De Finetti and the Dutch books

- In gambling a Dutch book is a set of odds and bets which guarantees a profit, regardless of the outcome of the gamble
- Consider the situation where the DM chooses a portfolio or a gamble whose value depends on the realized state s in some finite set S
- S is now the "set of states" and uncertainty relates to the element of S that will materialize
- That is, the gamble is an element x in $\mathbb{R}^S$ (denote a generic coordinate by s)
- Which gamble should the DM choose?

- Let $\succsim \subset \mathbb{R}^S \times \mathbb{R}^S$ be a binary relation that represents the DM's preferences

F1 (Weak order) $\succsim$ is complete and transitive

F2 (Continuity) $\{y : y \succsim x\}$ and $\{y : x \succsim y\}$ are closed for all x

- That is, the graph of $\succsim$ is closed

F3 (Additivity) $x + z \succsim y + z$ if and only if $x \succsim y$, for all x, y, z

- Additivity is a version of independence
- Additivity implies neutrality to wealth - the DM's choice between two portfolios x and y is idependent of the underlying, 'old', portfolio

F4 (Monotonicity) $x_s \geq y_s$ for all $s \in S$ implies $x \succsim y$

Theorem

$\succsim$ satisfies F1-F4 if and only if there is a unique probability vector p (i.e., $p \in \mathbb{R}^S_+$ and $\sum_{s \in S} p(s) = 1$) such that

$$x \succsim y \quad \text{if and only if} \quad \sum_{s \in S} p(s)x(s) \geq \sum_{s \in S} p(s)y(s)$$

Proof.

To prove the sufficiency, note that, by additivity, $x \succsim y$ if and only if $x - y \succsim 0$. Let $A = \{y \in \mathbb{R}^S : y \succsim 0\}$ and $B = \{y \in \mathbb{R}^S : 0 \succ y\}$. We show that A and B are convex sets. First note that $x \succsim y$ implies $x \succsim (x+y)/2 \succsim y$ by additivity and transitivity. Hence, by repeatedly applying this property, $x \succsim \ell 2^{-k} x + (1 - \ell 2^{-k}) y \succsim y$, for all $\ell, k \in \mathbb{N}$. By continuity, then, $x \succsim \lambda x + (1-\lambda) y \succsim y$ for all $\lambda \in (0,1)$ (since $(0,1)$ is dense in binary rationals). Thus A and B are convex sets.
Since A and B are convex sets, and $A \cap B = \emptyset$, there is a separating hyperplane $b \in \mathbb{R}^S \setminus \{0\}$ and $c \in \mathbb{R}$ such that $x \in A$ iff

$$\sum_{s \in S} b(s) x(s) \geq c.$$

Since $0 \in A$, $c \leq 0$. By monotonicity, $-\varepsilon \in B$ for all $\varepsilon > 0$ which implies that $c \not< 0$. By choosing x that has positive entry at s only if $b(s) < 0$, we obtain that $x \in B$ whenever such $b(s)$s exist. However, by monontonicy, $x \in A$. Thus $b \in \mathbb{R}^S_+$. $\square$

Proof.

[Cont.] Since $x \succsim y$ if

$$\sum_{s \in S} b(s)(x(s) - y(s)) \geq 0,$$

and since $b \in \mathbb{R}^S$, it follows that p such that

$$p(s) = \frac{b(s)}{\sum_{s' \in S} b(s')}, \quad \text{for all } s \in S,$$

is the desired probability vector. For uniqueness, since $A \cup B = \mathbb{R}^S$, there can be at most one hyperplane that separates them. $\square$

- Three interpretations
  1. Definition of subjective probabilities: the ones that induce expected payoff maximizing betting behavior
  2. Compelling axioms lead to expected payoff maximization even under subjective uncertainty
  3. Elicitation procedure

- If $x \succsim y$, then the DM would be willing sell y for a lower price than x, and buy x for a higher price than y
- A Dutch book is a collection of pairs of portfolios $(x^1, y^1), ..., (x^m, y^m) \in \mathbb{R}^S$ such that (i) $x^i \succsim y^i$ for all $i = 1, ..., m$, and such that (ii) $\sum_{i=1}^m x^i(s) < \sum_{i=1}^m y^i(s)$, for all $s \in S$
- That is, a Dutch book $(y^1, ..., y^m)$ can be traded against $(x^1, ..., x^m)$ with the DM that (i) would not require extra funding for the trader and (ii) generates profit with certainty in the future

Corollary

If $\succsim$ satisfies F1-F4, then a Dutch book does not exist

Proof.

Suppose that $x^i \succsim y^i$ for all $i = 1, ..., m$. By the theorem, there is p such that

$$\sum_{s \in S} p(s) x^i(s) \geq \sum_{s \in S} p(s) y^i(s), \text{ for all } i.$$

Thus

$$\sum_{i=1}^{m} \sum_{s \in S} p(s) x^i(s) \geq \sum_{i=1}^{m} \sum_{s \in S} p(s) y^i(s)$$

or

$$\sum_{s \in S} p(s) \left(\sum_{i=1}^{m} x^i(s) - \sum_{i=1}^{m} y^i(s) \right) \geq 0,$$

which violates part (ii) of the definition of a Dutch book. □

Subjective vs. objective probabilities

- vNM theory presumes objective probabilities, i.e. that beliefs that governs the DM's behavior problem has a well defined and symmetric meaning also to outsiders - to us
- In what sense can beliefs be objective?
- Classical statistics gives beliefs a frequentist interpretetion - probabilities are just limit frequencies of i.i.d trials

- But this presumes i.i.d. probabilites, cannot be used to define them
- Most of the relevant choice scenarios cannot be isolated
- But any behavior can, in principle, be justified by subjective beliefs - or can it?
- By logical positivism, subjective beliefs have meaning only if they can be tested

- Can we put restrictions on the observables, i.e. behavior, that allows us to deduce the the beliefs?
- If we know how beliefs are used we may deduce, by backwards engineering, the beliefs
- Testing the decision making procedure is feasible

Subjective probability

- There is set S of states of the world, an exhaustive list of scenarios that might unfold - $s \in S$ answers to all questions the DM may have (precluding DM's behavior, by the assumption of free will)
- There is a (finite) set X of outcomes or consequences X that specifies all that is relevant from the point of view of the well-being of the DM, alongside with the materialized state s
- The decision problem is to choose an act f from the set $S^X = \{f : S \to X\}$

- That is, the act specifies what the DM would choose under all states of the world, would this information be avialable to her
- Preferences $\succsim$ of the DM are defined on F, i.e. $\succsim \subset S^X \times S^X$
- Detailed information is typically not available to the DM, however
- There is set 2^S of possible events, i.e. the set of all subsets of S
- Interprete x as the constant act that implements x in all states

S1 (Weak order) $\succsim$ is complete and transitive

S2 (Relevance) $x \succ y$ for some $x, y \in X$

- The next axiom implictly assumes implictly that different acts do not affect differentially on the probabilities of certains states within an event

S3 (Sure thing) For every $f, g, f', g' \in F$ and for every $A \in 2^S$, if

$$f(s) = f'(s) \text{ and } g(s) = g'(s) \text{ for all } s \in A$$
$$f(s) = g(s) \text{ and } f'(s) = g'(s) \text{ for all } s \notin A$$
$$\text{then } f \succsim g \text{ if and only if } f' \succsim g'$$

- The implication of the sure thing principle is clearly true if $A = \emptyset$ or if $A = S$

S4 (Acts do not affect probabilities) For all $x, y, x', y' \in X$ with $x \succ y$ and $x' \succ y'$, and for all $A, B \in 2^S$, if

$f(s) = x$ and $f'(s) = x'$ for all $s \in A$ and $f(s) = y$ and $f'(s) = y'$ for all $s \notin A$

$g(s) = x$ and $g'(s) = x'$ for all $s \in B$ and $g(s) = y$ and $g'(s) = y'$ for all $s \notin B$

then $f \succsim g$ if and only if $f' \succsim g'$

- That is, $f \succsim g$ implies that A is more likely than B which in turn implies $f' \succsim g'$, and vice versa

- We write $f \succsim_A g$ if $f \succsim g$ and $f(s) = g(s)$ for all $s \notin A$
- An event A is null if there is no f, g such that $f \succ_A g$, .i.e. how two acts differ in A will never affect the preferences between the acts
- If an event is null, then it should never affect the preferences of the DM

S5 (Monotonicity) For every nonnull event $A \in 2^S$ and $x, y \in X$,
if $f(s) = x$ and $g(s) = y$ for all $s \in A$
then $f \succsim_A g$ if and only if $x \succsim y$

S6 (Continuity) For all $f, g \in F$ such that $f \succ g$, and for all $x \in X$, there is a finite partition $\{A_i\}$ of S such that for every A_i, if

$f'(s) = x$ for all $s \in A$ and $f(s) = f'(s)$ for all $s \notin A$, then $f' \succ g$

$g'(s) = x$ for all $s \in A$ and $g(s) = g'(s)$ for all $s \notin A$, then $f \succ g'$

- Continuity implies that each singleton event $\{s\}$ is a null
- Together with monotonicity this requires that S is an infinite set
- Before stating the result, we need to define a probability distribution

- A probability measure μ, defined on some measurable space, is finitely additive if

$$\mu(\cup_{i=0}^{n} A_i) = \sum_{i=0}^{n} \mu(A_i)$$

  where $\{A_i\}$ is a collection of disjoint sets
- A finitely additive measure μ is nonatomic if for every event A with $\mu(A) > 0$ and for every $r \in [0, 1]$ there is an event $B \subseteq A$ such that $\mu(B) = r\mu(A)$
- Judgements are captured by a probability distribution p on 2^S such that $p(S) = 1$ and $p(a \cup b) = p(a) + p(b)$ if $a \cap b = \emptyset$

Theorem

Let X be finite. Then binary relation $\succsim \subset F \times F$ and satisfies S1-S6 if and only if there is a function a nonatomic finitely additive probability measure μ on S and a function $u : X \to \mathbb{R}$ such that

$$\int_S u(f(s))d\mu(s) \geq \int_S u(g(s))d\mu(s) \quad \text{if and only if} \quad f \succsim g.$$

Moreover, μ is unique and the function u is unique up to positive linear transformation

Example

(Ellsberg) Let there be an urn of 90 balls in three colours, red, blue and yellow. We know 30 of balls are red but nothing of the composition of the remaining balls is known.

- A ball is drawn from the urn
- There are two choice scenarios:
    1. A bet between the ball being red or being blue
    2. A bet between the ball not being red or not being blue

Example

- Typical choices 1: $red \succ blue$ and 2: $not - red \succ not - blue$
- This implies, letting $p \in [0, 69/90)$ be the subjective belief of the proportion of blue balls in the urn, that

$$1/3 > p \quad \text{and} \quad 2/3 > 1 - p$$

  which is not possible
- Thus the decision makers do not have subjective beliefs - their choices violate the sure thing principle

- What kind of beliefs are feasible without the sure thing principle?

Subjective probability compunded with objective probability - Anscombe and Aumann

- Let there be a (finite) set S of the possible states of the world, reflecting the DM's uncertainty
- There is also a (finite) set X of prizes or outcomes that the DM cares about
- An outcome could be a winner of a horse race, the weather, stock price etc.
- Think there being a set L of (objective) lotteries over X
- DM's preferences are now defined over **acts** $S^L = \{ f : S \rightarrow L\}$
- An act is a function that specifies a lottery conditional on the realized state, e.g. a probability distribution over possible bets

- Conditions on preferences $\succsim S^L \times S^L$ are analogous to the vNM axioms

AA1 (Weak order) $\succsim$ is a complete and transitive

AA2 (Continuity) For all $f, g, h \in S^L$, if $f \succ g \succ h$, then there are $\lambda, \mu \in (0, 1)$ such that $\lambda \cdot f + (1 - \lambda) \cdot h \succ g \succ \mu \cdot f + (1 - \mu) \cdot h$

AA3 (Independence) For all $f, g, h \in S^L$ and $\lambda \in (0, 1)$, if $f \succsim g$ then $\lambda \cdot f + (1 - \lambda) \cdot h \succsim \lambda \cdot g + (1 - \lambda) \cdot h$

- Denote by f_s the lottery that the act f chooses in state s
- (Almost) from the vNM characterization we obtain that $\succsim$ satisfies AA1-AA3 if and only if there are functions $u_s : X \to \mathbb{R}$, $s \in S$, such that

$$\sum_{s\in S}\sum_{x\in X} f_s(x)u_s(x) \geq \sum_{s\in S}\sum_{x\in X} g_s(x)u_s(x) \text{ if and only if } \quad f \succsim g.$$

- Moreover, the functions u_s are unique up to positive linear transformation

AA4 (Monotonicity) If $f_s \succsim g_s$ for all $s \in S$, then $f \succsim g$

Theorem

$\succsim$ satisfies AA1-AA4 if and only if there is a function $u : X \to \mathbb{R}$ and a probability distribution p on S such that

$$\sum_{s \in S} p(s) \sum_{x \in X} f_s(x) u(x) \geq \sum_{s \in S} p(s) \sum_{x \in X} g_s(x) u(x) \quad \text{if and only if} \quad f \succsim g.$$

Moreover, p is unique and u is unique up to positive linear transformations.