Lecture notes 4: Equilibrium analysis

- Markets are the place where exchange of goods takes place
- The objective of a market to enhance transfer of goods between parties in a welfare enhancing way
- Coordination problem: what should be transferred from who to whom?
- Requires much information concerning preferences, technologies etc.
- Central coordinating device: the **price mechanism** which prices communicates (in certain sense) how much parties benefit from consumption of a good and how much they have to sacrifice resources in order to produce one
- The way to solve the outcome of the market is called **the equilibrium**
- Adam Smith's "invicible hand"

Detour: Partial equilibrium (in a single market)

- Consider one market and keep the interaction in other markets as given
- Let there by (for simplicity) "many" identical consumers and identical sellers
- In a market, the firm produces good y and trades it with the consumer against m at the rate p (price of the good)
- Assume that the buyer's utility function is of the quasilinear form $u(x) + m$ where $m \in \mathbb{R}_+$ respresnts the consumption of the other goods (m="money"), and u is a continuous, concave, differentiable function
- Assume also that $\lim_{x\to\infty} u'(x) = 0$, i.e. the marginal utility from consumption tends to 0

- Normalizing the price of m equal to 1, and assuming that the budget constraint of the consumer does not bind, the consumer's problem reduces to one of maximizing $u(x) - px$ with respect to x
- The (Marshallian) demand $x(p)$ is then determined by

$$u'(x(p)) = p$$

- Since u is concave, $x(\cdot)$ is continuosly decreasing in p $=>$ **demand function**

- Denoting by $c(y)$ the continuous, convex cost function of the firm, and by $py - c(y)$ its profit function, the optimal supply $y(p)$ is given by condition

$$c'(y(p)) = p$$

- Assume $c'(0) = 0$, i.e. the cost of producing first marginal unit is 0
- Since c is convex, $y(\cdot)$ is continuosly increasing in $p =>$ **supply function**

- Since $x(\cdot)$ is continuously decreasing and $y(\cdot)$ continuously increasing, the **excess demand** $x(p) - y(p)$ is a continuously decreasing function in p
- Since $c'(0) = 0$ and $\lim_{x\to\infty} u'(x) = 0$, also $x(0) - y(0) > 0$ and $\lim_{p\to\infty} x(p) - y(p) < 0$
- Since te excess demand function $x(\cdot) - y(\cdot)$ is continuous, by the Intermediate Value Theorem there is p^* such that

$$x(p^*) - y(p^*) = 0,$$

i.e. demand equals supply

Conclusion

An **equilibrium price** p^* *such that* $x(p^*) = y(p^*)$ *does exists*

Welfare properties of equilibrium

- Given the quasilinear utility specification, one way to measure the welfare gains from production y is by the function

$$u(y) - c(y)$$

- This function is maximized at x^* such that

$$u'(x^*) - c'(x^*) = 0$$

- Given that, in equilibrium,

$$u'(x(p^*)) = p^* = c'(y(p^*))$$

the equilirium also maximizes the social welfare

Conclusion

All gains from trade are exhausted when the firm produces $y(p^*)$ and trades it with the consumer to the amount $x(p^*)$ of the other good m

- We now argue that the (partial) market equilibrium price p^* and equilibrium trade $x(p^*) = y(p^*)$ maximize the sum of consumer's and producer's joint surplus and is also **Pareto-efficient**: no other allocation makes **both** parties better off
- Suppose that there is an allocation z and a compensation m such that

$$\begin{aligned} u(z) - m &\geq u(x(p^*)) - p^* x(p^*) \\ m - c(z) &\geq p^* x(p^*) - c(x(p^*)) \end{aligned}$$

with at least one inequality

- But then

$$u(z) - c(z) > u(x(p^*)) - c(x(p^*))$$

which contradicts the observation that the equilibrium maximizes the social welfare

- Quasilinear utility specification permits us to reflect with precision the generated surplus at equilibrium
- The welfare at level of production x and price p can be decomposed into **consumer's surplus** $u(x) - px$ and **producer's surplus** $px - c(x)$
- By the Fundamental Theorem of Calculus,

$$
\begin{aligned}
u(x) - u(0) &= \int_0^x u'(x)dx \\
c(y) - c(0) &= \int_0^y c'(y)dy
\end{aligned}
$$

- Since $u(0) = 0$, consumer's surplus at price p is

$$u(x(p)) - px(p) = \int_0^{x(p)} [u'(x) - p]dx$$

Similarly, since $c(0) = 0$, producer's surplus at price p is

$$py(p) - c(y(p)) = \int_0^{y(p)} [p - c'(y)]dy$$

- Thus the overall surplus is reflected by the area between the demand and supply curves, the consumer surplus the area between demand curve and equilibrium price p^*, and consumer surplus the area between supply curve and equilibrium price p^*

Signifigance of price taking

- It is crucial that bith the firm and the consumer are price takers
- Suppose, on the contrary, that the firm sets the price, i.e. has **monopoly** power
- The firm's objective optimization problem is of the form

$$\max_{p \geq 0} px(p) - c(x(p))$$

- Profit from from a marginal price increase is

$$\frac{\partial[px(p) - c(x(p))]}{\partial p} = x(p) + x'(p)p - x'(p)c'(x(p))$$

- Thus, the first order condition for the optimal solution p^{**} is

$$p^{**} + \frac{x(p^{**})}{x'(p^{**})} = c'(x(p^{**}))$$

- Since $x' < 0$, it follows that the left hand side is strictly positive, or

$$p^{**} > c'(x(p^{**}))$$

- Since x is a monotonic and c is convex, this can only be true if $p^{**} > p^*$ and $x(p^{**}) < x(p^*)$

Conclusion

A monopoly produces less than a price taking firm in an equilibrium

Conclusion

The output under monopoly is not socially optimal under monopoly

- The **deadweight loss** associated to the monopoly is

$$\int_{x(p^{**})}^{x(p^*)} [u'(y) - c'(y)]dy$$

- Interpretation: being a price setter, the monopoly will be able to extract the surplus of the consumer but it is constrained by the law of one price; the optimal solution is not exhaust all the welfare gains

- But now we have kept the other markets as given by assuming quaslinear utility function and that the budget constraint of the consumer is not violated by the optimal consuption
- In general this does not hold but rather the consumption in market affects the demand for the goods in other margets, via the budgets constraint of the consumers (substitution and income effects)
- Hence the markets are **interconnected**
- Are there any grounds to extend the above obervations to the multimarket situation?
  $=>$ general equilibrium in which all markets are simultaneously in partial equilibrium

General Equilibrium in Exchange Economies

- In this lecture we consider transactions between individuals pursuing their own self interests in perfectly competitive markets
- We consider **exchange economies**, i.e., there are no producers but only consumers who transact their endowments
- Exchange economies display the key features of the **Walrasian equilibrium**, extension that takes production into account is conceptually equivalent

Exchange Economies

- Construct an economy from a number of consumers:
  - Consumers maximize their utilities at given prices
  - Prices determine consumer's budget set as they determine the cost of consumption and also the value of consumer's endowment $=>$ also income depends on prices
- In general equilibrium analysis:
  - Behavioral assumptions: individual optimization and price taking
  - Equilibrium concept: market clearing simultaneously in **each** market
- Endogenous variables: vectors of consumption and prices
- In equilibrium: prices balance supply and demand

- Exchange economy formally:
  - Consumers $h \in \{1, ..., H\}$
  - Commodities or distinct markets $\ell \in \{1, ..., L\}$
  - Consumer h's utility function: $u_h(x^h) : \mathbb{R}^L_+ \rightarrow \mathbb{R}$, representing h's monotonic, continuous and convex preferences
  - Consumer h's initial endowment $\omega^h \in \mathbb{R}^L_+$
- An exchange economy is completely specified by the list $(u_h, \omega^h)_{h=1,...,H}$

- Recall that the Marshallian demand depends on the income of the agetns as well as the prices of the goods
- Now the income of a consumer depends on the value of her initial endowemen
- Hence, the price in market ℓ affects, via the budet set of the consumers, to the demand in the market of good k
- Hence demand in one market affects the demand in other markets

Problem

Is there market price configuration under which all the markets are in equilibrium at the same time?

Problem

If such an equilibrium existrs, what are its effiency properties?

Problem

If such an equilirium exists, how should we get into that?

- **The Edgeworth box**
  We consider first basic concepts in the two-consumer case
- The Edgeworth box is a useful didactic device

- Back to the general case
- Given prices $p \in \mathbb{R}^L_+$ and initial endowement ω^h, consumer h's consumable income is $p \cdot \omega^h \in \mathbb{R}_+$
- Consumer's **optimization problem** is of the form

$$\max_{x^h} u_h(x^h)$$
$$\text{s.t.} p \cdot x^h \leq p \cdot \omega^h$$

- Let $x^h(p) \in \mathbb{R}^L_+$ be the optimal consumption of h at p, i.e. her Marshallian **demand** at p
- Note that, as consumer's income is determined by p, the demand depends only on p

- Denote **net demand** of a consumer by

$$z^h(p) = x^h(p) - \omega^h$$

  - Determines if the consumer is a net seller or buyer of each good $\ell = 1, ..., L$

- The **budget constraint** can be written as

$$p \cdot z^h(p) \leq 0$$

- Optimal consumption of h implies **Walras' law**

$$p \cdot z^h(p) = 0$$

- Summing over individuals we get the **aggregate net demand**

$$z(p) = \sum_h z^h(p)$$

where

$$z_\ell(p) = \sum_h z_\ell^h(p)$$

is the aggregate net demand, or excess demand, of good $\ell = 0, ..., L$

- Thus, the aggregate version of Walras' law is

$$p \cdot z(p) = 0$$

i.e. the value of the aggregate net demand is zero

- Implied by optimal consumption

- Markets clear under prices p if the demand equals supply in market ℓ if demand equals supply in this market

$$\sum_h x_\ell^h(p) = \sum_h \omega_\ell^h$$

- Equivalently, the aggregate net demand is zero in this market

$$z_\ell(p) = 0$$

Definition

A **Walrasian equilibrium** of an exchange economy $(u^1, \omega^1, ..., u^H, \omega^H)$ is a price vector $p^* \in \mathbb{R}^L_+$ and net demands $z(p^*) = (z^1(p^*), ..., z^H(p^*))$ such that

$$z_\ell(p) = 0, \text{ for all markets } \ell$$

- In a Walrasian equilibrium
  - all agents maximize their payoffs under the prices
  - given the demands, all markets clear
- A consistency condition

- Questions:
  - does a Walrasian equilibrium exist?
  - is it unique?
  - is it desirable, i.e. Pareto-optimal?
  - is it stable, i.e. can it be reached?

Existence

- Before stating the existence, we need an important mathematical tool:

Theorem (Brouwer Fixed Point)

Let S be a nonempty, compact, and convex set in $\mathbb{R}^L_+$. If f is a continuous function from S to S, then there is $x \in S$ such that $f(x) = x$, i.e. a **fixed point**.

- In one dimensional problems, reduces to the Intermediate Value Theorem

- Note that since x^h is homogenous of degree zero for all h, also z is
- Thus it is without loss of generality to **normalize** any prices p such that $\sum_{\ell}^{L} p_\ell = 1$ (for any price vector p', dividing each p'_ℓ with $\sum_{k}^{L} p'_k$ to obtain p_ℓ will not affect the demand by the homogeneity of degree 0, and the new price vector satisfies $\sum_{\ell}^{L} p_\ell = 1$)
- Only the **relative prices** matter on consumption, hence it is without loss of generality to restrict prices that belong to the $L-1$ dimensional unit simplex Δ - a nonempty, compact, and convex set in $\mathbb{R}^L_+$

Theorem (Existence: Arrow and Debreu 1954)

If z satisfies Walras' law and the individual demands continuous, then there exists a Walrasian equilibrium price p^* such that $z_\ell(p^*) = 0$ for all markets ℓ

Proof.

Define a function g_ℓ on Δ such that

$$g_\ell(p) = \frac{p_\ell + \max\{0, z_\ell(p)\}}{\sum_k (p_k + \max\{0, z_k(p)\})}. \tag{1}$$

Then function $g = (g_1, ..., g_L)$ is from Δ to itself. By Brouwer's Theorem, there is p^* such that

$$g(p^*) = p^*. \tag{2}$$

□

Proof.

(cont.) We claim that $z_\ell(p^*) = 0$ for all ℓ. By construction, for each ℓ,

$$\begin{aligned} p_\ell^* \sum_k (p_k^* + \max\{0, z_k(p^*)\}) &= p_\ell^* \sum_k p_k^* + p_\ell^* \sum_k \max\{0, z_k(p^*)\} \\ &= p_\ell^* + p_\ell^* \sum_k \max\{0, z_k(p^*)\}. \end{aligned}$$

By (1) and (2), for each ℓ,

$$p_\ell^* \sum_k (p_k^* + \max\{0, z_k(p^*)\}) = p_\ell^* + \max\{0, z_\ell(p^*)\}.$$

□

Proof.

(cont.) Thus

$$p_\ell^* \sum_k \max\{0, z_k(p^*)\} = \max\{0, z_\ell(p^*)\}$$

and, *a fortiori*,

$$z_\ell(p^*) p_\ell^* \sum_k \max\{0, z_k(p^*)\} = z_\ell(p^*) \max\{0, z_\ell(p^*)\}.$$

Summing over all ℓ, and using Walras' law,

$$0 = \sum_\ell z_\ell(p^*) \max\{0, z_\ell(p^*)\}.$$

Unless $z_\ell(p^*) = 0$ for all ℓ, this condition cannot hold. □

- Thus Walrasian equilibrium exists if z is continuous and satisfies Walras' law
- Since our assumptions concering consumer preferences guarantee Walras law' as well as the continuity of the demand function

Corollary

A Walrasian equilibrium exists

- It is worth emphasizing the general nature of this result; all that is needed is that the excess demand is continuous and satisfies Walras' law, the latter arising naturally from all reasonable models of economic decision making
- Uniqueness, however, not implied

Welfare properties

- To analyze the welfare properties of the Walrasian equilibrium, recall the general definition of economic efficiency: A feasible outcome a is **Pareto-efficient** if there is no other feasible outcome a' that all agents weakly prefer over a and at least one agent strictly prefers over a
- Specializing to the current setting

Definition

A consumption vector x is **Pareto-efficient** if there is no y such that $\sum_h y_\ell^h \leq \sum_h \omega_\ell^h$ for all markets ℓ and such that $u_h(y^h) \geq u_h(x^h)$ for all consumers h, with at least one strict inequality

- Many Pareto-efficient allocations
- Is the Walrasian equilibrium Pareto-efficient? (sufficient condition)
- If one has in mind a particular Pareto-efficient allocation that reflects "social desirability", is the price mechanism sufficient to deliver that?

Theorem (First Fundamental Theorem of Welfare Economics)

Every Walrasian equilibrium allocation is Pareto-efficient

Proof.

Let p^* be a Walrasian equilibrium price and x the corresponding equilibrium allocation. Suppose that y Pareto dominates x. If $p \cdot y^h < p \cdot x^h$ for some h, then h could increase consumption of all goods from y^h without violating her budget constraint. Since preferences are monotonic and since $u(y^h) \geq u(x^h)$, this contradicts the assumption that x^h is an optimal choice for h. Thus $p \cdot y^h \geq p \cdot x^h$ for all h with at least one strict inequality (h that strictly prefers y^h to x^h). Summing over h gives

$$p \cdot \sum_{h=1}^{H} y^h > p \cdot \sum_{h=1}^{H} x^h = p \cdot \sum_{h=1}^{H} \omega^h,$$

where the last equality follows from Walras' law. □

Proof.

(cont.) In other words,

$$\sum_{\ell=1}^{\ell} p_\ell \sum_{h=1}^{H} y_\ell^h > \sum_{\ell=1}^{\ell} p_\ell \sum_{h=1}^{H} \omega_\ell^h.$$

But then there must be a particular ℓ such that $p_\ell > 0$ and

$$p_\ell \sum_{h=1}^{H} y_\ell^h > p_\ell \sum_{h=1}^{H} \omega_\ell^h.$$

Thus y is not feasible. □

- A justification for the usefulness of the price mechanism
- Formalizes the "invicible hand"; market forces steer towards efficiency through prices
- The key insight: prices communicate information about other agents tastes and production, and help each agent to coordinate her behavior optimally with that
- As there are no externalities, and market for all goods exist, individual optimization leads to global efficiency
- Basic requirement: **property rights**
- But note that Pareto-efficiency is just a very weak notion of optimality; permits also extremely unequal allocations
- The First Welfare Theorem does not say anything about the equality

Theorem (Second Fundamental Theorem of Welfare Economics)

Let x be a Pareto-efficient allocation. Identify a Walrasian equilibrium (u, ω) emerging from the utility functions u and the initial endowments ω. Let the initial endowment ω be redistributed so that the new initial endowment equals x. Then, in fact, x consitutues a Walrasian equilibrium allocation emerging from (u, x).

Proof.

Let the Walrasian price under (u, x) be p^*, and let the corresponding Walrasian equilibrium allocation be y. Since x^h is in consumer h's budget set under p^*, it must be that $u_h(y^h) \geq u_h(x^h)$ for all h. Since y is feasible and x is Pareto-efficient, necessarily $u_h(y^h) = u_h(x^h)$ for all h. Since y^h is optimal for each h under p^*, also x^h is optimal for each h under p.Then x is a Walrasian equilibrium under p^*. □

- Implication: *All Pareto-efficient allocations are Walrasian equilibrium allocations for some initial endowments*
- To achieve **any** Pareto efficient alloation, one only needs to manipulate the initial allocation and let the price mechanism guarantee the outcome
- Note that existence of competitive equilibrium (supposed in theorem) is assumed in the theorem

- Overall:
  - Marginal rates of substitution for individuals must be equalized at Pareto-optimal allocations
  - At a competitive equilibirium (x, p), interior optimality implies that
$$MRS^h_{\ell k} = \frac{\partial u_h(x)/\partial x_k}{\partial u_h(x)/\partial x_\ell} = \frac{p_k}{p_\ell}$$
    for each individual h and goods ℓ and k.
  - Watch out for corner solutions!

Example

Cobb-Douglas Economy Let $H = L = 2$. Utility functions take the form

$$u^h(x_1, x_2) = x_1^{\alpha_h} x_2^{1-\alpha_h}$$

where $0 < \alpha_h < 1$. Let initial endowments be given by $\omega^1 = (1, 0)$, $\omega^2 = (0, 1)$.

Example

(cont.) At h's optimum,

$$\left(\frac{\alpha_h}{1-\alpha_h}\right) p_2 x_2^h = p_1 x_1^h.$$

By Walras' law,

$$\begin{aligned} p_1 x_1^1 + p_2 x_2^1 &= p_1 \\ p_1 x_1^2 + p_2 x_2^2 &= p_2. \end{aligned}$$

Example

(cont.) Thus

$$\frac{p_1 x_1^1}{\alpha_1} = p_1$$
$$\frac{p_2 x_2^2}{1-\alpha_2} = p_2.$$

Market clearing implies

$$x_1^1 + x_1^2 = 1$$
$$x_2^1 + x_2^2 = 1.$$

Example

(cont.) Now there are 4 equations from which the 4 unkonwns (prices) can be solved: equilibrium allocations are

$$(x_1^1, x_2^1) = (\alpha_1, \alpha_2) \text{ and } (x_1^2, x_2^2) = (1 - \alpha_1, 1 - \alpha_2).$$

The corresponding equilibrium prices satisfy

$$\frac{\alpha_2}{1 - \alpha_1} = \frac{p_1}{p_2}$$

Bargaining theory of the markets

- The theory of competitive markets is salient about *how* the equilibrium is reached
- Intuitively, the terms of trade are determined via some kind of a bargaining process
- In its rudimentary form, bargaining could be interpreteted as process, where the agents wander around and agree tentatively witht he other agents the terms of trade
- What are the reasonable outcomes of such a game?
- The **Core** was the first attempt to formalize this

- Underlying is the idea that as the economy grows, individual agents are less able to affect on prices and the Walrasian equilibrium prevails
- It seeks to explain how the equilibrium is reached through **coalitional negotiation**

- As above let $I = \{1, ..., H\}$ be the set of agents with initial endowments $(\omega^h)_{h \in I}$
- A subset S of agents I is called a **coalition**
- An allocation $x_S = (x^h)_{h \in S}$ is feasible for coalition S if

$$\sum_{h \in S} x^h \leq \sum_{h \in S} \omega^h$$

- If x^h is feasible for I, then we simply say that it is feasible

Definition

A coalition $S \subseteq I$ **blocks** a feasible allocation x if there is an allocation y_S^h, feasible for S, such that $u_h(y^h) > u_h(x^h)$ for all $h \in S$.

- A feasible allocation that is blocked by a coalition is not agreeable since all the coalition members are better if they reject the allocation and form a subsystem where they reallocate their andowments among themselves
- The Core relies on a counterfactual argument that is typical for economics: unreasonable outcomes, that are in conflict with (group) rationality, should never happen and hence can be removed from the set of *possible* preductions

Definition

The **Core** is the set of feasible allocations x that no coalition blocks

- The Core is unique (why?) but may contain many points
- Since I is a coalition, any allocation in the Core must be **Pareto efficient** (PO)
- Since $\{h\}$ is a coalition, any allocation in the Core must be **individually rational** (IR)
- Thus the Core allocations are contained by the **contract curve**, *i.e.* the set of PO and IR allocations

Proposition

Any Walrasian allocation x is in the Core

Proof.

For suppose a coalition S blocks it via allocation y_S. Let p be the Walrasian price. Then, since $u_h(y^h) > u_h(x^h)$ for all $h \in S$, must have

$$p \cdot y^h > p \cdot \omega^h, \text{ for all } h \in S$$

Summing both sides across h,

$$p \cdot \sum_{h \in S} y^h > p \cdot \sum_{h \in S} \omega^h.$$

□

Proof.

(cont.) Equivalently

$$\sum_{\ell=1}^{L} p_\ell \sum_{h\in S}\left(y_\ell^h - \omega_\ell^h\right) > 0$$

implying, since $p_\ell \geq 0$ for all ℓ, that

$$\sum_{h\in S}\left(y_\ell^h - \omega_\ell^h\right) > 0$$

for at least one ℓ, contradicting the hypothesis y_S is feasible for S. □

- Hence the Walrasian equilibrium is consistent with the idea that the outcome is reached via bargaining'
- This implies that also the Walrasian equilibrium is in the contract curve, i.e. is PO and IR
- Interpretation: interpret the trades as bilateral so that z_ℓ^{hi} denote the amount of good ℓ that agent h obtains from agent i
  - Then, for all ℓ

$$\sum_{i \in I} z_\ell^{hi} = z_\ell^h \text{ and } \sum_{h \in I} \sum_{i \in I} z_\ell^{hi} = 0$$

- Let p^* be a Walrasian equilibrium from the initial endowment ω and z the corresponding net trades

- Choose a coalition S and construct a new initial endowment $\omega(S)$ such that

$$\omega_\ell^h(S) = \omega_\ell^h + \sum_{i \in I \setminus S} z_\ell^{hi}, \text{ for all } h \in S$$

- That is, the $\omega_\ell^h(S)$ constitutes a hypothetical endowment for each h in S that contains not only their initial endowments but also the trades of h with the agents outside S
- Now p is still a Walrasian equilibrium price from the initial endowment $\omega_S(S)$ of the economy restricted to S
- Also, the Core restricted to the economy of S agents from the initial endowment $\omega_S(S)$ contains all the allocations than the Core of the original economy - for suppose an allocation is not in the core

Large markets

- Intuitively, in markets with large number of participants, the role of a single agent becomes small
- Hence, in large markets, as agents are not able to influence price, they take it as given $=>$ *Walrasian equilibrium*
- Can this intuition be vierified by using our model of (coalitional) bargaining?
- That is, what happens to the Core when the economy grows?

- Increase the size of the economy by **replicating** the agents and their endowments for n times: $nI = \{11, ..., 1H, 21, ..., 2H, ..., n1, ..., nH\}$ each jh possessing endowment ω^h
- The n times replicated economy is called the n-**replica** of the original economy
- Allocation $(x^{jh})_{jh \in nI}$ has the **equal treatment property** if $x^{jh} = x^{ih}$ for all $j, i \in I$
- An allocation $(x^{jh})_{jh \in nI}$ that has the equal treatment property induces the same consumption for the all the similar agents
- An allocation with the equal treatment property can be expressed in terms of a feasible allocation of a *single* generation problem $(x^h)_{h \in I}$
- Any allocation in the Core meets equal treatment if preferences are convex and strongly monotonic

Theorem (Core Convergence)

Let preferences be strictly convex and strongly monotonic. Then the feasible allocation $(x^h)_{h \in I}$ is in the Core of the $n-$replica economy for all $n = 1, 2, \ldots$ **only if** *it is a Walrasian equilibrium allocation.*

Proof.

In the $H = 2$ case, let y not be a Walrasian allocation but in the Core for all replications $n = 1, 2, \ldots$ Since y it is feasible in the $n = 1$ case,

$$y^1 - \omega^1 = y^2 - \omega^2.$$

Since y is in the Core, it is Pareto-optimal. □

Proof.

(cont.) Since y is PO and distinct from x, there is, by strong monotonicity, an agent, say 2, such that $u_2(y^2) > u_2(x^2)$ and an agent, say 1, such that $u_1(y^1) < u_1(x^1)$. By the strict convexity of preferences, there is a rational number $n/m \in (0, 1)$, where n and m are integers, such that

$$u_1\left(\frac{n}{m}y^1 + \frac{n-m}{m}\omega^1\right) > u_1\left(y^1\right).$$

□

Proof.

(cont.) Take a coalition S consisting of n type 2 agents and m type 1 agents. Reallocate the coalition resources so that each of the type 2 agents still gets y^2 and each of the type 1 agents gets $\frac{n}{m}y^1 + \frac{m-n}{m}\omega^1$. Since the contribution of goods of type 1 agents to the coalition is

$$\left(\frac{n}{m}y^1 + \frac{m-n}{m}\omega^1 - \omega^1\right)m = (y^1 - \omega^1)n,$$

and since the type 2 receive the amount

$$(y^2 - \omega^2)n,$$

the reallocation is feasible for the coalition S. But then the coalition blocks y. □

- Thus we conclude that in a large market, the Walrasian equilibrium is the only reasonable prediction of a (coalitional) bargaining procedure
- As the market becomes large, the bargaining power of a single agent vanishes and so does her ability to influence the terms of trade $=>$ Walrasian equilibrium
- Intuition: in large market, if one agent receives more surplus than the other similar agents, whe can be replaced with them
- The Walrasian equilibrium need not be unique, though

Production Economies

- An obvious omission the exchange economy model is the lack of **production** and firms
- With production, firms' profits are channeled back to the owners $=>$ profits affect the owners' budget constraints
- Also the input prices w are determined in the market
- How is the equilibrium formed?

Concept check: Robinson Crusoe economy

- Consumption good $x \geq 0$
- Input good $y \geq 0$ and consumer's initial endowment $\omega > 0$ of the input good ($y =$labor work per day, $\omega =$24hrs)
- Consumer with a strictly quasi-concave, increasing, continuous utility function $u(x, \ell)$ (convex, monotonic, continuous preferences), where $\ell = \omega - y$ ($\ell =$leisure)
- Firm with strictly concave production function $f(y) = x$
- The consumer owns the firm

Definition

A Walrasian equilibrium in the Robinson Crusoe economy consists of a price vector $(p, w) \in \mathbb{R}^2_+$ such that:

1. (Optimization) Firm's choice $y(p, w)$ solves

$$\max_y pf(y) - wy$$

and consumer's choice $x(p, w), \ell(p, w)$ solves

$$\max_{x,\ell} u(x, \ell)$$
$$\text{s.t. } px + w\ell \leq w\omega + \underbrace{[pf(y(p, w)) - wy(p, w)]}_{\text{firm's profit}}$$

2. (Market clearing) Supply equals demand in both markets

$$\begin{aligned} x(p, w) &= f(y(p, w)) \\ y(p, w) &= \omega - \ell(p, w) \end{aligned}$$

- At the consumer's optimum (assuming interior solution)

$$\begin{aligned}
\frac{\partial}{\partial x} u\left(x(p,w),\ell(p,w)\right) &= \lambda p \\
\frac{\partial}{\partial \ell} u\left(x(p,w),\ell(p,w)\right) &= \lambda w \\
px(p,w) + w\ell(p,w) &= w\omega + \left[pf\left(y(p,w)\right) - wy(p,w)\right]
\end{aligned}$$

where λ is the Lagrangean multiplier

- Hence

$$\frac{\frac{\partial}{\partial \ell} u\left(x(p,w),\ell(p,w)\right)}{\frac{\partial}{\partial x} u\left(x(p,w),\ell(p,w)\right)} = \frac{w}{p}$$

- At the firm's optimum (assuming interior solution)

$$f'(y(p,w)) = \frac{w}{p}$$

- Hence the Walrasian equilibrium is characterized by the unique point $\bar{x}, \bar{y}$ such that

$$\frac{\frac{\partial}{\partial \ell} u\left(\bar{x}, \omega - \bar{y}\right)}{\frac{\partial}{\partial x} u\left(\bar{x}, \omega - \bar{y}\right)} = f'(\bar{y})$$

- Existence and welfare properties can be established along similar lines to the discussion in exchange economies

Discussion

- The consumer and the firm are price takers even if the consumer owns the firm
- The aim of this assumption is to demonstrate the power of price mechanism
- $=>$ prices are an effective method to steer consumption **and** production
- What are the welfare effects of the Walrasian equilibrium in this production economy?

- Since the consumer owns the production technology, his welfare maximization program is:

$$\max_{x,y} u(x, \omega - y)$$
$$\text{s.t.} \quad f(y) = x$$

or

$$\max_{y} u(f(y), \omega - y)$$

- Solution to this is y^* such that

$$f'(x^*)\frac{\partial}{\partial x}u(f(y^*), \omega - y^*) - \frac{\partial}{\partial \ell}u(f(y^*), \omega - y^*) = 0$$

- Hence $(f(y^*), y^*)$ satisfies the same condition as the Walrasian equilibrium allocation $(\bar{x}, \bar{y})$ => Walrasian equilibrium is Pareto-efficient (consumer optimal)
- Conversely, choosing prices p, w such that

$$f'(x^*) = \frac{w}{p},$$

  an equilibrium can be constructed => a Walrasian equilibrium exists
- The model can be generalized for multiple consumers (with hetergenous preferences and initial endowments), multiple firms (with hetergenous production technologies), multiple consumption and input goods, arbitrary ownership structures (as long as the owners are also consumers)
- => A general model of the economy and of the **"invicible hand"**