Mechanism Design view to the Nash Program

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Introduction

- A group of people often have to choose collectively an outcome in a situation in which unanimity about the the best outcome is lacking
- Need to negotiate under conflicting interests
- How we as outside observers should see the situation?
  - How is the outcome determined?
  - Is the outcome objectively good?
  - How do the extrenal factors affect the outcome?
  - Where does the bargaining power come from?
  - How will the number of participants affect the outcome?

- Two leading approaches, both inititated by Nash 1953
  - Cooperative: evaluate the outcome directly in terms of the conditions, "axioms", that a plausible outcome will satisfy
  - Non-cooperative: apply non-cooperative game theory to analyse strategic behavior, and to predict the resulting outcome
- An advantage of the strategic approach is that it is able to model how specific details of the interaction may affect the final outcome
- A limitation, however, is that the predictions may be highly sensitive to those details

Nash Program

- Nash (1953): use cooperative approach to obtain a solution via normative or axiomatic reasoning, and justify this solution by demonstrating that it results in an equilibrium play of a non-cooperative game
- Thus the relevance of a cooperative solution is enhanced if one arrives at it from very different points of view
- Similar to the microfoundations of macroeconomics, which aim to bring closer the two branches of economic theory, the Nash program is an attempt to bridge the gap between the two counterparts of game theory (axiomatic and strategic)
- Aumann (1997): The purpose of science is to uncover "relationships" between seemingly unrelated concepts or approaches

Axiomatic approach

- Let there be n players, a joint utility profile from a utility set $U \subseteq \mathbb{R}^n_{++}$
- U comprehensive, compact, and convex
- Collection $\mathcal{U}$ of all utility sets
- Solution $f : \mathcal{U} \to \mathbb{R}^n_{++}$ such that $f(U) \in U$

- Denote the (weak) Pareto frontier by $P(U) = \{u \in U : u' \geq u \text{ implies } u' \notin U\}$

Pareto optimality (PO): $f(U) \in P(U)$, for all $U \in \mathcal{U}$

- Use the notation $aU = \{(a_1 u_1, \ldots, a_n u_n) : (u_1, \ldots, u_n) \in U\}$, for $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$

Scale Invariance (SI): $f(aU) = af(U)$, for all $a \in \mathbb{R}^n_{++}$, for all $U \in \mathcal{U}$.

Independence of Irrelevant Alternatives (IIA): $f(U') \in U$ and $U \subseteq U'$ imply $f(U') = f(U)$, for all $U, V \in \mathcal{U}$.

- Thus if pair $f(U)$ is "collectively optimal" in U, and feasible in a smaller domain, then it should be optimal in the smaller domain, too
- Let π be a permutation $\pi : \{1, \ldots, n\} \to \{1, \ldots, n\}$

Symmetry (SYM): If $\pi(U) = U$, for any permutation π, then $f_i(U) = f_j(U)$ for all i, j

Theorem

A bargaining solution f satisfies PO, SI, IIA; and SYM on $\mathcal{U}$ if and only if f is the Nash bargaining solution:

$$f(U) = \arg\max_{u \in U} u_1 \cdots u_n$$

Strategic approach

- There is a set $1, \ldots, n$ of agents, distributing a pie of size 1
- Time preferences of i has the representation $u_i(x_i)\delta^t$, where x_i is i's consumption at period t, and u_i is increasing, concave, and continuously differentiable and $\delta \in (0, 1)$
- Unanimity bargaining game Γ: At any stage $t = 0, 1, 2, \ldots,$
  - Player $i(t) \in N$ makes an offer $x \in S$, and players $j \neq i(t)$ accept or reject the offer in the ascending order
  - If all $j \neq i(t)$ accept, then x is implemented, if j is the first who rejects, then j becomes $i(t+1)$
  - $i(0) = i$

- Focus on the *stationary subgame perfect equilibria* where:

1. Each $i \in N$ makes the same proposal $x(i)$ whenever he proposes.
2. Each i's acceptance decision in period t depends only on x_i that is offered to him in that period.

- Define a function v_i such that

$$u_i(v_i(x_i)) = u_i(x_i)\delta, \quad \text{for all } x_i$$

- By construction, $v_i(x_i) < x_i$, for all x_i
- By the concavity of u_i, $u_i'(x_i)/u_i(x_i)$ is decreasing, strictly positive under all $x_i > 0$, and hence

$$v_i'(x_i) \in (0, 1)$$

Lemma

There is unique $x \in \mathbb{R}^n_{++}$ and $d > 0$ such that

$$\begin{aligned} x_i &= v_i(x_i + d), \quad \textit{for all } i \\ \sum_{j=1}^{n} x_j &= 1 - d \end{aligned}$$

- Let
$$c_i(x_i) := v_i^{-1}(x_i) - x_i, \quad \text{for all } x_i$$
- $c_i(\cdot)$ continuous and monotonically increasing and hence there is $c_i^* \in (0, \infty]$ such that
$$\sup_{x_i \geq 0} c_i(x_i) = c_i^*$$

- Since $c_i(\cdot)$ is continuous and monotonically increasing, also its inverse

$$x_i(y) := c_i^{-1}(y) = v_i(x_i(y) + y), \qquad \text{for all } y \in [0, c_i^*],$$

  is continuous and monotonically increasing
- Moreover, since $0 = x_i(0)$ and $\infty = x_i(c_i^*)$, there is, by the Intermediate Value Theorem, a unique $d > 0$ such that

$$\sum_{i=1}^{n} x_i(d) = 1$$

Theorem

A stationary equilibrium of Γ exists. Moreover, it is unique

- In a stationary SPE all proposals are accepted
- Time does not matter: i's offer $(x_1(i), ..., x_n(i))$ is accepted by j if
$$x_j(i) \geq v_j(x_j(j)), \quad \text{for all } j \neq i$$
- Player i's equilibrium offer $x(i)$ maximizes his payoff with respect to this and the resource constraint

- At the optimum, all constraints bind:

$$x_j(i) = v_j(x_j(j)), \quad \text{for all } j \neq i,$$

and

$$\sum_{i=1}^{n} x_i(j) = 1, \quad \text{for all } j$$

- Since i's acceptance not dependent on the name of the proposer, there is $x_i = x_i(j)$ for all $j \neq i$

- Define d such that

$$d = 1 - \sum_{i=1}^{n} x_i$$

- Since

$$x_i(i) = 1 - \sum_{j \neq i} x_j = x_i + d$$

it follows that

$$\begin{aligned} x_i &= v_i(x_i + d), \quad \text{for all } i \\ \sum_{j=1}^{n} x_j &= 1 - d \end{aligned}$$

Uniqueness

- Stationarity needed for the result when $n \geq 3$
  - For high enough δ, any allocation x can be supportted in SPE
  - Any deviant player is punished rejecting his offer
  - Rejection rewarded by giving the whole pie to to the rejecting player
- Rubinstein (1982): in $n = 2$ case, stationarity not needed

- Krishna-Serrano (1996)
  - Allow accepting players leave the game with their endowment
  - Solution must be consistent (Lensberg 1983): the equilibrium outcome for $1, ..., k$ remains unchanged when $k+1, ..., n$ leave with their equilibrium shares
  - -> Since stationary not needed for 2-player problems, by consistency, it is not needed for 3-player problems, etc.
- Stationarity can be motivated by complexity considerations (Chatterjee-Sabourian 2000)

Relationship to the Nash solution

- For any i,

$$u_i(x_i + d) \prod_{i \neq j} u_j(x_j) = \delta^{-1} \prod u_i(x_i)$$

- Thus, depending on the initial proposer i, all stationary SPE outcomes lie in the same hyperbola
- Binmore-Rubinstein-Wolinsky (1986): Since $d \longrightarrow 0$ as $\delta \longrightarrow 1$:

Theorem

The stationary SPE outcome of Γ converges to the Nash solution as $\delta \longrightarrow 1$

General utility set

- Let bargaining take place in a comprehensive utility set $U \subseteq \mathbb{R}^n_{++}$
- Let boundary of U be smooth
- Britz-Herings-Predtetchinski (2010), Kultti-Vartiainen (2010): stationary SPE converge to the (asymmetric) Nash solution

Time preferences

- Utility functions represent preferences
- But what kind of preferences do the intertemporal utilities represent? Does the Nash solution have an interpretation in terms of them?
- Let pie be divided at any point of time $T = \mathbb{R}_+$ and denote by $X = \{(x_1, x_2) \in \mathbb{R}_+ : x_1 + x_2 \leq 1\}$ the possible allocations of the pie
- Let (complete, transitive) preferences over $X \times T$ satisfy, for all $x, y \in S$, for all $i \in N$, and for all $s, t \in T$, satisfy (Fishburn and Rubinstein, 1982):

A1. $(x, t) \succeq_i (\mathbf{0}, 0)$

A2. $(x, t) \succeq_i (y, t)$ if and only if $x_i \geq y_i$

A3. If $s > t$, then $(x, t) \succeq_i (x, s)$, with strict preference if $x_i > 0$

A4. If $(x^k, t^k) \succeq_i (y^k, s^k)$ for all $k = 1, ...,$ with limits $(x^k, t^k) \to (x, t)$ and $(y^k, s^k) \to (y, s)$, then $(x, t) \succeq_i (y, s)$

A5. $(x, t) \succeq_i (y, t + \Delta)$ if and only if $(x, 0) \succeq_i (y, \Delta)$, for any $t \in T$, for any $\Delta \geq 0$

- Under A1-A5, there is function v_i such that $(y, 0) \sim_i (x, t)$ if $v_i(x_i, t) = y_i$, for all x, y
- For any $\delta \in (0, 1)$ there is also u_i such that $(y, 0) \sim_i (x, t)$ if and only if $u_i(y_i) = u_i(x_i)\delta^t$

A6. $x_i - v_i(x_i, t)$ is strictly increasing in x_i

- Alternative interpretation of the Nash solution (cf. Rubinstein-Safra-Thomson 1992): x is the Nash solution if for any y and for any $t > 0$ it holds true that $v_i(y_i, t) > x_i$ implies $v_j(x_j, t) \geq y_j$
- Under A1-A6, the Nash solution does exist and is equivalent with there being a maximizer of the product $u_1(x_1)u_2(x_2)$, where (u_1, u_2) is the representation of the intertemporal preferences
- Note that this does not require that u_i concave -> A1-A6 weaker set of assumptions than the concavity of u_is, usually assumed in the literature
- Convergence without additional assumptions concerning the utility representation

Exact implementation

- But the convergence result approximate: only holds when $\delta \to 1$ (or the time span betnween offers vanishes)
- Exact implementation of the Nash solution: Howard (1992)
- Implementing the other solutions
    - Kalai-Smorodinsky: Moulin (1984)
    - Shapley: Gul (1989), Perez-Castrillo-Wettstein (2001)
    - The Core: Serrano-Vohra (1997), Lagunoff (1994)
    - Bargaining set: Einy-Wettstein (1999)
    - Nucleoulus: Serrano (1993)
    - etc...
- Do strategic considerations put any restrictions on what can be implemented?

Implementation foundations

- The most natural notion of strategic interaction: the Nash equilibrium
- Which solutions can be implemented in Nash equilibrium?
- Implementation theory: studies general conditions under which an outcome functions - e.g. a bargaining solution - can be implemented non-cooperatively

Nash implementation - impossibility

- Let $n = 2$
- There is a pie of size 1, to be shared among the two players with $x \in [0, 1]$ denoting a typical share of player 1, and $1 - x$ the share of player 2
- U comprises all continuous and strictly increasing vNM utility functions $u_i : [0, 1] \to \mathbb{R}$ normalized such that $u_i(0) = 0$ for all $u_i \in U$
- Denote the set of lotteries on $[0, 1]$ by Δ
- Expected payoff from a lottery $p \in \Delta$

$$
\begin{aligned}
u_1(p) &= \int_{[0,1]} p(x) u_1(x) dx \\
u_2(p) &= \int_{[0,1]} p(x) u_2(1 - x) dx
\end{aligned}
$$

- Bargaining solution (BS) $f : U^2 \longrightarrow [0,1]$ specifies an outcome for each pair of utility functions where $f(u)$ is the share of player 1 and $1 - f(u)$ the share of 2 under profile $u = (u_1, u_2)$
- A *game form* $\Gamma = (M_1 \times M_2, g)$ consists of *strategy sets* M_1 and M_2, and an *outcome function* $g : M_1 \times M_2 \longrightarrow \Delta$
- Given $u = (u_1, u_2)$, the pair (Γ, u) consititues a *normal form game* with the set of Nash equilibria $NE(\Gamma, u)$
- Mechanism Γ *Nash implements* bargaining solution f if, for all $u \in U^2$,

$$g(NE(\Gamma, u)) = f(u)$$

- Denote the *lower contour set* of i at $q \in \Delta$ under $u \in U^2$ by

$$L_i(q, u) = \{p \in \Delta : u_i(q) \geq u_i(p)\},$$

- BS f is *Maskin monotonic* if for all pairs u, u', if $x \in f(u')$ and $L_i(x, u') \subseteq L_i(x, u)$, for $i = 1, 2$, then $x \in f(u)$
- Maskin (1999, first version 1977): f Nash implementable only if it is Maskin monotonic
- Which bargaining solutions are Maskin monotonic?

- Maskin monotonicity implies that there has to be a preference reversal from u to u' if $x \in f(u) \backslash f(u')$
- BS f is *scale invariant* if $f(u) = f(\alpha u)$, for all $\alpha \in \mathbb{R}^2_{++}$, for all $u \in U^2$

Lemma

Any Maskin monotonic BS f is scale invariant

- Thus BS f Nash implementable only if it scale invariant

- BS f is *symmetric* if $u_1(f(u_1, u_2)) = u_2(1 - f(u_1, u_2))$ whenever $(w_1, w_2) = u(x)$ for some x implies that there is x' such that $(w_2, w_1) = u(x')$
- Note that, as we require that no pie is wasted, our BS f is automatically Pareto optimal: $f_1(u) + f_2(u) = 1$ for all $u \in U^2$
- Nash bargaining solution

$$f^N(u) = \arg\max_{[0,1]} u_1(x)u_1(1 - x)$$

Lemma

Let f be a (Pareto optimal and) symmetric BS. If f can be Nash implemented, then $f^N(u) = f(u)$ for all $u \in U^2$.

- Proof: Given that f must be scale invariant, replace IIA with Maskin monotonicity in the proof of Nash's theorem (see Vartiainen 2007 for details)
- Can the Nash bargaining solution be Nash implemented?

- Let $u_1(x) = x$ and $u_2(1-x) = 1-x$
  - Then $f^N(u) = 1/2$
  - Perform a Maskin monotonic transformation of 1's utility by choosing $u_1^\varepsilon(x) = x$ for $x \in [0, 1/3]$, and $u_1^\varepsilon(x) = 1/3 + \varepsilon(x - 1/3)$ for $x \in (1/3, 1]$
  - For small enough $\varepsilon > 0$, $f^N(u_1^\varepsilon, u_2) = 1/3$

Lemma

The Nash bargaining solution f^N cannot be Nash implemented

Theorem

No Pareto optimal and symmetric BS f can be Nash implemented

Virtual implementation - possibility

- We construct a canonical mechanism (cf. Moore-Repullo 1988; Dutta-Sen1988)) that Nash implements any strictly individually rational BS
- Let $\Gamma^* = (M^*, g^*)$ satisfy $M_1^* = M_2^* = U^2 \times \Delta \times \mathbb{N}$ with typical elements (u^1, q^1, k^1) and (u^2, q^2, k^2), respectively, and
  1. $g^*(m_1, m_2) = f(u)$ if $u^1 = u^2 = u$
  2. $g^*(m_1, m_2) = q^i$ if $q^i \in L_i(f(u^j), u^j)$, $u^1 \neq u^2$, and $k^i > k^j$
  3. $g^*(m_1, m_2) = 1$ if $k^1 > k^2$ and $g^*(m_1, m_2) = 0$ if $k^2 > k^1$
  4. $g^*(m_1, m_2) = (0, 0)$, in all other cases

- We claim that Γ^* Nash implements any Maskin monotonic BS
- Let $u = (u_1, u_2)$ be the true utility profile
  - It cannot be the case that (1) holds under $u^1 = u^2 = u' \neq u$ and $k^1 = k^2 = 0$ since, by (2), there would be $q^i \in L_i(f(u'), u') \backslash L_i(f(u), u)$ such that $k^i > 0$ that would consistute a profitable deviation for i
  - It cannot be the case that (2) holds since, by (3), $k^j > k^i$ would consititute a profitable deviation for j
  - It cannot be the case that (3) holds since one of the players would have a profitable deviation
  - It cannot be the case that (4) holds since, as a strictly individually rational BS chooses $f(u) \in (0, 1)$ and, hence, by (1) i would have a profitable deviation $u^i = u^j$

- Thus the only possible Nash equilibrium is $u^1 = u^2 = u$ which implements $f(u)$

Lemma

Any strictly individually rational BS f can be Nash implemented if it is Maskin monotonic

- Take any strictly individually rational f and let f^{ε} satisfy

$$
\begin{aligned}
f^{\varepsilon}(u) &= \text{implement } f(u) \text{ with probability } 1-\varepsilon \\
&\quad \text{and the uniform lottery over } [0,1] \text{ with prob. } \varepsilon
\end{aligned}
$$

- The expected payoff to i

$$
\begin{aligned}
u_1(f^{\varepsilon}(u)) &= (1-\varepsilon)u_1(f(u)) + \varepsilon \int_{[0,1]} u_1(x)dx \\
u_2(f^{\varepsilon}(u)) &= (1-\varepsilon)u_2(f(u)) + \varepsilon \int_{[0,1]} u_2(1-x)dx
\end{aligned}
$$

- We argue that f^{ε} is Maskin monotonic for any $\varepsilon > 0$

- Since any implementable BS is scale invariant, we may normalize the situation such that $u_i(0) = 0$ and $u_i(1) = 1$
- Take $u_1 \neq u_1'$ and find an open interval $(a, b) \subseteq [0, 1]$ such that $u_1(x) > u_1'(x)$ for all $x \in (a, b)$, or $u_1(x) < u_1'(x)$ for all $x \in (a, b)$
- Assume, for simplicity, that $a = 0$ and $b = 1$ (otherwise, modify f^{ε} only under (a, b) and not under $(0, 1)$)
- We shall show that $L_1(f^{\varepsilon}(u), u) \backslash L_1(f^{\varepsilon}(u), u_1', u_2)$ is not empty, implying that f^{ε} automatically satisfies Maskin monotonicity
- There are two cases to consider

Case $u_1 > u_1'$: Find $\xi \in (0,1)$ such that

$$\int_{[0,1]} u_1(x)dx = u_1(\xi)$$

- Modify f^ε by constructing a lottery

$$q_\xi = \text{implement } f(u) \text{ with prob. } 1-\varepsilon \text{ and } \xi \text{ with prob. } \varepsilon$$

- By construction $q_\xi \in L_1(f^\varepsilon(u), u)$ and $q_\xi \notin L_1(f^\varepsilon(u), u_1', u_2)$

Case $u_1 < u_1'$: Find $\pi \in (0,1)$ such that

$$\int_{[0,1]} u_1(x)dx = \pi$$

- Construct a lottery

$$q_\pi = \text{implement } f(u) \text{ with prob. } 1-\varepsilon \text{ and } 1 \text{ with prob. } \pi\varepsilon\}$$

- By construction $q_\pi \in L_1(f^\varepsilon(u), u)$ and $q_\pi \notin L_1(f^\varepsilon(u), u_1', u_2)$

- Thus in either case, f^{ε} satisfies Maskin monotoncity
- Since $\varepsilon > 0$ is arbitrariliy small, any strictly individually rational BS can be virtually implemented - with arbitrary precision
- Problems:
  - Optimally small deviation from exact implementation?
  - Machanism uses an integer construction, and is hence "unreasonable"

Exact implementation with reasonable mechanism

- Miyagawa (2002): simple mechanism that implements a large class of solutions
- Define a solution f^W by

$$f^W(u) = \arg\max_{x \in X} W(u_1(x), u_2(x))$$

  where $W : [0,1]^2 \longrightarrow \mathbb{R}$ is continuous, monotonic and quasi-concave
- The set of functions W satisfying these conditions is denoted by $\mathcal{W}$
- The function W may be interpreted as the objective function of the arbitrator
- E.g. Nash, Kalai-Smorodinsky

- Mechanism Γ^W

1. In stage 1, agent 1 announces a vector $p \in [0,1]^2$ such that $p_1 + p_2 \geq 1$
2. Having observed p, agent 2 makes a counter-proposal $p' \in [0,1]^2$ such that $W(p_1, p_2) = W(p'_1, p'_2)$
3. The agent who moves in the next stage, i, is then determined based on whether 2 agrees $(p = p')$ or disagrees $(p \neq p')$
   - If 2 agrees, then he moves next $(i = 2)$
   - Otherwise, 1 moves next $(i = 1)$
4. Agent i then chooses either "quit" or "stay," and then announces a lottery a_i
   - If he chooses to "quit," then the game ends with $p' a_i$ as the outcome
   - If agent i chooses to "stay," then agent $j \neq i$ either "accepts" a_i, in which case the outcome is a_i, or he selects another lottery a'_j in which case the outcome is $p'_j a'_j$

Theorem

For each $W \in \mathcal{W}$, game form Γ^W implements solution f^W in subgame-perfect equilibrium.

- Thus any reasonable solution can be implemented
- The true test is not whether a solution is consistent with rational play, but whether its implementation can be justified with a intuitively appealing (= simple, used in the real world,...) mechanism
- But then the question of finding a good solution is changed to one finding a good mechanism - do the problems really differ