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Valuating payoff streams under unequal discount factors[☆]

Hannu Salonen [a], Hannu Vartiainen [b,c,*]

[a] University of Turku, Finland
[b] Yrjö Jahnsson Foundation, Ludviginkatu 3-5, 00130 Helsinki, Finland
[c] Turku School of Economics, Finland

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Abstract

We study repeated prize allocation when the discount factors of the agents are unequal. The feasible set of payoffs behaves badly: the Pareto frontier is everywhere discontinuous and there is an open subset of (high) discount factors under which the feasible set is totally disconnected.
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1. Introduction

A handy way to model infinitely repeated interaction is to assume discounting. The special case of equal discount factors is well studied in the literature. An often cited and useful property of the feasible set of payoffs is that it coincides with the convex hull of stage game payoffs. While one justification for this might be public randomization, the more fundamental one relies on the repeated structure itself. Fudenberg and Maskin (1991) and Sorin (1986) demonstrate that any convex combination of stage game payoffs can be induced by alternating pure actions with the appropriate frequencies. This result constitutes a building block of the folk theorem by Fudenberg and Maskin (1986, 1990): Any individually rational feasible payoff configuration can be implemented in subgame perfect Nash equilibrium, even without public randomizing device.

Lehrer and Pauzner (1999) show that, when discount factors differ and *when players have access to a public randomizing device*, the set of feasible payoffs is larger than the convex hull of stage game payoffs.[1] They also assert that public randomization is without loss of generality — as it is under equal discount factors. We demonstrate that this assertation is not true.

The aim of this note is to problematize the unequal discount factors case. We allocate a fixed prize infinitely many times between a patient and an impatient player, and abstract from strategic issues.

The feasible set of payoffs is difficult to characterize. As after any finite history the continuation feasible set is a discounted version of the period zero feasible set, there is no finitary method to completely describe the feasible set — as opposed to the case of equal discount factors (cf. Fudenberg and Maskin, 1991).[2] However, we show that in a large class of discount factors the feasible set is *totally disconnected*. Whether this holds for all games under unequal discount factors remains an open question.

Pareto-optimality is the key criterion of successive collective decision making. The problem is that with any unequal discount factors the Pareto frontier of the feasible set is *nowhere continuous*. It is not obvious how bargaining takes place over Pareto domain that has such complex characteristics.

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* Corresponding author. Yrjö Jahnsson Foundation, Ludviginkatu 3-5, 00130 Helsinki, Finland. Tel.: +358 40 7206808; fax: +358 9 605 002.
E-mail address: hannu.vartiainen@yjs.fi (H. Vartiainen).

[1] The reason for this is that player's trade-off between consuming today versus later differ when their discount factors differ. Hence there may exist mutually beneficial "payoff trades".

[2] The feasible set of a repeated game is a fractal whose self-similar components happen to overlap.

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We assert that our results extend to a general class of repeated games. Some simulation excercises concerning the feasible set of other repeated games are provided in the final section.

2. The set up

There is an infinite sequence of indivisible prizes of consumption value unity, to be allocated either to player P (patient) or I (impatient). The discount factors of the two players are δ_p and δ_I, respectively, with $1>\delta_p>\delta_I>0$. Denote by $s_k\in\{0,1\}$ the stage k allocation with $s_k=1$ if the prize goes to P and 0 otherwise. A stream is denoted by $s=(s_0,s_1,\ldots)$, and the set of streams by $S=\{0,1\}^\infty$.

Normalized payoffs from stream s to P and I are, respectively,

$$u_P(s) = (1-\delta_p)\sum_{t=0}^{\infty} s_t\delta_p^t,$$

$$u_I(s) = (1-\delta_I)\sum_{t=0}^{\infty} (1-s_t)\delta_I^t.$$

Denote the value of s at period t by $u(s{:}t)$. That is,

$$u_P(s:t) = (1-\delta_p)\sum_{\tau=t}^{\infty} s_\tau\delta_p^{\tau-t},$$

$$u_I(s:t) = (1-\delta_I)\sum_{\tau=t}^{\infty} (1-s_\tau)\delta_I^{\tau-t}.$$

For any $A\subseteq S$, denote $u[A]=\{u_p(s),u_I(s)): s\in A\}$, and the i-projection of $u[A]$ by $u_i[A]$, for $i=I, P$. By our normalization, $u[S]\subset[0,1]^2$. Denote the set of Pareto-optimal sequences by

$$PO = \{s : u_i(s')>u_i(s)\Rightarrow u_j(s')<u_j(s),\ \text{for all}\ s'\in S\}.$$

The P- and I-maximal payoffs are generated by streams $1=(1,1,\ldots)$ and $0=(0,0,\ldots)$, respectively. Denote by by $1_t = (\underbrace{0,\ldots,0}_{t-1},1,0,0,\ldots)$ the sequence that gives P only the t'th prize. Use the notation $s-s'=(s_0-s_0', s_1-s_1',\ldots)$ if $s_t-s_t'\in\{0,1\}$ for all t. Then $1-s$ is the mirror image of s: it gives the prize to P whenever s gives it to I and vice versa. Moreover, $s\pm 1_t$ is the stream that differs from s only in that it changes the allocation of t'th prize in s.

2.1. Characterization

The set $S=\{0,1\}^\infty$ is a compact metric space ("the Cantor space"). Hence the function $u{:}S\rightarrow[0,1]^2$, for $u(s)=(u_I(s),u_p(s))$, is a continuous function on a compact metric space and, consequently, $u[S]$ is a compact subset of $[0,1]^2$.

Proposition 1. *Let $\delta_p<1/2$. Then $S=PO$.*

Proof. First we claim that $s'\neq s$ implies $u_p(s)\neq u_p(s')$ and $u_I(s)\neq u_I(s')$. Let t be the least index such that $s_t\neq s_t'$, and, without loss of generality, $s_t=1$, $s_t'=0$. If, say, $u_p(s)=u_p(s')$, then also $u_p(s{:}t)=u_p(s'{:}t)$. In the extreme case, $s_{t'}=0$ and $s'_{t'}=1$ for all $t'=t+1,t+2,\ldots$, i.e. $u_p(s{:}t)\geq 1-\delta_p$ and $u_p(s'{:}t)\leq\delta_p$. But since $\delta_p<1/2$, we have $1-\delta_p>\delta_p$. Thus $u_p(s{:}t)>u_p(s'{:}t)$, a contradiction.

Take any $s\in S$, and suppose that $s'\in S$ Pareto dominates s. By the previous paragraph, s' is strictly better for both P and I than s. Let t be the least index such that $s_t\neq s'_t$. By the argument of the previous paragraph $s_t=1$ implies that $u_p(s)>u_p(s')$, which is not possible. Hence $s_t=0$. But then, by the same argument, $u_I(s)>u_I(s')$, a contradiction. □

The following result is well known (e.g. Fudenberg and Maskin, 1991; Sorin, 1986).

Lemma 2. *If $\delta_i\geq 1/2$ then $u_i[S]=[0,1]$, for $i=I, P$.*

This does not yet say much about the feasible set $u[S]$ in $[0,1]^2$. However, in the special case of $\delta_p=\delta_I$ the set $u[PO]$ coincides with the interval $[(1,0),(0,1)]$ the convex hull of the "stage game" payoffs. We now argue that this does not generalize to the $\delta_p>\delta_I$ case.

A closed subset U of an Euclidean space is *connected* if it cannot be partitioned into two disjoint closed sets. The maximal connected subsets of U are called *components* of U. Now U is *totally disconnected* if all its components are one point sets.

The recursive structure of $u[S]$ implies that it is *self-similar*. Define $u[S{:}\emptyset]=u[S]$ and let, for all $t=0,1,\ldots$,

$$u[S:s_0,\ldots,s_t] = \{(x+s_t(1-\delta_p),y+(1-s_t)(1-\delta_I)) : (x,y)\in u[S:s_0,\ldots,s_{t-1}]\}.$$

Then, for all $t=0,1,\ldots$,

$$u[S:s_0,\ldots,s_{t-1}] = \cup_{s_t\in\{0,1\}}u[S:s_0,\ldots,s_{t-1},s_t].$$

Proposition 3. *For any $\delta_p>1/2$ there is d such that $u[S]$ is totally disconnected if $\delta_p\geq d>\delta_I>1/2$.*

Proof. First we show that $u[S]$ is totally disconnected when $\delta_p>\delta_I=1/2$. By the self-similarity of $u[S]$, it suffices to show that $u[S{:}0]$ and $u[S{:}1]$ are disjoint. Thus we need that $u(1,s_1,s_2,\ldots)\neq u(0,s_1',s_2',\ldots)$, for all $s_1,s_2,\ldots$ and $s_1',s_2',\ldots$.

Suppose, to the contrary, that $u(1,s_1,s_2,\ldots)=u(0,s_1',s_2',\ldots)$ for some $s_1,s_2,\ldots$ and $s_1',s_2',\ldots$. Then, since $u_I(1,s_1,s_2,\ldots)=u_I(0,s_1',s_2',\ldots)$ we have

$$\delta_I = \sum_{t=1}^{\infty} \delta_I^t(s_t'-s_t).$$

Since $\delta_I=1/2$, necessarily $s_t=0$ and $s_t'=1$ for all $t=1,2\ldots$. But this implies, since also $u_p(1,s_1,s_2,\ldots)=u_p(0,s_1',s_2',\ldots)$, that $\delta_p=1/2$, a contradiction.

That the desired $d>1/2$ exists follows from the continuity of $u_I(\cdot)$ in δ_I. □

Fig. 1a below depicts a fractal that is induced by discount factors $\delta_p=0.7$ and $\delta_I=0.55$. It is easy to see that $u[S]$ is disconnected. By the self-similarity of the components, it is also totally disconnected. Fig. 1b depicts $u[S]$ when $\delta_p=0.8$ and $\delta_I=0.7$. Now it is no longer clear whether $u[S]$ is connected or not.

We leave open the question of when exactly is the feasible set is totally disconnected. What is clear is that there are discount factors under which the feasible set is connected (when the discount factors are equal and above 1/2). Whether this holds in general when the discount factors are close and/or high is a difficult problem.

2.2. Pareto frontier

We first argue that a Pareto-optimal stream has a stage from which onwards all prizes are given to the impatient player only if the stream gives *all* the prizes to him.

Lemma 4. *Let $\delta_p \geq 1/2$. If $s \in PO|\{0\}$, then s allocates infinitely many prizes to P.*

Proof. Suppose, to the contrary, that there is t such that $s_t=1$ and $s_{t'}=0$ for all $t'>t$. Then $u_I(s{:}t)=\delta_I$, and $u_p(s{:}t)=1-\delta_p$. Let, without loss of generality, $t=0$. By Lemma 2, there is s' such that $s'_0=0$ and $u_p(s'{:}1)=(1-\delta_p)/\delta_p$. Since $s_t\delta_p \geq (1-s_t)\delta_I$, for all $t=0,1,\ldots$, with strict inequality when $s_t=1$ it follows that $u_p(s'{:}1)>u_I(1-s'{:}1)$. Thus, since $\delta_p>\delta_I$, we have

$$\begin{aligned} 1-\delta_I &> 1-\delta_p \\ &= u_p(s':1)\delta_p \\ &> u_I(1-s':1)\delta_I \\ &= u_I(1-s':0), \end{aligned}$$

where the final equality follows by $1-s'_0=1$ By construction, $\delta_I<1-u_I(1-s'{:}0)=u_I(s'{:}0)$. Thus, since move from s to s' keeps P indifferent but increases I's payoff, we have $s \notin PO$. □

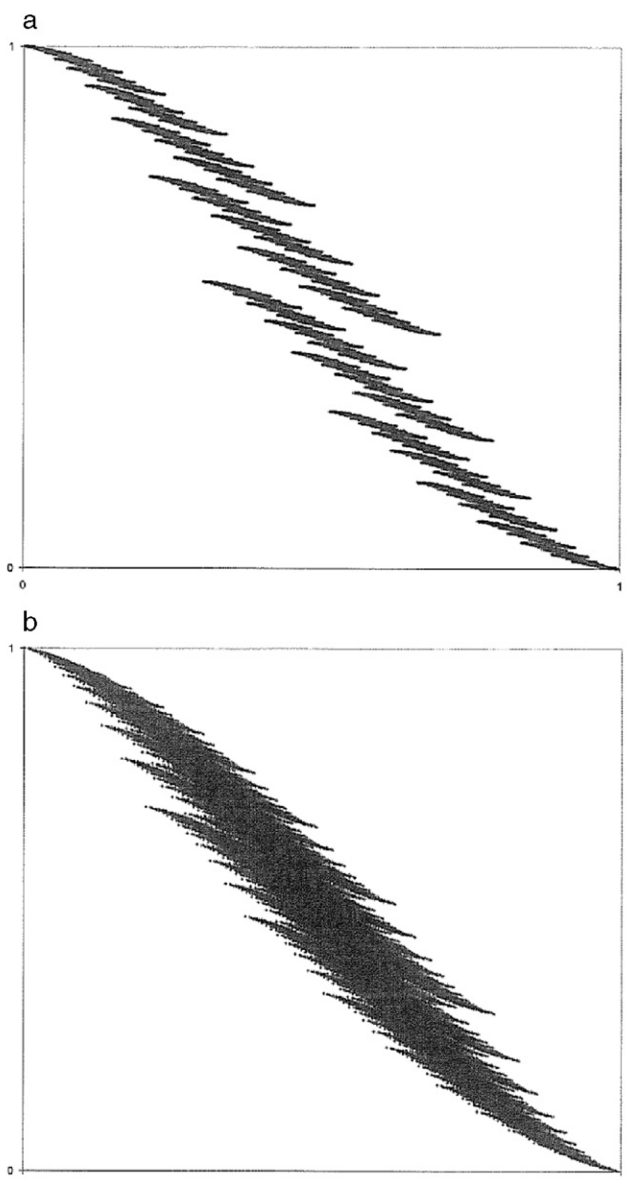

Fig. 1.

Thus any Pareto-optimal stream (other than 0) gives the prize infinitely many times to the patient player. From this it follows that any utility level of player P can be induced by some Pareto-optimal stream.

Proposition 5. *Let $\delta_p \geq 1/2$. Then $u_p[PO]=[0,1]$.*

Proof. Take any closed interval $[x,1] \subset [0,1]$, $x<1$. By Lemma 2, there is a nonempty, compact subset A of S such that $u_p(A)=[x,1]$. Since A is compact and u_I continuous, the set $B=\text{argmax}_{s\in A} u_I(s)$ is nonempty, compact subset of A, and hence there is s^* such that $s^* \in \text{argmax}_{s\in B} u_p(s)$. By construction, $s^* \in PO$. It suffices to show that $u_p(s^*)=x$. Suppose that $u_p(s^*)-x>0$. By Lemma 4, there is big enough t such that $s_t^*=1$ and $u_p(s^*)-x>(1-\delta_p)\delta_p^t$. But then $s^*-1_t \in A$ while $u_I(s^*-1_t)>u_I(s^*)$, which contradicts the assumption that $s^* \in B$. □

By Proposition 5, there is a function $f{:}[0,1]\to[0,1]$ whose graph coincides with the Pareto frontier, i.e. $f(u_p(s))=u_I(s)$ for all $s \in PO$ Then $f(u_p(s)) \geq u_I(s)$ for all $s \in S$. By construction, f is a strictly decreasing function. Function f is nowhere continuous if there is no open interval on which f is continuous.

Proposition 6. *f is nowhere continuous.*

Proof. Let, to the contrary of the proposition, there be an open interval $X \subset [0,1]$ on which f is continuous. Since f is strictly decreasing on X, the set of points where f is not differentiable has Lebesque measure zero. Let f be differentiable on a set $D \subset X$ that has strictly positive Lebesgue measure. Denote by $S(1_t)$ the set of sequences such that $s_{t'}=1$ for all $t' \geq t$. Since $\cup_{t=0}^{\infty} S(1_t)$ consists of countably many elements, set $u_p[\cup_{t=0}^{\infty} S(1_t)]$ as zero measure. Thus we may assume $D \subset u_p[S \backslash \cup_{t=0}^{\infty} S(1_t)]$.

Take $x \in D$, and find $s \in PO \backslash \cup_{t=0}^{\infty} S(1_t))$ such that $u(s)=(x,f(x))$. By construction, there are infinitely many periods t such that $s_t=0$ Thus, $s+1_t \in S$, for all these periods t. For any such t,

$$u_p(s+1_t) = u_p(s) + (1-\delta_p)\delta_p^t,$$
$$u_I(s+1_t) = u_I(s) - (1-\delta_I)\delta_I^t.$$

Choosing $\{x_t\}=\{u_p(s+1_t)\}$, we have, by the definitions of derivative and f,

$$\begin{aligned} f'(x) &= \lim_{t\to\infty} \frac{f(u_p(s)) - f(u_p(s+1_t))}{u_p(s)-u_p(s+1_t)} \\ &\geq \lim_{t\to\infty} \frac{u_I(s)-u_I(s+1_t))}{u_p(s)-u_p(s+1_t)} \\ &= \lim_{t\to\infty} -\left(\frac{\delta_I}{\delta_p}\right)^t \left(\frac{1-\delta_I}{1-\delta_p}\right) \\ &= 0 \end{aligned} \tag{1}$$

But (1) contradicts the fact that f is strictly decreasing. □

However, f also possesses some continuity properties.

Proposition 7. *Let $\delta_I \geq 1/2$. Then f is left-continuous.*

Proof. Let $\{x_n\}$ be an increasing sequence on $[0,1]$ converging to x. Then $\{f(x_n)\}$ is a decreasing sequence bounded below by 0. Hence $\{f(x_n)\}$ converges to y. Since $u[S]$ is compact, (x,y) is in $u[S]$ by definition of f. Since $(x_n,f(x_n))$ is a Pareto optimal utility allocation for each n, we cannot have $f(x)>y$. So $f(x)\leq y$, and since (x,y) is in $u[S]$, we have $f(x)=y$, by definition of f. □

We conclude from Proposition 4 that it is nonproblematic for the patient agent to find his personal maximal subject to impatient agent's reservation payoff. However, by Proposition 6, this is not the case for the impatient agent: his maximal payoff is very sensitive to the patient agent's payoff, and hence finding it is computationally very hard.

3. Discussion

This paper studies the consequences of differentiated discount factors on players' payoffs from prize streams. To see how differentiated discount factors affects the feasible set of a repeated game, consider the case of prisoners' dilemma:

	C	D
C	3,3	0,4
D	4,0	1,1

As demonstrated by Lehrer and Pauzner (1999), the induced feasible set under randomization has a smooth boundary, and contains the convex hull of stage game payoffs as a proper subset. However, without randomization the feasible set of prisoners' dilemma is a proper subset of the feasible set with randomization. The boundary is everywhere non-smooth and contains caves. Fig. 2 depicts the feasible set under $\delta_p=0.9$ and $\delta_I=0.7$. Our results still apply: the Pareto-frontier is is everywhere discontinuous. The shape of the feasible set is sensitive to the size of the discount factors.

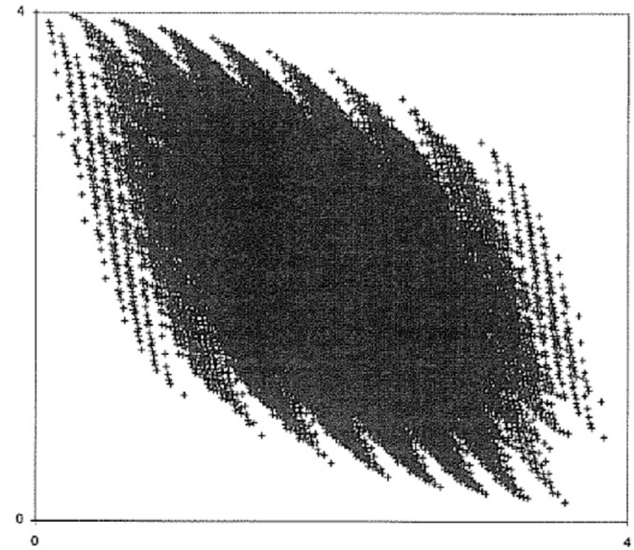

Fig. 2.

These observations have some relevance from the viewpoint of the Folk theorem. For example, the equilibria in Fudenberg and Maskin (1986, 1990) rely on the assumption that continuation payoffs can be matched with incentives not to deviate. When mixed strategies are used, exact match of continuation payoffs may be important. The problem is that when discount factors are unequal, the feasibe set is far from convex, and it is no longer clear (to us) that one can always find the needed continuation payoffs.

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