# Dynamic stable set as a tournament solution*

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**Abstract**

We define a notion of dynamic (vNM) stable set for a tournament relation. Dynamic stable set satisfies (i) an external direct stability property, and (ii) an internal indirect stability property. Importantly, stability criteria are conditioned on the histories of past play, i.e. the dominance system has memory. Due to the asymmetry of the defining stability criteria, a dynamic stable set is stable both in the direct and in the indirect sense. We characterize a dynamic stable set directly in terms of the underlying tournament relation. A connection to the covering set of Dutta (1988) is established. Using this observation, a dynamic stable set exists. We also show that a maximal implementable outcome set is a version of the ultimate uncovered set.

*Keywords*: vNM stable set, history dependence, covering set.
*JEL*: C71, C72.

# 1 Introduction

Von Neumann-Morgenstern (vNM) stable set is a solution with pedigree. Since the solution can be stated by assuming remarkably little structure on the underlying set up - a binary dominance relation on a set of feasible alternatives suffices - it can be usefully applied to a wide range of choice problems.[1] An attractive property

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[1]Recent contributions include Anesi (2006, 2010, 2012), Anesi and Seidmann (2013), Diamantoudi and Xue (2007), Ehlers (2007), Kultti and Vartiainen (2007), Penn (2008), Mauleon et al (2009). Greenberg (1990) provides a useful taxonomy of the stable set -approaches.

of the vNM stable set compared to, e.g., the *core* is that it tests alternatives *only* against those that are themselves deemed stable. Hence the stable set implicitly acknowledges that some blockings may not be credible. This feature of the solution reflects neatly strategic sophistication. It also guarantees the existence of the solution in circumstances where the core is empty.

However, the stable set has some well known deficiencies. The first is related to its existence. Existence problems are particularly acute when the underlying dominance is a tournament relation (total and antisymmetric) - a characteristic feature of many models of political decision making.[2] The three-cycle $xPy$, $yPz$, and $zPx$, where $P$ is the underlying dominance relation on the set $\{x, y, z\}$ of social alternatives, serves as the canonical counterexample.

Another problem of the stable set is conceptual. As most solution concepts that are not articulated in a full-fledged noncooperative language, it fails to take adequately into account the underlying dynamics, in particular the *history dependence* of blockings. This implies, on the one hand, that the stable set may be needlessly restrictive since it gives up an important degree of freedom in constructing the blocking strategies. On the other hand, letting blockings to be history dependent allows one to incorporate strategic sophistication into the solution concept.

This paper develops a new version of the stable set concept that permits history dependent blockings. We focus on the tournament context. Our solution concept, dubbed as the *strong dynamic stable set*, is as credible as it can possibly be: we require robustness against *both* direct *and* indirect blockings. Hence our solution is free from the critiques of Harsanyi (1974) and Xue (1998).[3]

Our main result is a compact characterization of the strong dynamic stable set which shows that it has an immediate relationship to some already well known tournament solutions. In particular, we show that a strong dynamic stable set can always be build on a covering set (Dutta 1988).[4] The existence of the solution follows directly from this observation. Furthermore, we show that the ultimate uncovered set constitutes the maximal set of outcomes that are implementable via a strong dynamic stable set. We also demonstrate that the *standard* stable set can be interpreted as a Markovian version of the strong dynamic stable set. Hence the

[2]For discussions on tournament solutions, see Moulin (1986), Dutta (1988), and Laslier (1997).

[3]Harsanyi (1974) points out that the standard stable set fails to take farsightedness appropriately into account since after a blocking the play may revert back to the stable set in a way that is beneficial to the blocking party. Xue (1999) argues that indirect dominance, where the agents block an alternative keeping in mind the outcome that is eventually implemented, is not a satisfactory basis for a stable set either. Nothing prevents an intermediate mover along the dominance sequence from deviating from the plan.

[4]Extending the solution to a more general framework à la Chwe (1994) should, however, be doable along the avenues of Vartiainen (2011).

standard stable set, whenever it exists, also possesses the strong stability properties that define the strong dynamic stable set. This feature explains why the standard stable set characterizes so successfully noncooperative strategic behavior in many political choice scenarios (see the discussion in the next subsection).

More specifically, we define the vNM stability criteria on the set of feasible *histories* that we take to be the set of all finite sequences of outcomes. A history is interpreted as the route of status quos via which the current status quo has been achieved. A history is then deemed stable (or terminal) if its final element is not blocked. A history directly dominates another history if the former can be constructed from the latter by adding to it an outcome that is strictly preferred to its last element. A history indirectly dominates another history if the former can be constructed from the latter by adding to it a dominance *sequence* such that the final outcome in the sequence is strictly preferred to any outcome in the middle.

A *strong dynamic stable set* is now a subset of histories that satisfy the following two stability conditions:

1. (External stability) Any history outside the set is directly dominated by a history in the set.

2. (Internal stability) Any history inside the set is *not in*directly dominated by a history inside the set.

The final element of a history in the strong dynamic stable set is interpreted as an outcome that can be implemented. In the tournament context, this version of stability is very strong in the sense that it implies *both* direct and indirect stability when applied to the set of histories. This implies that the solution is free from *both* the criticisms of Harsanyi (1974) - since no deviation from the final status quo of a history in a strong dynamic stable set can trigger a process that eventually implements a desirable outcome - and that of Xue (1998) - since from the final status quo of any history outside a strong dynamic stable set there is a direct and profitable deviation back to the stable set.

Without history dependence it would be clear that the condition is too strong to guarantee the existence (think of the three-cycle above). The key point of the paper is to demonstrate that the degree of freedom that comes with the history dependence allows us to circumvent the existence problem.

We characterize the strong dynamic stable sets in terms of the outcomes that are implementable via histories in them – implementable outcomes are the main object of the analysis. Our characterization specifies a necessary and sufficient condition for the set outcomes to be implementable via a stable set. Such a set of outcomes, which we dub a *consistent choice set* following Vartiainen (2013), is described directly in terms of the underlying dominance relation.[5] We then show that the any covering set of Dutta (1988) is also a consistent choice set. A fortiori, a strong dynamic stable set exists.[6] Furthermore, we demonstrate that the ultimate uncovered set is the maximal set of outcomes (in terms of set inclusion) that can be implemented via any strong dynamic stable set. Finally, we argue that any consistent choice set is a consistent set of Chwe (1994) in a coalitional game that is based on majority coalitions. Hence, in this game, the largest consistent set contains the ultimate uncovered set, and every covering set is a consistent set.

**Example** To see how a strong dynamic stable set works, consider again the above 3-cycle. Partition the set of histories $H = \cup_{k=0}^{\infty}\{x, y, z\}^k$ into three *phases* $H_x, H_y$, and $H_z$ recursively as follows. Initial step: $\emptyset \in H_w$ for some $w \in X$. Inductive step: for any $h \in H$,

- if $h \in H_x$, then $(h, x) \in H_x$ and $(h, y), (h, z) \in H_y$,
- if $h \in H_y$, then $(h, y) \in H_y$ and $(h, z), (h, x) \in H_z$,
- if $h \in H_z$, then $(h, z) \in H_z$ and $(h, x), (h, y) \in H_x$.

Proceeding step by step, each history in $H$ becomes allocated into exactly one of the phases $H_x, H_y$, and $H_z$ (each of the choice of the initial phase $H_w$ generates a different partition $\{H_x, H_y, H_z\}$). We now construct a strong dynamic stable set such that if a history belongs to $H_w$, there is no credible way to move to a path that implements an outcome that dominates $w$. This objective in mind, choose $V \subset H$ such that, for any $h \in H$,

- if $h \in H_x$, then $(h, x), (h, y) \in V$ and $(h, z) \notin V$,
- if $h \in H_y$, then $(h, y), (h, z) \in V$ and $(h, x) \notin V$,
- if $h \in H_z$, then $(h, z), (h, x) \in V$ and $(h, y) \notin V$.

To see formally that the constructed $V$ is a strong dynamic stable set, let, say, $h \in H_x$. Then $(h, y), (h, x) \in V$ and $(h, z) \notin V$.

Since $(h, x)$ is arbitrarily chosen, to show that $V$ is internally stable it suffices that neither $(h, y)$ nor $(h, x)$ are not indirectly dominated by any element in $V$. By the assumption concerning preferences, any history that indirectly dominates $(h, y)$ is of the form $(h, y, ..., y, x)$. Since $(h, y) \in H_y$, it follows that $(h, y, ..., y) \in H_y$. Thus $(h, y, ..., y, x) \notin V$, as desired. Further, any history that indirectly dominates $(h, x)$ is of the form $(h, x, ...x, z)$. Since $(h, x) \in H_x$ it follows that $(h, x, ..., x) \in H_x$ and thus $(h, x, ..., x, z) \notin V$, as desired.

Similarly, to show that $V$ is externally stable it suffices that $(h, z)$ is directly dominated by some element in $V$. Now $(h, z, y)$ directly dominates $(h, z)$ and, since $(h, z) \in H_y$ it follows that $(h, z, y) \in V$, as desired.

[5] Vartiainen (2013) establishes further connections of the concept of a consistent choice set to models of dynamic decision making.

[6] The uncovered set is due to Fishburn (1977), and Miller (1980).

## 1.1 Literature on dynamic solution concepts in political choice

The strong dynamic stable set solution developed in the paper is not the first tournament solution that takes dynamic considerations into account. Broadly, a common objective in the literature on dynamic solutions in collective choice is to understand farsighted blocking behavior. We now discuss the parts of this literature that relate to the vNM stable sets. From the perspective of the current paper, a good way to classify the solutions is by how they treat history dependence.

**Markovian solutions** Following von Neumann and Morgenstern (1953), the standard approach to internal and external stability has been to apply them directly on the set of outcomes. In the current set up, this corresponds to viewing the blocking process as *Markovian*, dependent only on the outcome on the table. An important question in the social choice literature has been to understand the strategic underpinnings of the Markovian stable set. Indeed, the target of many of the papers is to provide a justification for the stable set as a solution concept. Despite the criticism cast by Harsanyi (1974) and Xue (1998), this programme has been quite successful.

Anesi (2010) constructs a fully noncooperative model of a natural legislative bargaining where, at each stage, a randomly drawn committee member may propose a social alternative to replace the current status quo. The status quo is implemented if it wins a voting contest against the proposed alternative and otherwise the proposed alternative becomes the new status quo. Assuming a voting rule that is represented by an asymmetric dominance relation, Anesi (2010) shows, roughly, that any outcomes supported by the respective vNM stable set can also result as a Markov perfect equilibrium outcome of the game. Anesi and Seidmann (2013) generalize and strengthen this result by establishing the coincidence of the (maximal) stable set and the equilibrium outcomes in a larger class of environments, allowing for infinite and potentially not well ordered policy spaces. Further, Diermeier and Fong (2012) show an example of a set up in which the Markov perfect equilibria of the legislative bargaining coincides with the stable set but the proposing agent is not drawn randomly from the set of committee members.[7]

Anesi (2012) develops a model of repeated electoral competition that uses the same logic to generate an even a stronger result: any vNM stable set of the underlying policy game characterizes precisely the absorbing states Markov perfect equilibria of the electoral game. The paper thus provides a noncooperative foundation for the stable set.

[7] See Duggan (2012) and Duggan and Kalandrakis (2012) for the existence and other results in the general class of stochastic games that also contains the legislative bargaining models discussed here.

Penn (2008) studies an infinite agenda amendment procedure in the divide-the-dollar context and finds out that the minimax strategies entertain precisely the outcomes supported by the associated vNM stable set.[8]

Acemoglu et al. (2011) study another model of dynamic electoral competition. Making some restrictions on the economy that guarantee that the underlying social preferences are acyclic, they are able to establish the existence and uniqueness of their solution, the Markov voting equilibrium. Further, they show that in this framework the absorbing states of the Markov voting equilibria correspond the associated stable set.

The big storyline of the above contributions is that, when it exists, the vNM stable set characterizes Markov perfect equilibria of a large class of legislative bargaining models. This, on the one hand, provides a credible justification for the stable set as a solution concept. One the other hand, it also raises the question of why exactly is the stable set such a strong solution concept in the political choice set ups, given the criticisms by Harsanyi (1974) and Xue (1998). A contribution of the current paper is to provide insight into the latter question. We demonstrate that the usual version of the stable set - the one employed in the above literature - has remarkably strong stability properties. By our Proposition 3, a standard stable set, when it exist, constitutes also a Markovian version strong dynamic stable set. Hence the standard stable set is *both* indirectly and directly stable. The crucial assumption behind this result is that the strategic tendencies of the political choice problem are captured by a dominance relation. This requires that the physical blocking options are *in*dependent on the underlying status quo. Whenever this is the case, for example in the context of a committee game, the stable set provides a useful prediction of what to expect.

The question is what happens under the many scenarios when a vNM stable set, and hence the associated Markovian equilibrium structure, does not exist. A contribution of the current paper is to develop a version of the vNM stable set that exists under all circumstances. That the legislative bargaining games entertain a corresponding history dependent equilibria is left for future work.

**History dependent solutions** History dependence is important since it permits much larger space of coalitional strategies from which the stable set can looked from. This paper is not, however, the first one that makes this observation.[9] Bernheim and Slavov (2009) study a policy programs that specify an infinite stream of social states and is robust against one-time deviations by any majority coalition.

[8] The result may be sensitive to the details of the underlying model, see Kalandrakis (2004).

[9] Mariotti (1997) is a precursor in this literature, analysing coalition formation in normal form games. To the best of our knowledge, his paper is the first one to formally study history dependence in coalition formation.

Their solution permits history dependence in a natural way. However, as their solution concept, the dynamic Condorcet winner, is quite weak, history dependent policy processes do not permit policy paths beyond those supported by Markovian policy programs. In particular, existence does not hinge on history dependence. It is critical for their results that society cannot commit to the status quo or "implement" it.

Also the solutions concept of Konishi and Ray (2003) permits history dependence. While they remark that history dependence could be used to enlarge the set of feasible policy paths, they do not analyze the problem in detail. In fact, Anesi (2006) shows that the absorbing states of the Markovian version of the Konishi-Ray solution coincides with the corresponding stable set, when it exists.

The current paper is closely related to Vartiainen (2011) and (2013), which study coalitional behavior when blockings may be history dependent. The solution concept there is a version of the usual one-deviation restriction which, in Vartiainen (2011), can be interpreted as a version of Konishi and Ray (2003). The key result in Vartiainen (2011) is to show that the existence of a history dependent solution is guaranteed in a general model of coalition formation (Chwe 1994). Vartiainen (2013) applies the same solution concept to a model of voting. Also there, the existence hinges on history dependence. In these papers, the main insight is to combine the ideas of history dependence and that of one-deviation restriction in a general coalitional framework.

The current paper complements Vartiainen (2011) and (2013) by demonstrating that history dependence can be naturally and powerfully associated also to the vNM stable set modeling approach. That the characterizations of implementable outcomes in Vartiainen (2013) and in here coincide shows that the vNM stable can be equivalently seen as a one-deviation restriction on feasible coalitional strategies. This is one of the key insights of the current paper. Compared to Vartiainen (2013) what is also new here is the construction of the maximal set of implementable outcomes. Since Vartiainen (2013) focuses on more general set up, and hence uses a more tight covering criterion to characterize the maximal set of implementable outcomes, the results there do not imply that the ultimate uncovered set constitutes the maximal set of implementable outcomes.

It is also noteworthy that the strong dynamic stable set analyzed in this paper is stronger than Vartiainen (2013) in an important sense. Vartiainen (2013) demands that a one-step deviation cannot lead to an *equilibrium* play that eventually implements a preferred outcome. Here, however, there is no "equilibrium path". Rather internal stability rules out *all* indirect dominance chains between two stable outcomes. Thus the solution here is stronger as it tested against larger set of objects. For the same reason, it leaves no room for the Harsanyi critique discussed above. Moreover, here the external stability implies that there is a one-step playpath that leads to a stable outcome. In contrast, Vartiainen (2013) imposes no comparable restriction; the equilibrium path following a deviation can be arbitrarily long. This aspect makes the current solution more reliable, and fully robust against the Xue critique. The same comparisons can be against Vartiainen (2011) whereas there the is even bigger due to the differences in the frameworks.

This paper is organized as follows. Section 2 defines the strong dynamic stable set solution and develops a motivation for it. Section 3 characterizes the solution in terms of the implementable outcomes. Section 4 shows the connection of the solution to the covering set, and proves the existence. Section 5 concludes the paper with some discussion.

# 2 The set up

There is a *finite* set $X$ of alternatives and a *tournament* relation (antisymmetric and total) $R$ over $X$, reflecting payoff dominance (e.g., majority preferences). Denote the asymmetric part of $R$ by $P$, the *upper set* at $x$ by $P(x) = \{y \in X : yPx\}$, and the *weak upper set* at $x$ by $R(x) = P(x) \cup \{x\}$.

Dominance reflects which of the two outcomes would be chosen would only them be the available choices. Depending on the set up, the dominance relation may reflect individual or collective preferences (e.g., the majority relation).

## 2.1 Motivation

It is useful to think decision making as a concrete agenda setting procedure. There is an outcome $x_0 \in X$ serving as the initial status quo. Given a status quo, a decision maker(s) may challenge the status quo by demanding a new outcome. If, after a sequence $x_0, ..., x_{k-1}$ of challenged status quos, the current status quo $x_k$ is challenged with a new outcome $y$, then $y$ becomes the new status quo, i.e. $y = x_{k+1}$. If $x_k$ is not challenged, then $x_k$ is implemented.[10]

Call a finite set of successive non-implemented status quos as a finite *history*. Denote by $H = \cup_{k=0}^{\infty} X^k$ the set of finite histories, i.e. of all finite sequences $(x_0, ..., x_k)$. A typical element of $H$ is denoted by $h$ or $(h, x)$, where $x$ is the current status quo and $h$ is the past history.

We interpret the final outcome $x$ of the history $(h, x) \in H$ to be stable (or terminal) if $x$ is not deviated from. Profitability of a deviation depends on the consequences of the deviation, i.e. what outcome eventually becomes implemented. The novel element of the construction is that the outcome that becomes implemented after a deviation from $x$ may now depend on the past history $h$.

[10]For example, if $R$ reflects the majority rule, then $x_k$ may be challenged with $y$ by any majority coalition of individuals.

There is no problem in combining the idea of history dependency with the notion of stable set. We say that $(h,x) \in H$ is *directly dominated* by $(h,x,y) \in H$ if $yPx$. The set $V \subset H$ is now *directly dynamically stable* if it meets the following two conditions:

1. (External direct stability) If $h \notin V$, then there is an element in $V$ that *directly* dominates $h$.

2. (Internal direct stability) If $h \in V$, then there is *no* element in $V$ that *directly* dominates $h$.

Harsanyi (1974) criticized the direct notion of stability for its lack of farsightedness. The above notion of dynamic stability is vulnerable to Harsanyi's critique. Extrapolating on his argument, once an internally stable history is deviated to a history outside the stable set, the new history will also be deviated to, and this time to a history inside the stable set. The problem is that now nothing prevents the final stable history, to which the play eventually converges, from being more profitable than the first one, contradicting its stability. Indirect and profitable deviation of this sort is not restricted by internal stability.

Harsanyi's suggested remedy is to use *in*direct dominance as the dominance criterion.[11] Let us consider Chwe's (1994) formalization of this idea: $(h,x) \in H$ is *indirectly dominated* by $(h,x_0,...,x_K) \in H$ if $x = x_0$ and $x_K P x_k$, for all $k = 0,...,K-1$.[12] That is, a history indirectly dominates another if the former can be constructed from the latter by adding to a finite sequence of alternatives such that the final element payoff dominates all the interim elements in the sequence. The dominance property guarantees that it is indeed more profitable to continue along the path than to stop in the middle.

We may now define a history dependent stable set that is based on indirect dominance. Set $V \subseteq H$ is a *indirectly dynamically stable* if:

1. (External indirect stability) If $h \notin V$, then there is an element in $V$ that *indirectly* dominates $h$.

2. (Internal indirect stability) If $h \in V$, then there is no element in $V$ that *indirectly* dominates $h$.

However, as argued by Xue (1998), indirect dominance is not a satisfactory criterion for the stable set either. The problem is that an indirect dominance path

[11] In fact, Harsanyi (1974) seems to advocate *strict* stable set, comparable to history independent version of Definition 1.

[12] We adopt the weak version of indirect dominance -relation (also used by Konishi and Ray 2003, and Ray 2007). The original definition due to Chwe (1994) assumes *strict* relations.

may not be credible: nothing in the notion of indirect dominance guarantees that at the interim stage of the deviation path the decision maker(s) should not deviate to some *other* path. Hence the deviating decision maker cannot be sure that the deviation path will be followed as conjectured. But this implies that it is no longer optimal for the first deviating decision maker to originate the deviation path. Because of the implicit optimism embedded into the notion of indirect dominance, indirect strong dynamic stable set is not a reliable solution concept.

Note that $(h, x)$ may be indirectly dominated by an element $(h, x, ..., y)$ in a set without being directly dominated by any element is this set (since $(h, x, y)$ may not be in the set). This means that, in the presence of history dependence, internal direct stability does not imply internal indirect stability and external indirect stability does not imply external direct stability - unlike in the standard scenario where dominance is defined on outcomes. Thus neither of the above versions of dynamically stable sets implies the another.

We conclude that both the dynamic versions of the stable set - direct and indirect - are vulnerable to criticism. However, we also observe that the criticisms are not exchangeable. The stable set that is based on direct dominance is not subject to Xue's (1998) critique since there is no middle-of-the-path to deviate from, and the stable set that is based on *in*direct dominance is not subject to Harsanyi's (1974) critique since it accounts for long term impacts of a blocking. Thus, an obvious remedy to the problem would be to demand that the solution satisfies *both* direct and indirect stability criteria at the same time.

## 2.2 Strong dynamic stable set

Given the above motivation, our solution concept is defined as follows:[13]

**Definition 1 (Dynamic stability)** *A* strong dynamic stable set $V \subset H$ *is defined by:*

1. (External direct stability) *If* $h \notin V$, *then there is an element in* $V$ *that* directly *dominates* $h$.

2. (Internal indirect stability) *If* $h \in V$, *then there is* no *element in* $V$ *that* indirectly *dominates* $h$.

On the one hand, since direct dominance implies indirect dominance, and indirect *un*dominated implies direct *un*dominance, it is clear that this specification of strong dynamic stable set encompasses both the direct *and* indirect dynamic stability notions described in the previous subsection. Since the other direction is automatically true, we state the following fact.

[13] Following Harsanyi (1974), the solution might be called dynamic *strict* stable set.

**Remark 2** *A set $V \subseteq H$ is directly* and *indirectly dynamically stable if and only if $V$ is a strong dynamic stable set meeting Definition 1.*

Thus a strong dynamic stable set is free from the criticisms of Harsanyi (1974) since an outcome that will eventually implemented after a deviation from the final status quo of a stable history can never dominate the status quo. It is also free from the criticism of Xue (1998) as the final status quo of a nonstable history can be deviated to a preferred outcome that will is the immediately implemented. This double-stability feature makes the strong dynamic stable set a credible solution concept.

Our main interest is in outcomes that can be implemented via a strong dynamic stable set. The final outcome $x$ of a history $(h, x)$ of a strong dynamic stable set $V$ is implemented once the history $(h, x)$ is reached. Denote by

$$\mu(V) := \{x : (h, x) \in V, \text{ for some } h \in H\},$$

the set of final elements of the histories in $V$, i.e. outcomes that are implementable via histories in $V$. The set $Y$ of outcomes is said to be *implementable via the strong dynamic stable set* $V$ if $Y = \mu(V)$. We will give a characterization of strong dynamic stable sets directly in terms of sets that are implementable via them (i.e., without reference to the history constructions).

## 2.3 Relationship to the standard stable set

In this subsection we explore the relationship between the strong dynamic stable set and the version of the stable set that is typical in the literature, here referred as the *standard stable set*. We show that this notion coincides with the *history independent* version of the strong dynamic stable set. Since the latter is defined with respect to our tight stability criteria, the result shows that also the usual version of the stable set, when it exists, possesses these properties.

A strong dynamic stable set $V \subset H$ is *Markovian* if $(h, x) \in V$ if and only if $(h', x) \in V$, for all $h, h' \in H$, for all $x \in X$.

Note that a Markovian strong dynamic stable set still requires direct external stability and indirect internal stability, and hence is dynamically stable both in the direct and indirect sense.

A set $Y$ of alternatives constitutes a *standard stable set* if it meets the following two conditions:

1. (External stability) If $x \notin Z$, then there is $y \in Z$ such that $yPx$.

2. (Internal stability) If $x \in Z$, then there is *no* $y \in Z$ such that $yPx$.

We now argue that the two notions coincide.

**Proposition 3** *Set $Z$ of alternatives is implementable via a Markovian strong dynamic stable set if and only if $Z$ is a standard stable set.*

**Proof.** "If": Let a strong dynamic stable set $V$ be Markovian. We show that $\mu(V)$ is a standard stable set. Let $x \notin \mu(V)$. Since $V$ is Markovian, $(h, x) \notin V$ for any $h$. To verify external stability, by direct external stability there is $y$ such that $(h, x, y) \in V$, and hence $y \in \mu(V)$, such that $yPx$, as required. Let $x \in \mu(V)$. We check internal stability. Since $V$ is Markovian, $(h, x) \in V$ for any $h$. By internal indirect stability, $(h, x, y) \notin V$, and hence $y \notin \mu(V)$ for any $y$ such that $yPx$, as required.

"Only if": Let $Z$ be a standard stable set. We construct a Markovian strong dynamic stable set $V$ such that $\mu(V) = Z$. Let $V$ be defined by the following property: $(h, x) \in V$ if and only if $x \in Z$. Clearly $V$ is Markovian. To see that it satisfies external direct stability, let $(h, x) \notin V$. Then also $x \notin Z$ and by external stability there is $y \in Z$ such that $yPx$. Thus $(h, x, y) \in Z$ directly dominates $(h, x)$. To check external indirect stability, let $(h, x) \notin V$ and suppose that $(h, x, y_0, ..., y_K)$ indirectly dominates $(h, x)$. By the definition of indirect dominance, $y_K P x$. Since $x \in Z$, and since $Z$ is a simply stable set, $y_K \notin Z$. Thus $(h, x, y_0, ..., y_K) \notin V$. ■

Note that nothing in the above proof appeals to the assumption that the underlying dominance relation is a tournament; neither antisymmetry nor totality of $R$ is needed. Thus whenever the underlying strategic tendencies are describable by a binary relation, the standard stable set has remarkably strong stability properties; it is externally stable in the sense of direct dominance and internally stable in the sense of indirect dominance. Hence, paralleling Remark 2, the standard stable set is *both* directly and indirectly stable. The result explains why the standard stable set, when it exists, has the attractive strategic properties demonstrated in the literature on legislative bargaining (e.g. Acemoglu et al. 2011; Anesi and Seidmann 2014; Anesi 2010, 2012; Diermeier and Fong 2012; Penn 2008).

Of course, the problem is that the standard stable set often does not exist. The next two sections show that the same concerns are not warranted with the strong dynamic stable set in the tournament context.

# 3 Characterization

Given $B \subseteq X$, we say that $y$ *covers* $x$ *in* $B$ if (i) $x, y \in B$, (ii) $yPx$, and (iii) $zPy$ implies $zPx$, for all $z \in B$. Since $P$ is a tournament relation, we can state this more succinctly: $y$ covers $x$ in $B$, $x, y \in B$, if

$$\{y\} \cup P(y) \cap B \subseteq P(x) \cap B. \qquad (1)$$

Note that the third outcomes this relation takes into account are restricted to the set $B$.[14]

**Definition 4 (Consistent choice set)** *A set $C \subseteq X$ is a* consistent choice set *if any $x$ in $C$ is not covered in $C \cup \{y\}$ by any $y$ in $X$.*

That is, if $C$ is a consistent choice set, then, for any $x \in C$ and for any $y \in X$ there is $z \in C$ such that $z \in (\{y\} \cup P(y)) \backslash P(x)$. A consistent choice set meeting Definition 4 is related to the consistent set by Chwe (1994), as we will see later.

Now we establish that any strong dynamic stable set is outcome equivalent to a consistent choice set.

**Lemma 5** *$V$ is a strong dynamic stable set only if $\mu(V)$ is a consistent choice set.*

**Proof.** Let $V$ be a strong dynamic stable set. We show that $x \in \mu(V)$ is not covered in $\mu(V) \cup \{y\}$ by $y$. Suppose, on the contrary, that $y$ covers $x$ in $\mu(V) \cup \{y\}$. Then necessarily $y \in P(x)$. Identify $h \in H$ such that $(h, x) \in V$. Since $y \in P(x)$, $(h, x, y)$ directly dominates $(h, x)$. By internal stability, $(h, x, y) \notin V$. By external stability, there is $(h, x, y, z) \in V$ such that $(h, x, y, z)$ directly dominates $(h, x, y)$. Thus $z \in \mu(V)$ and $z \in P(y)$. By internal stability, however, $(h, x, y, z)$ cannot indirectly dominate $(h, x)$, implying - since it does directly dominate $(h, x, y)$ - that $z \notin P(x)$. Thus $z \in \mu(V) \cap P(y) \backslash P(x)$ which implies that $y$ does not cover $x$ in $\mu(V) \cup \{y\}$. ■

Now we show the converse, that for any consistent choice set there exists a strong dynamic stable set that is outcome equivalent to the consistent choice set. Interpret $C$ as an index set and construct recursively a partitioning $\{H_x\}_{x \in C}$ of $H$ as follows. Initial step: choose $x_0 \in C$ and $\emptyset \in H_{x_0}$. Inductive step: If $h \in H_x$, then, for any $y \in X$,

$$(h, y) \in \begin{cases} H_y, & \text{if } y \in C \backslash P(x), \\ H_x, & \text{if } y \notin C \backslash P(x). \end{cases} \tag{2}$$

By induction, each history $h$ becomes allocated to exactly one element of $\{H_x\}_{x \in C}$. An element of the partition $\{H_x\}_{x \in C}$ is called a *phase*. A phase summarizes all the relevant information contained in the history in a sense that all histories in a particular phase are treated in a similar way when allocating histories to the stable set or its complement. That is, to decide whether $(h, y)$ is consider stable one only needs to know which phase $h$ belongs to. More specifically, construct a set $V^C \subseteq H$ such that

$$V^C = \{(h, y) : h \in H_x \text{ and } y \in C \backslash P(x)\}. \tag{3}$$

[14] For different versions of covering, see Duggan (2012).

Then,

$$
\begin{aligned}
\mu(V^C) &= \{y : (h, y) \in V^C, \text{ for some } h \in H\} \\
&= \{y : y \in C \backslash P(x), \text{ for some } x \in C\} \\
&= C. \qquad (4)
\end{aligned}
$$

Thus, elements in the consistent choice set $C$ are implementable with $V^C$. We now show that $V^C$ is a strong dynamic stable set.

**Lemma 6** *$V^C$ is a strong dynamic stable set.*

**Proof.** *External stability*: Take any $(h, y) \notin V^C$ such that $h \in H_x$. By the construction of $V^C$, $y \notin C \backslash P(x)$. By the construction of $C$, there is $z$ such that $z \in C \cap R(y) \backslash P(x)$. Since $y \notin C \backslash P(x)$, necessarily $y \neq z$. Thus $z \in P(y)$ and hence $(h, y, z)$ directly dominates $(h, y)$. Since $(h, y, z) \in H_x$, and $z \in C \backslash P(x)$, we have $(h, y, z) \in V^C$, as desired.

*Internal stability*: Take any $(h, y) \in V^C$ such that $h \in H_x$. Suppose that there is $(h, y_0, ..., y_K) \in V^C$ with $y_0 = y$ that indirectly dominates $(h, y)$. Let $J < K$ satisfy $(h, y_0, ..., y_k) \in H_{y_J}$ for all $k = J + 1, ..., K$. Since $(h, y_0, ..., y_K) \in V^C$, such a $J$ exists. By the construction of $V^C$, $y_K \in C \backslash P(y_J)$. But this contradicts the assumption that $y_0, ..., y_K$ is a dominance chain. ■

By Lemma 5, a set $Z$ of alternatives is implementable via a strong dynamic stable set only if $Z$ is a consistent choice set. Conversely, by (4) and Lemma 6, outcomes of any consistent choice can be implemented via a stable set. We compound these observations in the following characterization.

**Theorem 7** *Set $Z$ of alternatives is implementable via a strong dynamic stable set if and only if $Z$ is a consistent choice set.*

This result does not, however, tell anything about the existence of a consistent choice set nor how it can be identified. These issues are theme of the next section.

# 4 Existence and the maximal set of implementable outcomes

Vartiainen (2013) proved the existence of a the consistent choice set and its relationship to the covering set (Dutta 1988) in a more general set up. In this section, we demonstrate the connection of consistent choice set to the covering set by appealing to a new property of a covering set. We also characterize the maximal consistent choice set, and tie our solution to the consistent set of Chwe (1994) and policy processes with one-deviation property of Vartiainen (2013).

## 4.1 Covering set

The covering relation (1) in $B \subseteq X$ is transitive. Since $X$ is a finite set, the set $uc(B)$ of maximal elements in the covering relation $B$ is nonempty for all $B \subseteq X$. The set $uc(B)$ is called the *uncovered* set of $B$ (cf. Fishburn, 1977; Miller, 1980).

Consider the following important extension of the uncovered set by Dutta (1988). Set $D \subseteq X$ is a *covering set* if it satisfies the following internal and external stability properties:

1. Any $x \in D$ is not covered in $D$ by any $y \in D$.

2. Any $x \in X \backslash D$ is covered in $D$ by some $y \in D$.

The ways a covering set and a consistent choice set differ can be summarized by the following two features. Let $D$ be a covering set and $C$ a consistent choice set.

- Some $x \in D$ may be covered in $D \cup \{y\}$ by some $y \in X \backslash D$ (but no $x \in C$ may be covered in $C \cup \{y\}$ by any $y \in X \backslash C$).

- Some $x \in X \backslash C$ may not be covered in $C$ by any $y \in C$ (but any $x \in X \backslash D$ must be covered in $D$ by some $y \in D$).

The first of the points describes the sense in which a covering set may not meet the properties of a consistent choice set, and second the other way around. We will now argue, however, that the first point is vacuous.

**Lemma 8** *Let $D$ be a covering set. Then any $x \in D$ is not covered in $D \cup \{y\}$ by any $y \notin D$.*

**Proof.** Suppose that $x \in D$ is covered in $D$ by $y \notin D$, i.e. $\{y\} \cup P(y) \cap D \subseteq P(x) \cap D$. By the definition of the covering set, there is $w \in D$ that covers $y$ in $D$, i.e. $\{w\} \cup P(w) \cap D \subseteq P(y) \cap D$ (see Fig. 1, where $y \to x$ means $x \notin P(y)$, etc.). Hence also $w \in P(x)$. Since $x$ is uncovered in $D$, there exists $z \in D$ such that $z \in \{w\} \cup P(w) \backslash P(x)$. Since $w$ covers $y$ in $D$, it follows that $z \in P(y)$. Hence $z \in P(y) \backslash P(x)$, as desired. ■

[Figure 1]

Hence a covering set $D$ meets all the properties of a consistent choice set. We conclude:

**Theorem 9** *Any covering set is a consistent choice set.*

However, the converse is not true, i.e. a consistent choice set may fail the external stability property of a covering set that any element outside the set must be covered by an element inside the set. The next example that this may indeed be the case. Additionally, the example demonstrates that there may not be a unique minimal consistent choice set, a property not shared by the covering set concept (see Dutta 1988)

Let $X = \{x_1, ..., x_5\}$ and let the dominance relation be as depicted in Fig. 2 (where, again, $x_i \rightarrow x_j$ means $x_i P x_j$)

[Figure 2]

No element is covered in $X$ and the unique covering set is $X$. However, there are many consistent choice sets in addition to $X$. In particular, $\{x_j, x_{j+1}, x_{j+2}\}$ is a consistent choice set for all $j = 1, ..., 5$ (modulo 5). This also implies that intersection of consistent choice sets is empty.

Dutta (1988) proves the existence of a covering set by using the iterated version of the uncovered set, the *ultimate* uncovered set $UUC$ (Miller, 1980; Dutta, 1988; Laslier, 1998). The $UUC$ is defined recursively as follows. Let $uc^{k+1}(X) = uc(uc^k(X))$, for all $k = 0, ....$ Then $UUC := uc^\infty(X)$. Since $X$ is a finite set, no element in $UUC$ is covered in $UUC$, *i.e.*, every arc spanned by $P$ in $UUC$ is an edge a three-cycle *in* this set.

Due to its recursive structure, the $UUC$ is uniquely defined. Importantly, Dutta (1988) shows that the $UUC$ is a covering set. This proves the existence of a covering set and, by Theorems 7 and 9, the existence of a strong dynamic stable set.

**Corollary 10** *A consistent choice set exists. Hence, a strong dynamic stable set exists.*

By Remark 1, this corollary also implies that a directly dynamically stable set and an indirectly dynamically stable set exist.

Dutta (1988) further shows that the $UUC$ is the unique *maximal* covering set (in the set of set inclusion). We now establish a corresponding result regarding consistent choice sets. Since a consistent choice set need not be a covering set, this is *not* implied by Dutta (1988). To prove that $UUC$ is the maximal consistent choice set, it suffices to show that no element that is removed in the iterations of of the uncovering operation can be supported in a consistent choice set.

**Theorem 11** *$UUC$ is the maximal consistent choice set.*

**Proof.** Let $C$ be a consistent choice set. We show that $C$ is a subset of $uc^{\infty}(X)$.

The proof is by induction. First, define recursively $B_k$ as the set of elements in $X \setminus \cup_{j=0}^{k-1} B_j$ that are covered in $X \setminus \cup_{j=0}^{k-1} B_j$, for $k = 0, 1, \ldots$. Since $X \setminus \cup_{j=0}^{k} B_j = uc^k(X)$ for all $k$, it suffices to show that $C \subseteq X \setminus \cup_{j=0}^{k} B_j$.

Clearly $C \subseteq X$. Suppose that $C \subseteq X \setminus \cup_{j=0}^{k-1} B_j$. By Definition 4, $x \in C$ is not covered in $X \setminus \cup_{j=0}^{k-1} B_j$ by any $y \in X \setminus \cup_{j=0}^{k-1} B_j$. Thus $C \cap B_k = \emptyset$, and hence $C \subseteq X \setminus \cup_{j=0}^{k} B_j$, as desired. ■

Vartiainen (2013) demonstrates a related result in a more general set up. There, the characterization of the maximal consistent choice set is based on a tighter notion of covering, and hence does not directly imply Theorem 11 in the tournament context

By Lemma 6, $V^{UUC}$ as defined in (2) - (3) is a strong dynamic stable set. Moreover, by Theorem 7, the outcomes induced by $V^{UUC}$ are the maximal set of outcomes induced by any strong dynamic stable set.

**Corollary 12** *$V^{UUC}$ is a strong dynamic stable set. Moreover, $UUC$ is the unique maximal set of outcomes that can be implemented via any strong dynamic stable set.*

Thus it is without loss of generality to focus on $UUC$ were one interested in the welfare consequences of dynamically stable coalitional bargaining with farsighted agents.

## 4.2 Consistent set and policy processes with the one-deviation property

One of the best known solution concepts in the literature of coalitional games is the *consistent set* of Chwe (1994). For our purposes, it suffices to give a narrow description of Chwe's solution to obtain a relationship between consistent sets and consistent choice sets. If a set $CS$ has the following property, then it is a consistent set: $x \in CS$ if and only if, for any $y$, either $y \notin R(x)$ or there is a chain $(x, y, z_0, \ldots, z_K)$ that indirectly dominates $(x, y)$ but not $(x)$ such that $z_K \in CS$.

**Remark 13** *A consistent choice set is a consistent set.*

To see this, let $C$ be a consistent choice set. If $x \in C$ and $y \in P(x)$, then there is $(x, y, z)$ that directly dominates $(x, y)$ but does not indirectly dominate $(x)$ such that $z \in C$. Hence $z \in P(y) \setminus P(x)$. The converse is not true, however.[15]

[15] Chwe (1994) gives an example where the consistent set is not contained by the uncovered set and, hence, by the ultimate uncovered set.

By Theorem 7 we conclude that under the maintained interpretation of the underlying coalitional set up, the set of outcomes implementable via a strong dynamic stable set is a consistent set. Interestingly, this observation also provides a credible justification for the consistent sets. The aim of Chwe (1994) is to define a weak solution concept, one that "eliminates with confidence". Hence any feasible outcomes should be contained in the largest consistent set. Our paper provides a justification for the inclusion to the other direction in cases where the consistent set is also a consistent choice set. Then any outcome in the consistent set can be implemented via a strong dynamic stable set.

Vartiainen (2013) studies policy processes à la Bernheim and Slavov (2009) that meet the standard one-deviation property and that include the option to "stop", i.e. to implement the status quo outcome. Moreo specifically, a policy process is a function $\sigma$ defined on all histories $H$, choosing to "stop" if the current status quo is to be implemented and $y \in X$ if the current status quo is to replaced with $y$. Thus the policy process $\sigma$ defines the policy path starting from any history $(h, x)$ as well as the outcome that is eventually implemented from this history. The one-deviation restriction requires that, at each history $(h, x)$, the eventually implemented outcome is not dominated by the outcome that is eventually implemented after a deviation from $\sigma(h, x)$ to $X \cup \{stop\} \backslash \sigma(h, x)$.

In a more general framework that studied in this paper, it is demonstrated that policy processes meeting the one-deviation property can be characterized by the consistent choice set. That the one-deviation property and the strong dynamic stable set reduce to the same characterization is interesting, there are also important differences in the modeling practices which make the two solutions quite different at the outset. The two solutions are different in terms of their internal logic.

The internal indirect stability requires that the history $(h, x)$ in a strong dynamic stable set is robust against *any* stable dominance paths. However, the one-deviation property only requires that $\sigma(h, x) = stop$ is robust against *the* path $\{\sigma^0(h, x, y), \sigma^1(h, x, y), ..., \sigma^K(h, x, y)\}$, where $\sigma^0(h, x, y) = \sigma(h, x, y)$, $\sigma^k(h, x, y) = \sigma(h, x, y, \sigma^0(h, x, y), ..., \sigma^{k-1}(h, x, y))$, for all $k = 1, ..., K$, and $\sigma^K(h, x, y) = stop$. Thus the strong dynamic stable set is tested against a larger set of dominance paths. Furthermore, external direct stability requires that a history $(h, x)$ is dominated by some $(h, x, y)$ in the strong dynamic stable set. The one-deviation property only requires that $\sigma(h, x) = y$ implies that there is some dominance path $\{\sigma^0(h, x, y), \sigma^1(h, x, y), ..., \sigma^K(h, x, y)\}$ that implements $\sigma^K(h, x, y)$. Hence the path following the deviation need not be of single step.

These two aspects show that the one-deviation property is uses more information to determine the acceptable play under all circumstance. Because of the flexibility that comes with the more elaborated structure, it is also more generally applicable. In fact the strong dynamic stable set cannot be extended without complications to the general case studied in Vartiainen (2013). The two notions coincide only in the finite tournament context. One interpretation is that the strong dynamic stable set characterizes the policy processes with one-deviation property in this context.

# 5 Conclusion

The von Neumann-Morgenstern (vNM) stable set solution can, in principle, be applied to any abstract decision theoretic situation with a proper dominance structure. The solution reflects sophisticated reasoning in a way that many of its competitors, e.g. the core, do not: outcomes are tested only against deviations that are themselves deemed stable. This aspect of the solution has intrinsic appeal and also enhances its applicability. However, the solution suffers from credibility problems (due to Harsanyi 1974 and Xue 1998). Moreover, the solution often fails to exist.

This paper argues that the problems with the standard vNM stable set solution are due to the fact that the solution fails to take fully into account the underlying dynamics. Our aim is to propose a remedy to this deficiency. We develop a new version of the solution that incorporates the idea of history dependence into the notion of blocking. Importantly, our solution allows the underlying social dynamics to have memory. Memory can be interpreted as an extension of the level of farsightedness of the players. This feature gives a new degree of freedom into formation of the solution, and permits strengthening the usual internal and external stability criteria without risking the existence of the solution. That is, to avoid the credibility problems identified by Harsanyi and Xue, we require that external stability is in the form if direct dominance whereas internal dominance is in the form of indirect dominance.[16]

Our focus is on a tournament set up (total, antisymmetric dominance relation) where the problems with the standard stable set solution are particularly acute. We first point out that the standard stable set can be interpreted as a Markovian version of the strong stable set, and hence possesses the same strong stability criteria. This observation explains why the standard stable set, when it exists, reflects correctly strategic behavior in typical models of legislative bargaining (see e.g. Anesi and Seidmann, 2014, and the literature therein).

We characterize the strong dynamic stable set directly in terms of the tournament relation. The stable set turns out have a close relationship with the covering set of Dutta (1988). More precisely, any covering set gives rise to a strong dynamic

[16] Due to history dependence, indirect and direct dominance need not coincide.

stable set (however, the converse it not true). Thus, since the ultimate uncovered set is a covering set, a strong dynamic stable set exists. We also show that the ultimate uncovered set coincides with the set of all outcomes that can be implemented via a strong dynamic stable set.

The definition of strong dynamic stable set can be extended outside the tournament structure by adding some structure to the underlying set up (e.g. in the manner of Chwe 1994). However, when "inducement relation" between the nodes is not complete, as it is in the context of a tournament, the existence of the solution cannot guaranteed without further restrictions. One possible avenue is to follow Vartiainen (2011) by allowing deviations also in the middle of histories.

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