + b \quad \text{and} \quad V(z) = aU(z) + b.$$

Then also

$$\begin{aligned} V(y) &= \lambda[aU(x)+b] + (1-\lambda)[aU(z)+b] \\ &= a[\lambda U(x) + (1-\lambda)U(z)] + b \\ &= aU(y) + b \end{aligned}$$

Since this was derived for an arbitrary ordered triple $\{x, y, z\}$, the conclusion holds, by replacing one element at a time, for all elements in Δ. Thus $V = aU + b$.

5. Show that there is a utility function $u : X \to \mathbb{R}$ and nonatomic probability measure μ on S display expected utility maximization, then the preferences drawn from them satisfy S1-S6.

   A: We only show that S3-S6 are met by u and μ.

   S3: For every $f, g, f', g' \in F$ and for every $A \in 2^S$, since $f(s) = g(s)$ and $f'(s) = g'(s)$ for all $s \notin A$ it follows that

$$\begin{aligned} \int_S [u(f(s)) - u(g(s))]d\mu(s) &= \int_A [u(f(s)) - u(g(s))]d\mu(s) \\ \int_S [u(f'(s)) - u(g'(s))]d\mu(s) &= \int_A [u(f'(s)) - u(g'(s))]d\mu(s) \end{aligned}$$

Since $f(s) = f'(s)$ and $g(s) = g'(s)$ for all $s \in A$, we have

$$\int_A [u(f(s)) - u(g(s))] d\mu(s) = \int_S [u(f'(s)) - u(g'(s))] d\mu(s).$$

Thus $f \succsim g$ if and only if $f' \succsim g'$.

S4: Take $A, B \in 2^S$ and let $f(s) = x$ and $f'(s) = x'$ for all $s \in A$ and $f(s) = y$ and $f'(s) = y'$ for all $s \notin A$ and $g(s) = x$ and $g'(s) = x'$ for all $s \in B$ and $g(s) = y$ and $g'(s) = y'$ for all $s \notin B$. Then

$$\begin{aligned}
\int_S u(f(s)) d\mu(s) &= u(x)\mu(A) + u(y)(1 - \mu(A)) \\
\int_S u(f'(s)) d\mu(s) &= u(x')\mu(A) + u(y')(1 - \mu(A)) \\
\int_S u(g(s)) d\mu(s) &= u(x)\mu(B) + u(y)(1 - \mu(B)) \\
\int_S u(g'(s)) d\mu(s) &= u(x')\mu(B) + u(y')(1 - \mu(B))
\end{aligned}$$

Let $u(x) > u(y)$ and $u(x') > u(y')$. Then $f \succsim g$ if

$$u(x)\mu(A) + u(y)(1 - \mu(A)) \geq u(x)\mu(B) + u(y)(1 - \mu(B))$$

i.e.

$$\mu(A) \geq \mu(B)$$

i.e.

$$u(x')\mu(A) + u(y')(1 - \mu(A)) \geq u(x')\mu(B) + u(y')(1 - \mu(B))$$

i.e. $f' \succsim g'$.

S5: Let $A \in 2^S$ be a nonnull event and and $f(s) = x$ and $g(s) = y$ for all $s \in A$. Then, if $f(s) = g(s)$ for all $s \notin S$,

$$\begin{aligned}
\int_S [u(f(s)) - u(g(s))] d\mu(s) &= \int_A [u(f(s)) - u(g(s))] d\mu(s) \\
&= \mu(A)(u(x) - u(y))
\end{aligned}$$

which is nonnegative if and only if $u(x) \geq u(y)$. Thus $f \succsim g$ if and only if $x \succsim y$.

S6: Let

$$\int_S [u(f(s)) - u(g(s))] d\mu(s) > 0. \tag{3}$$

For every $x, y \in X$, and for every $A \in 2^S$ it holds true that

$$\begin{aligned} \int_S [u(f(s)) - u(g(s))] d\mu(s) &= \int_{S \setminus A} [u(f(s)) - u(g(s))] d\mu(s) + \int_{A_i} [u(f(s)) - u(g(s))] d\mu(s) \\ &\geq \int_{S \setminus A_i} [u(f(s)) - u(g(s))] d\mu(s) + \inf_{x,y} [u(x) - u(y)] \mu(A_i) \quad (4) \end{aligned}$$

Since μ is a nonetomic measure, there is, for any $\varepsilon > 0$, a finite partition $\{A_i^\varepsilon\}$ of S such that $\mu(A_i^\varepsilon) \leq \varepsilon$, for every A_i^ε. Combining (3) and (4) there is, for every $\varepsilon > 0$, a finite partition $\{A_i^\varepsilon\}$ such that for all $A_i^\varepsilon \in \{A_i^\varepsilon\}$,

$$\int_{S \setminus A_i^\varepsilon} [u(f(s)) - u(g(s))] d\mu(s) + \inf_{x,y} [u(x) - u(y)] \mu(A_i^\varepsilon) > 0$$

6. Monty Hall -problem: In a famous TV quiz a contestant chooses one of the three doors, behind one of which a prize is hidden. After the door is chosen, the moderator (Monty Hall) opens one of the doors behind which there is no prize, and permits the contestant to change the choice of his door. Should the contestant do that? What does that mean in the language of Savage states?

   A: Let the contestant choose, say, door A, and Monty Hall open from the remaining doors one behind which there is no prize. We argue that the contestant should change his choice to the other remaining door. The probabilities of Monty Hall choosing particular doors given the contestant's choices are summerized by the following table:

		Monty Hall's choice			
		A	B	C	Σ
	A	0	1/6	1/6	1/3
True door	B	0	0	1/3	1/3
	C	0	1/3	0	1/3
	Σ	0	1/2	1/2	1

Conditional probability of the prize being behind door A given that Monty Hall opens door, say, B is

$$\Pr\{\text{True door} = A : \text{Monty Hall's choice } = B\} = \frac{1/6}{1/2} = \frac{1}{3}$$

$$\Pr\{\text{True door} = C : \text{Monty Hall's choice } = B\} = \frac{1/3}{1/2} = \frac{2}{3}$$

In the language of Savage, this means that states have to contain all the information that is relevant for decision making, including how the information is generated.