Lecture notes 3: Production

Producer theory

- Start with a single firm facing given prices
- Need to describe the technology of the firm
- **Exogenous**: prices
- **Endogenous**: output and input demands
- Aim to understand the optimal production decision of the firm
    - No attention to organizational nor stratgic aspects
    - Objective to have a model that can be transferred in it its pure form to general equilibrium framework
- Comparative statics involve only substitution effects

Primitives:

- Firm with one production good
- Input space $\mathbb{R}^K$
- **Production function**

$$f : \mathbb{R}^K_+ \to \mathbb{R}_+$$

summarizes of the technologically feasibe output/input combinations

Axiom

Production function f is continuous, strictly increasing and quasiconcave

Profit maximization problem

- With output price $p > 0$ and input prices $w = (w_1, ..., w_K)$, where $w_k > 0$ for all k, **profit** is

$$pf(y) - w \cdot y$$

  for any $y \in \mathbb{R}_+^K$ (use the dot product notation $w \cdot y = \sum_k w_k y_k$)
- The firm's objective is to **maximize profits,** i.e. to maximize the size of the owner's budget set

Optimal production

- The problem reduces to

$$\max_{y \in \mathbb{R}_+^{K-1}} pf(y) - w \cdot y \tag{1}$$

- Letting $y(p, w)$ denote the optimal choice(s) at prices (p, w),

$$\pi(p, w) = pf(y(p, w)) - w \cdot y(p, w)$$

  is the **profit function** of the firm
- Assuming an interior solution ($y_k > 0$ for all $k = 1, .., K$), FOCs of the optimum $y(p, w)$ can be written

$$\frac{\partial f(y(p, w))}{\partial y_k} = \frac{w_k}{p}, \qquad \text{for all inputs } k, j$$

- **Marginal rate of technological substitution**:

$$MRTS_{kj} = \frac{\partial f\left(y(p,w)\right)/\partial y_k}{\partial f\left(y(p,w)\right)/\partial y_j}, \qquad \text{for all inputs } k,j$$

  signifies the slope of the **isoquant** $\{y \in \mathbb{R}_+^{K-1} : f(y) = q\}$ at $q = f\left(y(p,w)\right)$

- At the optimum,

$$MRTS_{kj} = \frac{w_k}{w_j}, \qquad \text{for all inputs } k,j$$

- Suppose one observes the profit of the firm and prices, can we deduce the production function and the optimal production?
- Using the envelope argument:

Proposition (Hotelling's Lemma)

$$\frac{\partial \pi(p,w)}{\partial p} = f(y(p,w))$$

$$\frac{\partial \pi(p,w)}{\partial w_k} = -y_k(p,w), \quad \textit{for all inputs } k$$

- Hence the profit function $\pi(p,w)$ is decreasing in w_k, increasing in p

- A function $g : \mathbb{R}^K \rightarrow \mathbb{R}_+$ is convex if $tg(x) + (1-t)g(x') \geq g(tx + (1-t)x')$ for all $x, x' \in \mathbb{R}^K$, for all $t \in (0,1)$

Proposition

The profit function $\pi(p, w)$ is a convex function in (p, w)

Proof.

First, for any (p, w), (p', w') and (p'', w''),

$$\begin{aligned} \pi(p, w) &= pf\left(y(p, w)\right) - w \cdot y(p, w) \\ &\geq pf\left(y(p'', w'')\right) - w \cdot y(p'', w'') \end{aligned}$$

and

$$\begin{aligned} \pi(p', w') &= p'f\left(y(p', w')\right) - w' \cdot y(p', w') \\ &\geq p'f\left(y(p'', w'')\right) - w' \cdot y(p'', w'') \end{aligned}$$

$\square$

Proof.

(cont.) Hence

$$\begin{aligned} & t\pi(p', w') + (1-t)\pi(p', w') \\ \geq \quad & t[pf\left(y(p'', w'')\right) - w \cdot y(p'', w'')] \\ & +(1-t)[p'f\left(y(p'', w'')\right) - w' \cdot y(p'', w'')] \\ = \quad & [tp + (1-t)p']f\left(y(p'', w'')\right) \\ & -[tw + (1-t)w'] \cdot y(p'', w'') \end{aligned}$$

Since this holds for any (p'', w'') it holds in particular when $(p'', w'') = t(p, w) + (1-t)(p', w')$, which gives the result. □

- Using Hotelling's Lemma, by the concavity of π,

$$\begin{aligned}\frac{\partial f(y(p,w))}{\partial p} &= \frac{\partial^2 \pi(p,w)}{(\partial p)^2} \geq 0 \\ \frac{\partial y_k(p,w)}{\partial w_k} &= -\frac{\partial^2 \pi(p,w)}{(\partial w_k)^2} \leq 0, \quad \text{for all } k = 1, ..., K-1\end{aligned}$$

- Interpretation:
  - If the price of an output increases, then the supply increases: "Law of Supply"
  - If the price of an input increases, the demand for the input decreases: "Law of Input Demand"

Cost minimization

- For each quantity of output, q, find the least costly input combination that yields q:

$$\min_{y \in \mathbb{R}_+^{K-1}} w \cdot y$$
$$\text{s.t. } q = f(y)$$

- Denote the solution by $z(w, q)$, i.e. the **conditional factor demand** function
- The value function, $c(w, q)$, is called the **cost function**

$$c(w, q) = w \cdot z(w, q).$$

- $z(w, q)$ is completely analogous to $h(p, u)$ in consumer theory and $c(w, q)$ is analogous to $e(p, u)$

Proposition

The cost function $c(w,q)$ is increasing in q, concave in w, increasing in p, and homogenous of degree zero.

Proof.

We show that c is increasing in q. Note that $c(w,q)$ minimizes the Lagrangian

$$\mathcal{L}(w,q) = w \cdot y - \lambda[f(y) - q].$$

Hence

$$c(w,q) = w \cdot z(w,q) - \lambda[f(z(w,q)) - q].$$

By the envelope theorem

$$\frac{\partial c(w,q)}{\partial q} = \lambda$$

□

Proof.

(cont.) Since, at the optimum,

$$w_k = \lambda \frac{\partial f(z(w,q))}{\partial y_k}$$

it follows that

$$\frac{\partial c(w,q)}{\partial q} = \left(\frac{\partial f(z(w,q))}{\partial y_k}\right)^{-1} w_k \geq 0$$

$\square$

...back to optimal production

- Given the notion of cost function, the problem of the firm simplies remarkably: just find the optimal level of output!
- That is, given p and w, the firm's objective is to solve

$$\max_{q \in \mathbb{R}_+} pq - c\left(w, q\right)$$

- The first order condition for this the familiar:

$$p = \frac{\partial c\left(w, q\right)}{\partial q}$$

i.e., at the firm's optimum, the marginal cost equals the price

- Once the cost minimizing input is determined, the problem of optimal production becomes one dimensional
- We can write by $q^S(p)$ the optimal output of the firm, given the output price p (and the input prices w)
- Then $q^S(\cdot)$ is the firm's supply function

Difference between consumer and producer theory:

- The utility function u only represents preferences $\succsim$ and cannot be observed even in principle: multiple respresentations
- Production function f is a unique description of the technology and, in principle, observable
- Conclusion: Not only ordinal but also cardinal differences have meaning under f, e.g. concavity of f matters!