FDPE, Mechanism Design Spring 2013
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Problem set with sketches of solutions:

1. Explain, without making reference to the Gibbard-Satterthwaite theorem why the following SCF is not implementable in dominant strategy equilibrium: X contains at least three elements, all (linear) preference profiles are possible, and

$$f(\succsim) = \begin{cases} x, & \text{if } x \succ_i y \text{ for all } i \in N \text{ and all } y \in X \\ x^*, & \text{otherwise} \end{cases}$$

(That is, x is implemented if it is a unanimously agreed top alternative, otherwise x^* is implemented). Can this f be implemented in Nash equilibrium? Can it be *fully* implemented in Nash equilibrium?

Answer: Choose preference rankings

	$\succsim_{-i}$		$\succsim_i$	$\succsim_i'$
x	$\cdots$	x	x	y
$\vdots$		$\vdots$	y	x
			$\vdots$	$\vdots$
x^*	$\cdots$	x^*	x^*	x^*

By construction, $f(\succsim_{-i}, \succsim_i) = x$ and $f(\succsim_{-i}, \succsim_i') = x^*$. Hence

$$f(\succsim_{-i}, \succsim_i) \succ_i' f(\succsim_{-i}, \succsim_i'),$$

violating strategy-proofness.

For Nash equilibrium, construct (g, S) such that $S_i = X$ and

$$g(s) = \begin{cases} x, & \text{if } s_i = x \text{ for all } i \in N \\ x^*, & \text{otherwise} \end{cases}$$

Then the strategy $\sigma = (\sigma_1, ..., \sigma_n)$ from the set of preference profiles to X such that

$$\sigma_i(\succsim) = \begin{cases} x, & \text{if } x \succ_i y \text{ for all } i \in N \text{ and all } y \in X \\ x^*, & \text{otherwise} \end{cases}$$

constitutes a Nash equilibrium and implements x whenever x is top ranked by all i, and x^* otherwise.

Any fully Nash implementable social choice function ϕ is Maskin monotonic: if $\{y : \phi(\succsim) \succsim_i y\} \subseteq \{y : \phi(\succsim) \succsim_i' y\}$ for all i, then $\phi(\succsim) = \phi(\succsim')$. By the example above, f is *not* monotonic (why?). Hence not fully Nash implementable.

2. Median voter theorem. Let X be an interval in $\mathbb{R}$, and let there be n agents i, n odd, with single peaked preferences on X. Demonstrate that the social choice function f^M that always chooses the median voter's ideal point is implementable in dominant strategy equilibrium.

   Answer: Lecture notes.

3. Roommates problem. Suppose that there are n graduate students. At most two students are placed in a same office. Each student has strict preferences over the other $n-1$ students. Show that there may not exist a pairwise-stable matching even if not being matched is the worse outcome for every student.

   Answer: Let n be odd. Consider the following preferences:

$\succsim_1$	$\succsim_2$		$\succsim_n$
2	3	$\cdots$	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	1		$n-1$
1	2	$\cdots$	n

Every pairwise partition of the agents leaves one agent without a pair which is the worst of possible outcomes for that agent. As a result, the matching is unstable (why?).

In the case n is even, let the preferences be

$\succsim_1$	$\succsim_2$		$\succsim_{n-1}$	$\succsim_n$
2	3	$\cdots$	1	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	
$n-1$	1		$n-2$	$\vdots$
n	n		n	
1	2	$\cdots$	$n-1$	n

Every pairwise partition of the agents forces one agent in $1, ..., n-1$ to for a pair with the agent n, which is the worst of possible pairings for that agent. As a result, the matching is unstable (why?).

4. What are the maximal and minimal number of steps for the Deferred Acceptance algorithm to stop?

   Answer: Minimal number of steps: 1. Maximal number of steps: men proposing DA will stop when all women have an acceptable offer. Hence until the final step, there must be at least one woman who has not received any (acceptable) offers. After the first step, at least one man become rejected in each step. In the final step, no man gets rejected. Given that there is at least one woman who is not offered by any man, each of the n men can get maximum $n-2$ rejections before the final step and after the first step. Hence an upper bound on the number of steps is

$$1 + n(n-2) + 1.$$

   To see that this bound is tight, construct the following patterns of prerences. Men:

m_1	m_2	$\cdots$	m_{n-1}	m_n
w_1	w_2	$\cdots$	w_{n-1}	w_1
w_2	$\vdots$		w_1	w_2
$\vdots$	w_{n-1}		$\vdots$	$\vdots$
w_{n-1}	w_1	$\cdots$	w_{n-2}	w_{n-1}
w_n	w_n	$\cdots$	w_n	w_n

   Women:

w_1	w_2		w_{n-1}	w_n
m_2	m_3	$\cdots$	m_n	$\vdots$
$\vdots$	$\vdots$		m_1	
m_{n-1}	m_n		$\vdots$	$\vdots$
m_n	m_1	$\cdots$	m_{n-2}	
m_1	m_2	$\cdots$	m_{n-1}	$\vdots$

   The men proposing DA stops in $n^2 - 2n + 2$ steps (show this!).

5. Marriage market with dogs. Suppose that there are n men, n women, and n dogs. A matching is a set of 3-tuples (m, w, d) such that m is a man, w is a woman and d is a dog, such that every individual is included in one and only one triple. For any matching, if triple $(m, w, d) \in \mu$ then we say that m, w, and d are matched with each other under μ. Each individual and dog has preferences of pairs of partners. A matching is stable if there is no three-way deviation (i.e. no triple such that each of them would strictly like to be matched with

the other two rather than her current partners). Show that a stable matching may not exist. (Assume that being unmatched and being matched with one partner are strictly worse than being matched with two partners for each individual).

Answer: Let the there be twom men $\{m_1, m_2\}$, two women $\{w_1, w_2\}$, and two dogs $\{d_1, d_2\}$. Preferences of men are defined over $\{w_1, w_2\} \times \{d_1, d_2\}$, for women over $\{m_1, m_2\} \times \{d_1, d_2\}$, and for dogs over $\{m_1, m_2\} \times \{w_1, w_2\}$. Consider the following preferences

m_1, m_2	w_1, w_2	d_1	d_2
$w_2 d_2$	$m_2 d_2$	$m_1 w_1$	$m_1 w_1$
$w_2 d_1$	$m_1 d_2$	$m_1 w_2$	$m_2 w_2$
$w_1 d_2$	$m_1 d_1$	$m_2 w_2$	$m_1 w_2$
$w_1 d_1$	$m_2 d_1$	$m_2 w_1$	$m_2 w_1$

Every resulting match is is unstable (show this!).

6. Equilibria in marriage games. Let $(M, W, \succsim)$ be a marriage market. Consider the following strategic-form game: Every man and woman knows the preferences of the others. Every man simultaneously announces the name of a woman or his own name and at the same time every woman simultaneously announces the name of a man or her own name. If a man and woman announce each other's name, they are matched with each other, otherwise they remain unmatched. Let $\mu[s]$ be the matching generated by this game when each agent i announces s_i and $s = (s_i)_{i \in M \cup W}$.

   A strong Nash-equilibrium is a strategy profile s such that there is no coalition $C \in M \cup W$ and strategy profile s_C for these agents with $\mu[s_C, s_{-C}](i) \succ_i \mu[s](i)$ for all $i \in C$. Prove that the set of strong Nash equilibrium matchings is equal to the set of stable matchings. Does this set also contain all Nash equilibrium matchings?

   Answer: A strategy profile constitutes a strong NE if and only if it implements a match in the core. For if a strong NE match is not in the core, there is a subset of men and women that could deviate and be profitably rematched among themselves. For if a core allocation does not support a strong NE, there is a subset of men and women that will deviate by rematching among themselves. But this contradicts the assumption that the allocation was in the core.

   Not every NE is a strong NE. Construct a strategy where all men announce woman 1 and all women announce man 1. Only man 1 and woman 1 become matched. Moreover, no man or woman can be matched by unilaterally deviating since any other choice will not point to him/her back. Hence a unilateral deviation is not profitable.

7. Show that Deferred Acceptance algorithm coincides with the Serial Dictator when agents in one side of the market have the same preferences over the agents in the other side

   Answer: Men proposing DA with all the women ranking the men according to (say) their indeces $1, .., n$. At any stage, man 1 is guaranteed be accepted by his favourate woman, and man 2 by his his favourate woman except man 1's favourate woman, etc.

   Men proposing DA with all the men ranking the women according to their indeces $1, .., n$. Then in stage 1 woman 1 receives offers from all the men, accepts her favourate man, and rejects the rest. In stage 2 woman 2 receives offers from all the men except the one that was accepted by 1, accepts her favourate man in this group, and rejects the rest. Etc.