Lecture notes 2: Consumer choice and demand function

Consumer with preferences

- Up to now, all discussion concerning the economic agent has been completely general
- Now we turn to an economically important special case: the **consumer,** who makes choices over feasible combinations of commodities
- In these notes, we lie down the standard axioms imposed on consumer behavior and study their implications
- The model is used when we turn to modeling markets

- We take $X = \mathbb{R}^L_+$, the set of all possible combinations of L distinct commodities indexed by $\ell = 1, ..., L$
- An element $x_1, ..., x_L$ of X is called a **bundle,** where x_ℓ is the quantity of good ℓ
- In addition to those implied by rationality (transitivity, completeness), we impose some extra conditions on preferences that facilitate meaningful comparison between the bundles and guarantee the induced choice is "well behaved"

Monotonicity

- Monotonicity is the condition that gives the commodity the meaning of a "good": more is better

Axiom (Monotonicity)

Preferences $\succsim$ *are* **monotonic** *if, for all* $x, y \in X$,

$$\begin{aligned} x_\ell &> y_\ell, \text{ for all } \ell \text{ imply } x \succ y \\ x_\ell &\geq y_\ell \text{ for all } \ell \text{ imply } x \succsim y \end{aligned}$$

- It is important that monotonicity does not restrict preferences at all in cases where the quantity of at least one good decreases

- Graphically, monotonicity precludes the possibility that indifference set $I(x)$ such that

$$I(x) = \{y \in X : y \sim x\}$$

does not contain segment that "bends upward" and that $I(x)$ lies above $I(y)$ whenever $x \succ y$

Examples

Let $L = 2$. Monotonic preferences:

- $(x_1, x_2) \succsim (y_1, y_2)$ if $\min\{x_1, x_2\} \geq \min\{y_1, y_2\}$ (Leontief)
- $(x_1, x_2) \succsim (y_1, y_2)$ if $x_1^\alpha x_2^\beta \geq y_1^\alpha y_2^\beta$, $\alpha, \beta > 0$ (Cobb-Douglas)
- $(x_1, x_2) \succsim (y_1, y_2)$ if $v(x_1) + x_2 \geq v(y_1) + y_2$, for increasing $v(\cdot)$ (quasi-linear)

- A weaker axiom with similar spirit:

Axiom (Nonsatiation)

Preferences $\succsim$ *are* **locally nonsatiated** *if for all* $x \in X$ *and for all* $\delta > 0$*, there exists* $y \in X$ *such that*

$$\|y - x\| < \delta \text{ and } y \succ x.$$

- Local nonsatiation is implied by monotonicity but not vice versa
- Graphically, local nonsatiation implies that the indifference set $I(x) = \{y \in X : y \sim x\}$ is in fact a curve, containing no $L-$dimensional balls

Convexity

- Rationality, continuity, and monotonicity guarantee that indifference curves are downward sloping but may have kinks, i.e. moving towards a preferred bundle may actually make the agent worse off
- The next condition guarantees this will never happen

Axiom (Convexity)

Preferences $\succsim$ *are* **convex** *if for all* $x, y, \in X$ *and for all* $\lambda \in [0, 1]$,

$$x \succsim y \text{ implies } (\lambda x + (1 - \lambda)y) \succsim y$$

They are **strictly convex** *if for all* $x, y, \in X$ *and for all* $\lambda \in (0, 1)$,

$$x \succsim y \text{ implies } (\lambda x + (1 - \lambda)y) \succ y$$

Characterization

- Recall the notion of continuous preferences

Axiom (Continuity)

Preferences $\succsim$ *are* **continuous** *if, for all* $x \in X$, *the upper and lower contour sets* $\succsim(x)$ *and* $\precsim(x)$ *are closed*

- Continuity implies that the indifference curves are continuous
- Debreu's Theorem states the existence of a continuous utility function when preferences are continuous

- A useful way to construct a continuous utility function when preferences are also monotonic: find (exercise), by continuity, a function $t(x)$ such that $x \sim (t(x), ..., t(x))$ for all x

Proposition

Let rational preferences on X be continuous, convex and monotonic. Then $u(x) = t(x)$ for all x represents the preferences.

Proof.

Let $x \succsim y$. Then $(t(x), ..., t(x)) \succsim (t(y), ..., t(y))$. By monotonicity $t(x) \geq t(y)$, i.e. $u(x) \geq u(y)$. □

- In particular, the constructed u is also **continuous**

Consumer's problem

- We have constructed the consumer preferences on the set of consumption bundles $X = \mathbb{R}^L_+$
- We are mainly interested choices in "economic domains", where the consumer's feasible sets are characterized by his consumable income w, and prices $p_1, ..., p_L$ (nonnegative numbers) of the commodities
- Formally, given an income w of the consumer and a price vector $p = (p_1, ..., p_L) \in \mathbb{R}^L_+$, the **budget set** of the consumer is defined by

$$B(w, p) = \left\{ x \in \mathbb{R}^L_+ : \sum_{\ell=1}^{L} p_\ell x_\ell \leq w \right\}$$

- $B(w, p)$ is a compact (= closed and bounded) and convex set

- The task of finding the $\succsim$ −optimal consumption bundle from $B(w,p)$ is referred as the **consumer's problem**
- When the preferences are monotonic, then any optimal consumption bundle x^* satisfies **Walras' Law**: lies in the budget line $\sum_\ell p_\ell x_\ell^* = w$

Remark

If $\succsim$ is continuous, then there **is** *an optimal consumption bundle x in $B(w,p)$*

Proof.

$\succsim(x) \cap B(p,w)$ is nonempty compact set for all $x \in B(w,p)$ and $\succsim(x) \cap B(p,w) \subseteq \succsim(y) \cap B(p,w)$ for all $x \succsim y$. If $\bigcap_{x\in B(p,w)} \succsim(x) \cap B(p,w)$ is empty, there a finite collection $x^1, ..., x^k$ of elements in $B(p,w)$ such that $\bigcap_{x\in\{x^1,...,x^k\}} \succsim(x) \cap B(p,w)$ is empty (this follow from $B(p,w)$ being a compact set). But then $\succsim(x^*) \cap B(p,w)$ would be empty for x^* that is $\succsim$ −maximal in $\{x^1,...,x^k\}$, a contradiction. □

Remark

If $\succsim$ is continuous, monotonic, and convex, then the set of optimal consumption bundles in $B(p, w)$ is **convex**

Proof.

Suppose that x and y are optimal, and hence $x \sim y$. Since preferences are convex, $\lambda x + (1 - \lambda)y \succsim x$. Since $\sum_\ell p_\ell x_\ell \leq w$ and $\sum_\ell p_\ell y_\ell \leq w$, also $\lambda \sum_\ell p_\ell x + (1 - \lambda) \sum_\ell p_\ell y_\ell \leq w$, and hence $\sum_\ell p_\ell [\lambda x_\ell + (1 - \lambda) y_\ell] \leq w$. Thus $\lambda x + (1 - \lambda)y \in B(p, w)$, implying that also $\lambda x + (1 - \lambda)y$ has to be optimal. □

Remark

If $\succsim$ is continuous, monotonic, and **strictly** *convex, then there is a* **unique** *optimal consumption bundle in $B(w, p)$*

- Denote by $x(w, p)$ the optimal consumption bundle in $B(w, p)$, referred as **Marshallian demand** under w, and p
- Further, $x(\cdot, \cdot)$ is the **Marshallian demand function**, specifying the optimal consumption bundle for each w and p
- An important property of the demand function is that it is not sensitive to small changes in the underlying environment, i.e. it is **continuous** in p and w

Proposition

Let $\succsim$ is continuous, monotonic, and strictly convex and $x(p, w)$ optimal consumption bundle under p, w. Then the Marshallian demand function $x(\cdot, \cdot)$ is continuous in p and w.

Proof.

[Sketch] Suppose that x is *not* continuous in p. Then there is p^k converging to p^* such that $x(p^k, w)$ converges to $y^* \neq x(p^*, w)$. Since, by continuity, $y^* \in B(p^*, w)$, and since $x(p^*, w)$ is an optimal choice in $B(p^*, w)$, $x(p^*, w) \succ y^*$. For sufficiently high k, also $x(p^*, w) \succ x(p^k, w)$. Choose a bundle z close to $x(p^*, w)$ such that $z \succ x(p^k, w)$ for all sufficiently high k and such that $\Sigma_\ell p_\ell z_\ell < w$. For sufficiently high k, also $\Sigma_\ell p_\ell^k z_\ell < w$. This contradicts the assumption that $x(p^k, w)$ is the optimal choice in $B(p^k, w)$. □

Optimal consumption

- Our next task is to *find* the optimal consumption bundle
- Let consumer preferences $\succsim$ be rational, monotonic, strictly convex, and continuous
- Then the consumer's problem can be represented as **utility maximization problem**

$$\max_{x \in B(p,w)} u(x).$$

or, equivalently,

$$\max_{x \geq 0} u(x)$$
$$\text{s.t. } \sum_{\ell=1}^{L} p_\ell x_\ell \leq w.$$

- By the remarks made in the previous slide, this problem has a **unique** solution $x(p,w)$

- How to derive the Marshallian demand $x(p, w)$?
- Construct a Lagrangean

$$\mathcal{L}(x, \lambda) = u(x) - \lambda \left(\sum_{\ell=1}^{L} p_\ell x_\ell - w \right),$$

where $\lambda \in \mathbb{R}$ is the **Lagrange multiplier**

- Let $x(p, w) := x^* > 0$ maximize the Lagrangean (assuming that u is differentiable)
- The first order conditions (FOC) are

$$\begin{aligned} \frac{\partial u(x^*)}{\partial x_\ell} - \lambda p_\ell &= 0, \text{ for all } \ell \\ \sum_{\ell=1}^{L} p_\ell x_\ell^* &= w \end{aligned}$$

- Thus

$$\frac{\partial u\left(x^{*}\right)/\partial x_{\ell}}{\partial u\left(x^{*}\right)/\partial x_{k}}=\frac{p_{\ell}}{p_{k}}$$

- The ratio p_{ℓ}/p_{k} is the **marginal rate of substitution** between goods ℓ and k at x^{*} : the rate at which x_{ℓ} should increase when x_{k} decreases (or vice versa) for the agent's utility to remain intact

Example

Let preferences be charcatcerized by a Cobb-Douglas utility function

$$u(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}, \text{ where } \alpha \in (0, 1)$$

We derive the Marshallian demand x_1^*, x_2^*. First order conditions for optimality:

$$\begin{aligned}
\alpha \left(\frac{x_2^*}{x_1^*} \right)^{1-\alpha} - \lambda p_1 &= 0 \\
(1-\alpha) \left(\frac{x_2^*}{x_1^*} \right)^{-\alpha} - \lambda p_2 &= 0 \\
p_1 x_1^* + p_2 x_2^* &= w
\end{aligned}$$

We have

$$\frac{x_2^*}{x_1^*} = \frac{(1-\alpha) p_1}{\alpha p_2}$$

Example

(cont.) and solving for x_1^* and x_2^*

$$x_1^* = \frac{\alpha w}{p_1}, \; x_2^* = \frac{(1-\alpha)w}{p_2}$$

Indirect utility

- We are mainly interested in understanding the effect of price changes on consumption and welfare
- Define the **indirect utility function** by

$$v(p, w) = u(x(p, w))$$

- What are the properties of $v(p, w)$ implied by utility maximization?

- Let u represent monotonic and continuous preferences $\succsim$
- Then the indirect utility function $v(\cdot,\cdot)$ is:
  i homogenous of degree 0 ($v(p,w) = v(tp,tw)$ for all $t > 0$)
  ii strictly increasing in w, strictly decreasing in p_ℓ
  iii continuous
- An important tool in analysing the indirect utility (or any value function resulting from maximization) is the **envelope theorem:** only the **direct** effect of a parameter change matters when evaluating the effects of changes in the environment

- To see this, recall first that by the FOC,

$$\frac{\partial}{\partial x_\ell}\mathcal{L}(x(p,w),\lambda) = \frac{\partial u\left(x(p,w)\right)}{\partial x_\ell} - \lambda p_\ell = 0, \text{ for all } \ell$$

- Thus

$$\begin{aligned}
\frac{\partial v\left(p,w\right)}{\partial p_\ell} &= \frac{\partial}{\partial p_\ell}\mathcal{L}(x\left(p,w\right),\lambda) \\
&= \frac{\partial}{\partial p_\ell}\left[u\left(x\left(p,w\right)\right) - \lambda\left(\textstyle\sum_\ell p_\ell x_\ell\left(p,w\right) - w\right)\right] \\
&= \sum_k \frac{\partial x_k\left(p,w\right)}{\partial p_\ell}\left(\frac{\partial u\left(x(p,w)\right)}{\partial x_k} - \lambda p_k\right) - \lambda x_\ell\left(p,w\right) \\
&= -\lambda x_\ell\left(p,w\right)
\end{aligned}$$

where the third equality follows by the chain rule and the last one from the FOC of the Lagrangean

- Similarly,

$$
\begin{aligned}
\frac{\partial v(p,w)}{\partial w} &= \frac{\partial}{\partial w}\mathcal{L}(x(p,w),\lambda) \\
&= \frac{\partial}{\partial w}\left[u(x(p,w)) - \lambda\left(\sum_\ell p_\ell x_\ell(p,w) - w\right)\right] \\
&= \sum_\ell \frac{\partial x_\ell(p,w)}{\partial w}\left(\frac{\partial u(x(p,w))}{\partial x_\ell} - \lambda p_k\right) + \lambda \\
&= \lambda
\end{aligned}
$$

  The second equality follows by the chain rule and the last one from the FOC of the Lagrangean

- Thus Lagrange multiplier λ gives the marginal (shadow) value of relaxing the constraint, i.e. the marginal value of wealth w

- Since

$$\frac{\partial v\left(p,w\right)}{\partial p_\ell} = -\lambda x_\ell\left(p,w\right) \quad \text{and} \quad \frac{\partial v\left(p,w\right)}{\partial w} = \lambda$$

we have:

Proposition (Roy's Identity)

The Marshallian demand $x\left(p,w\right)$ can be recovered from indirect utility function $v\left(p,w\right)$ by

$$x_\ell\left(p,w\right) = -\frac{\partial v\left(p,w\right)/\partial p_\ell}{\partial v\left(p,w\right)/\partial w}$$

Example

(cont.) With Cobb-Douglas utility function $u(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$, the Marshallian demand is

$$x_1(p, w) = \frac{\alpha w}{p_1}, \; x_2(p, w) = \frac{(1-\alpha)w}{p_2}$$

The indirect utlity is

$$\begin{aligned} v(p, w) &= x_1(p, w)^{\alpha} x_2(p, w)^{1-\alpha} \\ &= \left(\frac{\alpha w}{p_1}\right)^{\alpha} \left(\frac{(1-\alpha)w}{p_2}\right)^{1-\alpha} \\ &= w \left(\frac{\alpha}{p_1}\right)^{\alpha} \left(\frac{1-\alpha}{p_2}\right)^{1-\alpha} \end{aligned}$$

Example

With **quasilinear utility function** $u(x_1, x_2) = \phi(x_1) + x_2$, ϕ increasing, differentiable and concave, the optimization problem is

$$\max \phi(x_1) + x_2$$
$$\text{s.t. } \bar{p}x_1 + x_2 \leq \bar{w}$$

where $\bar{p}$ is the price ratio p_1/p_2 and $\bar{w}$ is the ratio w/p_1.Marshallian demand depends only on $\bar{p}$ and $\bar{w}$ and hence we may denote it $x(\bar{p}, \bar{w})$ (assume > 0). It satisfies

$$\begin{aligned} \phi'(x_1(\bar{p}, \bar{w})) &= \bar{p} \\ \bar{p}x_1(\bar{p}, \bar{w}) + x_1(\bar{p}, \bar{w}) &= \bar{w} \end{aligned}$$

Since ϕ is a concave function, $x_1(\bar{p}, \bar{w})$ is a decreasing function of $\bar{p}$ (why?)

Duality

- The problem with the utility function and the indirect utility functions is that they are not observable, only x, p and w are
- An important property called **duality of consumption** transforms the problem into language of the observables, and hence allows us to make emprically testable predictions
- Given the utility function $u(\cdot)$, denote by $h\left(p,u\right)$ the choice that solves the **expedinture minimizing** problem subject to the utility being at least u (a number):

$$\min_{x_\ell \geq 0} \sum_{\ell=1}^{L} p_\ell x_\ell$$
$$\text{s.t. } u\left(x\right) \geq \bar{u}.$$

- Notice that even though the feasible set is not bounded, the problem has a solution when $p \in \mathbb{R}^L_{++}$

- $h(p, \bar{u})$ is called the **Hicksian** or **compensated demand function**
- Denote the value function under the minimizer by $e(p, \bar{u})$, the **expenditure function**

$$e(p, \bar{u}) = \sum_{\ell=1}^{L} p_\ell h_\ell (p, \bar{u})$$

- Since $e(p, \bar{u})$ minimizes costs under the constraint that utility $\bar{u}$ is generated, and since $\bar{u}$ can be generated under w such that $\bar{u} = v(p, w)$, we have

$$w \geq e(p, v(p, w))$$

- Similarly,

$$u \leq v(p, e(p, \bar{u}))$$

- The duality between the indirect uitlity function $v(\cdot,\cdot)$ and the expenditure function $e(\cdot,\cdot)$ manifests itself in the following parity:

Proposition

Let preferences be continuous, monotonic, and strictly concave. For any price vector $p \in \mathbb{R}^L_{++}$,

$$w = e\left(p, v\left(p, w\right)\right) \text{ and } \bar{u} = v\left(p, e\left(p, \bar{u}\right)\right)$$

Proof.

Suppose that $w > e\left(p, v\left(p, w\right)\right)$. Then there is a less costly way to attain utility $u = v\left(p, w\right)$ than $x\left(p, w\right)$, say y. Thus $\sum_\ell p_\ell y_\ell < w$. But by strict convexity of preferences, $\lambda x\left(p, w\right) + (1-\lambda)y \succ x\left(p, w\right)$, for all $\lambda \in (0, 1)$. Moreover, since

$$\lambda \sum_\ell p_\ell x_\ell(p, w) \leq \lambda w \quad \text{and} \quad (1-\lambda) \sum_\ell p_\ell y_\ell < (1-\lambda) w$$

also

$$\sum_\ell p_\ell [\lambda x_\ell\left(p, w\right) + (1-\lambda) y_\ell] < w$$

and hence $\lambda x\left(p, w\right) + (1-\lambda)y$ belongs to the budget set. But this contradicts the assumption that $x\left(p, w\right)$ is an optimal choice. Similar argument rules out $\bar{u} < v\left(p, e\left(p, \bar{u}\right)\right)$. □

- Then we also get a parity between the Marshallian demand function $x(\cdot, \cdot)$ and the $h(\cdot, \cdot)$ Hicksian demand function

$$x(p, w) = h(p, v(p, w)) \text{ and } h(p, \bar{u}) = x(p, e(p, \bar{u}))$$

- The underlying force behind the duality is that any two disjoint convex sets can be separated by a hyperplane

- A counterpart of Roy's Identity can now be stated in the context of expedinture functions a Hicksian demand functions

Proposition (Shephard's Lemma)

The Hicksian demand $h(p, w)$ can be recovered from the expenditure function $e(p, \bar{u})$ by

$$h_\ell(p, \bar{u}) = \frac{\partial e(p, \bar{u})}{\partial p_\ell}$$

- To see this, observe that $e\left(p,\bar{u}\right)$ is the value of the Lagrangean

$$\mathcal{L}(x,\lambda)=\sum_{\ell=1}^{L}p_\ell x_\ell-\lambda[u(x)-\bar{u}]$$

at the minimizer $x=h\left(p,\bar{u}\right)$

- By FOC,

$$\begin{aligned} p_\ell-\lambda u'(h\left(p,\bar{u}\right)) &= 0, \text{ for all } \ell \\ u(h\left(p,\bar{u}\right))-u &= 0 \end{aligned}$$

- Thus, using again the envelope argument,

$$
\begin{aligned}
\frac{\partial e(p,w)}{\partial p_\ell} &= \frac{\partial \mathcal{L}(h(p,\bar{u}),\lambda)}{\partial p_\ell} \\
&= \frac{\partial \left\{ \sum_{\ell=1}^{L} p_\ell h_\ell(p,\bar{u}) - \lambda[u(h(p,\bar{u})) - \bar{u}] \right\}}{\partial p_\ell} \\
&= h_\ell(p,\bar{u}) + \sum_{k=1}^{L} \frac{\partial h_k(p,\bar{u})}{\partial p_\ell}[p_k - \lambda u'(h(p,\bar{u}))] \\
&= h_\ell(p,\bar{u})
\end{aligned}
$$

Example

(cont.2) With Cobb-Douglas utility function $u(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$, the FOC for the Hicksian demand h^* is

$$\begin{aligned} p_1 - \lambda\alpha \left(\frac{x_2^*}{x_1^*}\right)^{1-\alpha} &= 0 \\ p_2 - \lambda(1-\alpha)\left(\frac{x_2^*}{x_1^*}\right)^{-\alpha} &= 0 \\ (x_1^*)^{\alpha}(x_2^*)^{1-\alpha} - \bar{u} &= 0 \end{aligned}$$

We have

$$\frac{h_2^*}{h_1^*} = \frac{(1-\alpha)p_1}{\alpha p_2}$$

and solving for x_1^* and x_2^*

$$h_1^* = \left(\frac{\alpha p_2}{(1-\alpha)p_1}\right)^{1-\alpha} \bar{u}, \;\; h_2^* = \left(\frac{(1-\alpha)p_1}{\alpha p_2}\right)^{\alpha} \bar{u}$$

Example

(cont.3) With Hicksian demand

$$h_1(p,\bar{u}) = \left(\frac{\alpha p_2}{(1-\alpha)p_1}\right)^{1-\alpha}\bar{u},\ \ h_2(p,\bar{u}) = \left(\frac{(1-\alpha)p_1}{\alpha p_2}\right)^{\alpha}\bar{u}$$

The expedinture function

$$\begin{aligned} e(p,\bar{u}) &= p_1\left(\frac{\alpha p_2}{(1-\alpha)p_1}\right)^{1-\alpha}\bar{u} + p_2\left(\frac{(1-\alpha)p_1}{\alpha p_2}\right)^{\alpha}\bar{u} \\ & \left[\left(\frac{\alpha}{1-\alpha}\right)^{1-\alpha}\left(\frac{p_2^{1-\alpha}}{p_1^{-\alpha}}\right) + \left(\frac{(1-\alpha)}{\alpha}\right)^{\alpha}\left(\frac{p_1^{\alpha}}{p_2^{1-\alpha}}\right)\right]\bar{u} \\ &= \left(\frac{p_1}{\alpha}\right)^{\alpha}\left(\frac{p_2}{1-\alpha}\right)^{1-\alpha}\bar{u} \end{aligned}$$

- An interesting feature of the expedinture function is that it is **concave** in p

Proposition

For any prices p and p', and for any $\lambda \in (0,1)$,

$$\lambda e(p, \bar{u}) + (1-\lambda) e(p', \bar{u}) \leq e(\lambda p + (1-\lambda) p', \bar{u})$$

- Concavity implies that $\partial^2 e / (\partial p_\ell)^2 \leq 0$ for all ℓ

Proof.

Since $h(p, \bar{u})$ minimizes costs to achieve $\bar{u}$ under p and $h(p', \bar{u})$ minimizes costs to achieve $\bar{u}$ under p' we have, for any p'',

$$\begin{aligned} \sum_\ell p_\ell h_\ell(p, \bar{u}) &\leq \sum_\ell p_\ell h_\ell(p'', \bar{u}) \\ \sum_\ell p'_\ell h_\ell(p', \bar{u}) &\leq \sum_\ell p'_\ell h_\ell(p'', \bar{u}) \end{aligned}$$

Since the inequalities hold for any p'', they hold particular if $p'' = \lambda p + (1 - \lambda)p'$. Multiplying the first inequality with λ and the second with $(1 - \lambda)$ and summing side by side, □

Proof.

(cont.)

$$
\begin{aligned}
& \lambda \sum_\ell p_\ell h_\ell(p,\bar{u}) + (1-\lambda)\sum_\ell p'_\ell x_\ell(p',\bar{u}) \\
\leq \; & \sum_\ell p_\ell h_\ell(\lambda p + (1-\lambda)p', \bar{u})
\end{aligned}
$$

Since $\sum_\ell p_\ell h_\ell(p,\bar{u}) = e(p,\bar{u})$ etc., the result follows. □

Income and substitution effects

- We are ultimately interested in how the environment affects the behavior of the consumer, i.e. how the changes in income and prices modify her consumption
- Instinctively, we tend to think that a decrease of a good's price will increase its demand
- To verify whether this holds true, we need to decompose the effects of a price change into **substitution effect** and **income effect**

- Substitution effect: relatively cheaper goods become more attractive
- Income effect: increased income permits optimization, cannot say much about the direction for a particular good
    - *Normal* good's demand increases as income increases
    - *Inferior* good's demand decreases as income increases
- "Local" properties; may depend on the current level of prices, income and consumption

- Total effect of a price change on goods demand?

Proposition (Slutsky decomposition)

Let $\bar{u}$ be value of the indirect utility at (p, w). The effect of price change of good ℓ on good k can be written

$$\frac{\partial x_k(p,w)}{\partial p_\ell} = \underbrace{\frac{\partial h_k(p,\bar{u})}{\partial p_\ell}}_{\text{Substit. eff.}} - \underbrace{x_\ell(p,w)\frac{\partial x_k(p,w)}{\partial w}}_{\text{Income eff.}}$$

Proof.

By duality, for all p,

$$h(p, u^*) = x(p, e(p, u^*))$$

By differentiating both sides,

$$\frac{\partial h_k(p, \bar{u})}{\partial p_\ell} = \frac{\partial x_k(p, e(p, \bar{u}))}{\partial p_\ell} + \frac{\partial e(p, u)}{\partial p_\ell} \frac{\partial x_k(p, e(p, \bar{u}))}{\partial w}$$

Rearranging,

$$\frac{\partial x_k(p, e(p, \bar{u}))}{\partial p_\ell} = \frac{\partial h_k(p, \bar{u})}{\partial p_\ell} - \frac{\partial e(p, \bar{u})}{\partial p_\ell} \frac{\partial x_k(p, e(p, \bar{u}))}{\partial w}$$

□

Proof.

(cont.) Using Shephard's Lemma,

$$\frac{\partial x_k\left(p, e\left(p, \bar{u}\right)\right)}{\partial p_\ell} = \frac{\partial h_k\left(p, \bar{u}\right)}{\partial p_\ell} - h_\ell\left(p, \bar{u}\right)\frac{\partial x_k\left(p, e\left(p, \bar{u}\right)\right)}{\partial w}$$

Noting that, by duality, $h\left(p, \bar{u}\right) = x\left(p, e\left(p, \bar{u}\right)\right)$ and $e\left(p, \bar{u}\right) = w$, the result follows. □

- In particular,

$$\frac{\partial x_\ell(p,w)}{\partial p_\ell} = \frac{\partial h_\ell(p,\bar{u})}{\partial p_\ell} - x_\ell(p,w)\frac{\partial x_\ell(p,w)}{\partial w}$$

- Since e is a concave function in p, and by Shephard's Lemma

$$h_\ell(p,\bar{u}) = \frac{\partial e(p,\bar{u})}{\partial p_\ell}$$

  it follows that $\partial h_\ell(p,\bar{u})/\partial p_\ell = \partial^2 e(p,\bar{u})/(\partial p_\ell)^2$ must be nonpositive
- A good is normal if $\partial x_\ell(p,w)/\partial w$ is nonnegative and inferior otherwise

Proposition

(Law of Demand) If a good is normal, then its consumption increases as its price decreases. If the good's consumption decreases as its price decreases (= "Giffen good"), then the good must be inferior.