Uncertainty

Example (St Petersburg paradox, Bernoulli 1738)

How much would you pay for a gamble where one tosses a fair coin until one wins and after each toss, the prize money doubles?

- The paradox: one is usually(!) willing to pay only limited price for the lottery whereas the expected monetary value is

$$2 \cdot \frac{1}{2} + 2^2 \cdot \frac{1}{2^2} + 2^3 \cdot \frac{1}{2^3} + \ldots = 1 + 1 + 1 + \ldots = \infty$$

- ...hence one has to maximize the expected utility value of the money
- For example, with utility function $u(x) = \log_2 x$, then the value of the gamble is 1

- Under uncertainty, the decisions concern the feasible lotteries, and hence the preference relation should be defined over them
- How?
- The **lottery space** (assume finite X)

$$L = \left\{ p \in \mathbb{R}_+^{|X|} : \sum_{x \in X} p(x) = 1 \right\}$$

- For any two lotteries p and q, and parameter $\lambda \in [0,1]$, denote by $\lambda \cdot p + (1-\lambda) \cdot q$ is mixed lottery such that

$$(\lambda p + (1-\lambda)q)(x) = \lambda \cdot p(x) + (1-\lambda)q(x), \text{ for all } x \in X$$

- L is a mixture space is a mixture space: $p, q \in L$ implies $\lambda \cdot p + (1-\lambda) \cdot q \in L$ for all $\lambda \in [0,1]$
- Denote the degenerate lottery that puts all the probablity mass on choice x by 1_x
- The observable choices $\succsim$ are now defined over L, i.e. $\succsim \subset L \times L$

Expected utility maximization

- The von Neumann-Morgenstern axioms are

NM1 (Weak order) $\succsim$ is a complete and transitive

- That is, $\succsim$ is a preference relation on L

NM2 (Continuity) For all $p, q, r \in L$, if $p \succ q \succ r$, then there are $\lambda, \mu \in (0,1)$ such that
$\lambda \cdot p + (1-\lambda) \cdot r \succ q \succ \mu \cdot p + (1-\mu) \cdot r$

- Equivalently, the upper and lower contour sets of $\succsim$ are closed
- Continuity axiom is occasionally called the Archimedean axiom

- The following condition implies that such α is unique

NM3 (Independence) For all $p, q, r \in L$ and $\lambda \in (0,1)$, if $p \succsim q$ then $\lambda \cdot p + (1-\lambda) \cdot r \succsim \lambda \cdot q + (1-\lambda) \cdot r$

- A direct implication of independence is that if $p \sim q$, then $\lambda \cdot p + (1-\lambda) \cdot r \sim \lambda \cdot q + (1-\lambda) \cdot r$, for any $\lambda \in (0,1)$

Lemma

If $\succsim$ satisfies weak order (NM1) and independence (NM3), then

$$\begin{aligned} p &\succsim q \quad \textit{if and only if} \\ \lambda \cdot p + (1-\lambda) \cdot q &\succsim \lambda' \cdot p + (1-\lambda') \cdot q, \quad \textit{for all } \lambda \geq \lambda' \end{aligned}$$

For all $p, q, r \in L$ and $\lambda \in (0,1)$, if $p \succsim q$ then $\lambda \cdot p + (1-\lambda) \cdot r \succsim \lambda \cdot q + (1-\lambda) \cdot r$

- A direct implication of independence is that if $p \sim q$, then $\lambda \cdot p + (1-\lambda) \cdot r \sim \lambda \cdot q + (1-\lambda) \cdot r$, for any $\lambda \in (0,1)$

Lemma

If $\succsim$ satisfies weak order (NM1) and independence (NM3), then

$$\begin{array}{rcl} p & \succsim & q \quad \textit{if and only if} \\ \lambda \cdot p + (1-\lambda) \cdot q & \succsim & \lambda' \cdot p + (1-\lambda') \cdot q, \quad \textit{for all } \lambda \geq \lambda' \end{array}$$

Proof.

By independence, for any $\lambda \in (0,1)$

$$\lambda \cdot p + (1-\lambda) \cdot q \succsim \lambda \cdot q + (1-\lambda) \cdot q = q.$$

Applying this with respect to $\mu \in (0,1)$,

$$\begin{aligned} \lambda \cdot p + (1-\lambda) \cdot q &\succsim \mu \cdot (\lambda \cdot p + (1-\lambda) \cdot q) + (1-\mu) \cdot q \\ &= \mu\lambda \cdot p + (1-\mu\lambda) \cdot q. \end{aligned}$$

By choosing $\mu = \lambda'/\lambda$, the result ensues. $\square$

Lemma

If $\succsim$ satisfies weak order (NM1), continuity (NM2), and independence (NM3), then for any $p \succ q \succ r$ there is $\alpha \in (0,1)$ such that $\alpha \cdot p + (1-\alpha) \cdot r \sim q$

Proof.

Consider sets $\{\lambda \in [0,1] : \alpha \cdot p + (1-\alpha) \cdot r \succ q\}$ and $\{\lambda \in [0,1] : q \succ \alpha \cdot p + (1-\alpha) \cdot r\}$. By construction, the sets are disjoint. By the previous lemma, they are intervals and, by continuity, open. Hence they do not cover $[0,1]$, and there is a point $\alpha \in [0,1]$ not in $\{\lambda \in [0,1] : \alpha \cdot p + (1-\alpha) \cdot r \succ q\}$ or $\{\lambda \in [0,1] : q \succ \alpha \cdot p + (1-\alpha) \cdot r\}$. By construction, $\alpha \cdot p + (1-\alpha) \cdot r \sim q$. By the previous lemma, α is unique. $\square$

Theorem

Let X be a finite set. Then $\succsim$ satisfies NM1-NM3 if and only if there is a function $u : X \to \mathbb{R}$ such that

$$\sum_{x \in X} p(x)u(x) \geq \sum_{x \in X} q(x)u(x) \quad \textit{if and only if} \quad p \succsim q.$$

Moreover, the function u is unique up to positive linear transformation

Proof.

Identify x^* and x_* such that

$$1_{x^*} \succsim 1_x \succsim 1_{x*}, \quad \text{for all } x \in X$$

By Lemmata 4 and 3 there is, for any $x \in X$, a unique α_x such that

$$1_x \sim \alpha_x \cdot 1_{x^*} + (1 - \alpha_x) \cdot 1_{x_*}$$

Any lottery p can be rewritten as a mixture

$$p = p(x) \cdot 1_x + (1 - p(x)) \cdot p^x$$

where

$$p^x(y) = \frac{p(y)}{1 - p(x)}, \quad \text{for all } x \in X \backslash \{x\}$$

□

Proof.

[(cont.)] By independence,

$$p \sim p(x)[\alpha_x \cdot 1_{x^*} + (1-\alpha_x) \cdot 1_{x_*}] + (1-p(x)) \cdot p^x$$

By induction on the cardinality of X,

$$p \sim \sum_x p(x)[\alpha_x \cdot 1_{x^*} + (1-\alpha_x) \cdot 1_{x_*}]$$

or, equivalently,

$$p \sim \left(\sum_x p(x)\alpha_x\right) \cdot 1_{x^*} + \left(1 - \sum_x p(x)\alpha_x\right) \cdot 1_{x_*}$$

By choosing $u(x) = \alpha_x$ for all $x \in X$, and by Lemma 3,

$$p \succsim q \quad \text{if and only if} \quad \sum_x p(x)u(x) \geq \sum_x q(x)u(x).$$

$\square$

Proof.

[(cont.)] To see the uniqueness, note that for any representation v of preferences $\succsim$,

$$v(x) = \alpha_x v(x^*) + (1 - \alpha_x) v(x_*), \text{ for all } x \in X$$

Find $a \in \mathbb{R}_{++}$ and $b \in \mathbb{R}$ such that

$$v(x^*) = au(x^*) + b \quad \text{and} \quad v(x_*) = au(x_*) + b$$

Since also

$$u(x) = \alpha_x u(x^*) + (1 - \alpha_x) u(x_*), \text{ for all } x \in X$$

it follows that

$$\begin{aligned} v(x) &= \alpha_x [au(x^*) + b] + (1 - \alpha_x)[au(x_*) + b] \\ &= a[\alpha_x u(x^*) + (1 - \alpha_x) u(x_*)] + b \\ &= au(x) + b \end{aligned}$$

- Provides a definition of a utility function
- The cost is that identification of u requires many observations - all pairs of lotteries over X
- The proof suggests an elicitation method: ask the DM to identify the probability α_x of winning x^* relative to losing x_* under which he is indifferent with choice x
- Gives a justification for the expected utility maximization (rather than, say, median)
- Normative argument: if the axioms are accepted, there has to be a utility function

- In the proof we used an induction argument
- Without finiteness of X, the indictive step requires further assumption
- The sure thing principle: if lottery p is concentrated on a set A, and every prize in A is at least as good as lottery q, then p is a good as q
- This additional property (almost) restores the vNM theorem in any choice set

Example (Kahnemann and Tversky 1979)

(orignal idea due to Allais, 1953) There are two choice scenarios:

- 1. Choice between lotteries
    1. $0.33 \cdot 2500 + 0.66 \cdot 2400 + 0.01 \cdot 0$
    2. $1 \cdot 2400$

  2. Choice between lotteries
    1. $0.33 \cdot 2500 + 0.67 \cdot 0$
    2. $0.34 \cdot 2400 + 0.66 \cdot 0$
- Of the subjects, 82% chose 1a and 83% chose 2a which means that at least 65% chose both 1b and 2a

Example (cont.)

- However, there is no utility function u that is consistent with the choices

$$0.33 \cdot u(2500) + 0.66 \cdot u(2400) + 0.01 \cdot u(0) < u(2400)$$

and

$$0.33 \cdot u(2500) + 0.67 \cdot u(0) > 0.34 \cdot u(2400) + 0.66 \cdot u(0)$$

i.e.

$$0.66 \cdot (u(2400) - u(0)) < 0.66 \cdot (u(2400) - u(0))$$

Application: Risk Aversion

- Assume that $X = \mathbb{R}_+$, and interpret x as "money"
- Let vNM preferences $\succsim$ over simple lotteries L (with finite support, for simplicity) be represented by a utility function $u : \mathbb{R}_+ \to \mathbb{R}$
- What are plausible assumptions concerning u?
- More money is strictly better: $u(x) > u(y)$ if $x > y$

- Expected value of lottery p is

$$e(p) = \sum p(x)x \in \mathbb{R}_+$$

- Preferences $\succsim$ exhibit risk aversion if $u(e(p)) \geq \sum p(x)u(x)$ for all simple lotteries p, and he strict risk aversion if the inquality is strict for all simple lotteries p

Theorem

$\succsim$ exhibit (strict) risk aversion if the associated u is (strictly) concave

Proof.

By Jensen's inequality $\square$

- A certainty equivalent outcome $c(p) \in \mathbb{R}_+$ of a lottery p is defined by the condition

$$u(c(p)) = \sum p(x)u(x)$$

- If u is strictly increasing and concave, then $c(p)$ is well defined (exists, is unique)
- The risk premioum associated to lottery p is then defined by $R(p) = e(p) - c(p)$

- Preferences $\succsim$ are decreasingly (increasinly, constantly) absolute risk averse if $R(p+1_x)$ is nondecreasing (nonincreasing, constant) in x
- If $\succsim$ are constantly risk averse, then the DM's wealth level does not affect his risk behavior (why?)

Theorem

The DM is decreasingly (increasinly, constantly) absolute risk averse if the Arrow-Pratt measure $-u''(x)/u'(x)$ of risk aversion is nondecreasing (nonincreasing, constant) in x

- The family of utility functions exhibiting constant absolute (strict) risk aversion (CARA) is given by the following conditions: there is a constant λ such that $\lambda = -u''(x)/u'(x)$ for all x, and there are $a > 0$ and b such that for all

$$u(x) = -ae^{-\lambda x} + b$$

Example (Rabin's paradox)

Assume weak risk aversion, e.g. DARA with

$$\frac{1}{2}u(x-10) + \frac{1}{2}u(x+11) \leq u(x), \text{ for all } x \in \mathbb{R}_+$$

i.e. gamble of losing 10€ and winning 11€ with equal probability is weakly rejected at all wealth levels

- Often observed in calibration excercises

Example

- But then

$$u(x-10)-u(x) \leq u(x+11)-u(x)$$

or

$$\frac{u(x-10)-u(x)}{10} \leq \frac{10}{11} \cdot \frac{u(x+11)-u(x)}{11}$$

- Thus the marginal utiloity u' drops by at least $10/11$ in every 21€

Example

- By extrapolation, adding 1000€ to the DM's wealth would mean that the DM's marginal value of a € drops by proportion

$$\left(\frac{10}{11}\right)^{1000/21} = 0.012$$

Moreover, there is *no* compensating prize M that would render acceptable a gamble where one loses 100€ with probability $1/2$ and wins prize M with probability $1/2$!

$$\sum_{t=1}^{5}\left(\frac{10}{11}\right)^{t} \leq \sum_{t=6}^{T}\left(\frac{10}{11}\right)^{t}, \quad \text{for all } T$$

- The importance of reference dependence

De Finetti and the Dutch books

- In gambling, a Dutch book is a set of odds and bets which guarantees a profit, regardless of the outcome of the gamble
- Consider the situation where the DM chooses a portfolio or a gamble whose value depends on the realized state s in some finite set S
- That is, the gamble is an element x in $\mathbb{R}^S$ (denote a generic coordinate by s)
- Which gamble should the DM choose?

- Let $\succsim \subset \mathbb{R}^S \times \mathbb{R}^S$ be a binary relation that represents the DM's preferences

F1 (Weak order) $\succsim$ is complete and transitive

F2 (Continuity) $\{y : y \succ x\}$ and $\{y : x \succ y\}$ are open for all x

- That is, the graph of $\succsim$ is closed

F (Additivity) $x + z \succsim y + z$ if and only if $x \succsim y$, for all x, y, z

- Additivity implies neutrality to risk - the DM's choice between two portfolios x and y is idependent of the underlying, 'old', portfolio

Proof.

To prove the sufficiency, note that, by Additivity, $x \succsim y$ if and only if $x - y \succsim 0$. Let $A = \{y : y \succsim 0\}$ and $B = \{y : 0 \succ y\}$. We show that A and B are convex sets. First note that $x \succsim y$ implies $x \succsim (x+y)/2 \succsim y$ by Additivity and Transitivity. Hence, by repeatedly applying this property, $x \succsim \ell 2^{-k} x + (1 - \ell 2^{-k}) y \succsim y$, for all $\ell, k \in \mathbb{N}$. By Continuity, then, $x \succsim \lambda x + (1-\lambda) y \succsim y$ for all $\lambda \in (0,1)$ (since $(0,1)$ is dense in binary rationals). Thus A and B are convex sets.

Since A and B are convex sets, and $A \cap B = \emptyset$, there are $b \in \mathbb{R}^S \backslash \{0\}$ and $c \in \mathbb{R}$ such that $x \in A$ iff

$$\sum_{s \in S} b(s) x(s) \geq c.$$

Since $0 \in A$, $c \leq 0$. By Monotonicity, $-\varepsilon \in B$ for all $\varepsilon > 0$ which implies that $c \nless 0$. $\square$

Proof.

[(cont.)] Thus $x \succsim y$ if

$$\sum_{s\in S} b(s)(x(s) - y(s)) \geq 0.$$

Then p such that

$$p(s) = \frac{b(s)}{\sum_{s'\in S} b(s')}, \quad \text{for all } s \in S,$$

is the desired probability vector. $\square$

- If $x \succsim y$, then the DM would be willing sell y for a lower price than x, and buy x for a higher price than y
- A Dutch book is a collection of pairs of portfolios $(x^1, y^1), ..., (x^m, y^m) \in \mathbb{R}^S$ such that (i) $x^i \succsim y^i$ for all $i = 1, ..., m$, and such that (ii) $\sum_{i=1}^m x^i(s) < \sum_{i=1}^m y^i(s)$, for all $s \in S$
- That is, a Dutch book $(y^1, ..., y^m)$ can be traded against $(x^1, ..., x^m)$ with the DM that (i) would not require extra funding for the trader and (ii) generates profit with certainty in the future

Corollary

If $\succsim$ satisfies F1-F4, then a Dutch book does not exist

Proof.

Suppose that $x^i \succsim y^i$ for all $i = 1, ..., m$. By the theorem, there is p such that

$$\sum_{s \in S} p(s) x^i(s) \geq \sum_{s \in S} p(s) y^i(s), \text{ for all } i.$$

Thus

$$\sum_{i=1}^{m} \sum_{s \in S} p(s) x^i(s) \geq \sum_{i=1}^{m} \sum_{s \in S} p(s) y^i(s)$$

or

$$\sum_{s \in S} p(s) \left(\sum_{i=1}^{m} x^i(s) - \sum_{i=1}^{m} y^i(s) \right) \geq 0,$$

which violates part (ii) of the definition of a Dutch book. $\square$