Prospect theory

- Despite its elegance, the expected utility theory does not survive experimental tests
- Problems:
  1. People are bad with probabilities
  2. Decisions are reference dependent
- Prosect theory of Kahnemann and Tversky (1979) aims to descriptive plausibility
- Requires relaxation of the axioms - or meta-axioms - of the expected utility theory

Rank dependent utility

- Under expected utility model, risk aversion captured by the concavity of the utility function
- Concavity is a local phenomenon, reflected by the sensitivity of the DM to additional money
- But risk aversion seems to be something else than just psychophysics of money - it is related to optimism and pessimism
- Problem with the unreasonably large degree of risk aversion under small bets ($=>$ Rabin's paradox)

- Consider a preference elicitation procedure over outcomes $x \in [0, 100]$
- By the construction in the proof of the vNM theorem, choose $u(0) = 0$ and $u(100) = 1$, and let $u : [0, 100] \to [0, 1]$
- Then $u(x) = p$ reflects the probability p under which the DM is indifferent between x and a lottery $p \cdot 1_{100} + (1 - p) \cdot 1_0$

- Risk aversion is reflected by the concavity of u
- Equivalently, risk aversion is reflected by the convexity of the function $p \mapsto w(p)$ such that $w = u^{-1}$
- The new interpretation: the DM has risk neutral utility function but his probability assesement is distorted $=>$ as if risk averse behavior due to distorted probability assesements

Example

(Allais reconsidered) There are two choice scenarios:

1. Choice between lotteries
   1. $0.33 \cdot 2500 + 0.66 \cdot 2400 + 0.01 \cdot 0$
   2. $1 \cdot 2400$
2. Choice between lotteries
   1. $0.33 \cdot 2500 + 0.67 \cdot 0$
   2. $0.34 \cdot 2400 + 0.66 \cdot 0$

Example (cont.)

No function reflecting expected utility maximization is consistent with choices 1b and 2a - what about function w?

$$w(0.33) \cdot 2500 + w(0.66) \cdot 2400 < 2400$$

and

$$w(0.33) \cdot 2500 > w(0.34) \cdot 2400$$

i.e.

$$w(0.34) + w(0.66) < 1$$

which holds true for any strictly convex w

- But nonlinearity of w implies nonadditivity: there are p and q such that $w(p+q) \neq w(p) + w(q)$
- A problem: violation of the first-order stochastic dominance (i.e. monotonicity): shifting probability mass from an outcome to a preferred outcome may decrease the desirability of the lottery

Example

Let $p, q \in [0, 1]$ be such that $w(p+q) > w(p) + w(q)$

1. Consider the choice between lotteries
    1. $p \cdot 10 + q \cdot 10 + (1-p-q) \cdot 0$
    2. $(p+q) \cdot 10 + (1-p-q) \cdot 0$

Example

- The value of the lotteries are $10(w(p)+w(q))$ and $10w(p+q)$ and hence 1b is chosen
- Observe that lottery $p\cdot(10+\varepsilon)+q\cdot 10+(1-p-q)\cdot 0$ has value $(10+\varepsilon)w(p)+10w(q)$
- Since 1b is chosen it follows, for small enough ε, that also $p\cdot(10+\varepsilon)+q\cdot 10+(1-p-q)\cdot 0$ is inferior to $(p+q)\cdot 10+(1-p-q)\cdot 0$
- But this violates the first-order stochastic dominance as $p\cdot(10+\varepsilon)+q\cdot 10+(1-p-q)\cdot 0$ stochastically dominates $(p+q)\cdot 10+(1-p-q)\cdot 0$

- Adding risk aversion to the utility function would not change the conclusion: distortion of the probability weights opens the door for violation of the first-order stochastic dominance
- The key problem: probabilities and outcomes cannot be replaced 1-1 when one distorts probabilities rather than utilities - probabilities and wealth are evaluated in different scales as probabilities are atoms whereas wealth is a cumulative number
- The way to avoid the problem: focus on rank dependent utility
- Let the outcomes be drawn from a finite set $x_0 < x_1 < ... < x_n$
- Then expected value of lottery p can be written

$$\sum_{i=0}^{n} p_i x_i = \sum_{i=0}^{n} \left(\sum_{j=i}^{n} p_j \right) (x_i - x_{i-1})$$

- Denote the *rank* of lottery p at i by

$$r_i = \sum_{j=i}^{n} p_j$$

i.e., the probability of a reward at least x_i

- The expected value of a lottery p can then be written compactly

$$\sum_{i=0}^{n} r_i (x_i - x_{i-1})$$

- Generalizing this, define the *rank dependent utility* for a given probability weighting function w by

$$V(p) := \sum_{i=0}^{n} w(r_i)(x_i - x_{i-1})$$

- A lottery p first-order stochastically dominates p' if $r_i \geq r_i'$, for all $i = 0, ..., n$ (where r_i is the ith rank of p and r_i' is the ith rank of p')
- Rank dependent utility satisfies the first-order stochastic dominance -criterion: if p first-order stochastically dominates p', then

$$\begin{aligned} V(p) - V(p') &= \sum_{i=0}^{n} (w(r_i) - w(r_i'))(x_i - x_{i-1}) \\ &\geq 0 \end{aligned}$$

where the inequality follows since $x_{i+1} \geq x_i$ and w is increasing

- Note that $r_{i+1} = r_i - p_i$ and $r_n = 0$, $r_0 = 1$
- Writing the decision weight of an outcome i by

$$\pi_i(p) = w(r_i) - w(r_{i+1})$$

the rank dependent utility has the form

$$V(p) = \sum_{i=0}^{n} \pi_i(p) x_i$$

- When w is linear $\pi_i(p) = w(r_i) - w(r_{i+1}) = p_i$, and the rank dependent utility is simply the expected value of the lottery

$$V(p) = \sum_{i=0}^{n} p_i x_i$$

- With rank dependent utility function, we can capture behavioral tendencies that are not consistent with expected utility maximization
  - Optimism: differences in low ranks larger than in high ranks $=>$ concave w (e.g. $\sqrt{p}$)
  - Pessimism: differences in low ranks smaller than in high ranks $=>$ convex w (e.g. p^2)

- Common finding: w concave in $(0, 1/3)$, convex in $(1/3, 1)$
  - Possibility effect - overweighting the small probability events/ best outcomes
  - Certainty effect - overweighting the high probability events/ worst outcomes
- Explains the coexistence of gambling and insurance

- More general formulation: a utility function $u : \mathbb{R} \to \mathbb{R}$ and rank-dependent utility function V such that

$$V(p) = \sum_{i=0}^{n} \pi_i(p) u(x_i)$$

- Let $\succsim$ be a binary relation on the set L of simple lotteries on $\mathbb{R}$

RDU1 (weak ordering) $\succsim \subseteq L^2$ is transitive and complete

- First-order stochastic dominance means that shifting of probability mass from an outcome to a higher outcome

RDE2 (stochastic dominance) If p first-order stochastically dominates q, then $p \succsim q$

RDU3 (continuity) $\{q : q \succsim p\}$ and $\{q : p \succsim q\}$ are closed for all $p \in L$

RDU4 (consistency) Rank-tradeoff consistency

Theorem

Binary relation $\succsim \subseteq L^2$ *satisfies RDU1-RDU4 if and only there is aprobability weight function* w *and utility function* u *such that the induced* V *represents* $\succsim$

- Local properties of w can be tested by looking at small changes in p, and letting the rank r vary in the range $[0, 1]$

- The sure thing principle, requiring that if, under two lotteries (prospects), a certain outcome is chosen with the same probability, then this outcome will not affect the comparison of the two lotteries, is the most important assumption of Savage
- Any reasonable theory of choice under uncertainty should pass this test

Example

Take four prospects p, p' and q, q'such that $p_i = p'_i = q_j = q'_j$, and $p_k = q_k$ and $p'_k = q'_k$ for all $k \neq i, j$, then, by the sure thing principle, p is preferred to q if and only if p' is preferred to q'. Rank dependent utility does satisfy this.

Reference dependence

- The second aspect of prospect theory concerns reference dependence
- Models based on binary preferences, whose interpretation relies on the revealed preference argument, are *as if* -theories
- I psychology, descriptive models seek to predict choices that are made
- Theories which seek to simulate decision making processes $=>$ procedural theories
  - Agents draw on decision heuristics
  - Rules of thumb
- Which procedures are followed?

- Kahnemann and Tversky (1979): two stage procedure
    1. Prospects (lotteries) are edited using heuristics
    2. Prospects are evaluated against a reference point by using (some version of) the rank ordered utility - wins and lossess have distinct meaning
- Reference point can be thought as the status quo wealth level
- Framing effect

- Heuristics
  - Rank ordered utility fits the inverted *s*-shaped *w* function
  - Diminishing sensitivity and loss aversion
  - Reflection effect
  - Deminance heuristic
  - Etc.
- Wakker (2010): "More than half of the observed risk aversion has nothing to do with utility curvature or probability weighting - it is generated by loss aversion"

Example

Consider three choice scenarios

1. A choice between gain-prospects
   1. 50
   2. $0.5 \cdot 100 + 0.5 \cdot 0$
2. A choice between loss-prospects
   1. -50
   2. $0.5 \cdot 0 + 0.5 \cdot -100$
3. A choice between loss-propects, with a side payment
   1. $100 - 50$
   2. $100 + 0.5 \cdot 0 + 0.5 \cdot -100$

Example (cont.)

Majority of repondents choose 1a (exhibiting risk aversion), 2b (exhibiting risk seeking)

- 3b depends on the frame: if fixed income 100 is seprated from the rest of the problem, subjects behave as in $2(= \text{b})$, but in the subjects are consequentalists, then they choose as in $1(= \text{a})$
- Thus procedures matter

- Bounded rationality prevens from taking all relevant infomation into account
- Too much risk aversion or too much risk loving in reality, to be plausible
- Asset integration + isolation

- Loss aversion defined: there is a basic utility function u and a loss aversion parameter λ such that the overall utility v from (changes in) wealth x is $v(x) = u(x)$ if $x \geq 0$ and $v(x) = \lambda u(x)$ if $x < 0$
- u captures the intrinsic value of outcomes
- We assume that u is smooth and increasing, and use normalization $u(1) = 1$, $u(0) = 0$
- Loss aversion (gain seeking) holds if $\lambda > 1$
- Example: Rabin's paradox

- Reference point is the point from which changes in wealth are evaluated
- Given the reference point x^*, the expected value of a (finite) prospect p is

$$\sum p(x)v(x - x^*) = \sum_{x \geq x^*} p(x)u(x - x^*) + \lambda \sum_{x < x^*} p(x)u(x - x^*)$$

- To apply rank ordering to this model, let there be two probability weighting functions w^+ and w^-, the first desrcibing probability distortion conditional on $x - x^* \geq 0$, and the latter desrcibing probability distortion conditional on $x - x^* < 0$
- Denote by $\pi^+(p, r)$ and $\pi^-(p, r)$ the corresponding decision weight functions
- Then the value of a prospect p, given u, is

$$\begin{aligned} V_u(p) &= \sum \pi_i(p)(x) v(x - x^*) \\ &= \sum_{x \geq x^*} \pi_i(p)(x) u(x - x^*) + \lambda \sum_{x < x^*} \pi_i(p)(x) u(x - x^*) \end{aligned}$$

- Denoting $V_u^+(p)$ and $V_u^-(p)$ the value of the prospect conditional on postive and negative wealth changes, it follows that

$$V_u(p) = V_u^+(p) + V_u^-(p)$$

- Value of a prospect satisfies the first-order stochastic dominance criterion as well as the sure-thing principle