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Subgame perfect implementation of voting rules via randomized mechanisms

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Abstract We characterize completely ordinal and onto choice rules that are subgame perfect of Nash equilibrium (SPE) implementable via randomized mechanisms under strict preferences. The characterization is very operationalizable, and allows us to analyse SPE implementability of *voting rules*. We show that *no scoring* rule is SPE implementable. However, the *top-cycle* and the *uncovered* correspondences as well as *plurality with runoff* and *any strongly Condorcet consistent* voting rule can be SPE implemented. Therefore our results are favourable to majority based voting rules over scoring rules. Nevertheless, we show that many interesting Condorcet consistent but not strongly Condorcet consistent rules, such as the *Copeland* rule, the *Kramer* rule and the *Simpson* rule, *cannot* be SPE implemented.

1 Introduction

Abreu and Sen (1990) (hereafter AS) and Moore and Repullo (1988) (MR) characterize distinct necessary and sufficient conditions for choice rules to be implementable in subgame perfect Nash equilibrium (SPE). Vartiainen (2005) closes the remaining gap. In particular, Vartiainen (2005) shows that with *linear* preferences the gap is automatically closed. Moreover, for any choice rule that is *onto* the set of alternatives–a property that is met by any *neutral* voting rule on a rich domain of preferences–the AS characterization simplifies to the following form: a choice rule is SPE implementable if and only if there exists a sequence of outcomes $(a^0, \ldots, a^{K+1})$ and players $(i^0, \ldots, i^K)$ such that (1) $a^k P_{i^k} a^{k+1}$ for

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all $k = 0, \ldots, K$, (2) $a^{K+1} P_{i^K} a^K$, and (3) a^k is not top ranked by i^k under R' for any $k = 0, \ldots, K$, whenever $a^0 \in f(R') \setminus f(R)$, for any preference profiles R, R' (Vartiainen 2005, Corollary 1).

Randomization is known to be useful in implementation.[1] This paper analyses SPE implementability of voting rules via randomized mechanisms. We assume that preferences over pure outcomes are linear, and hence the sufficiency part of the above characterization is in force. The key effect of randomization is that it expands the number of potential AS sequences dramatically which allows more choice rules to be SPE implemented. Moreover, the expansion of AS sequences allows us to rewrite the AS characterization in a more operationalizable form: an ordinal choice rule f that is *onto* the set of social alternatives is SPE implementable if and only if there is $i \in N$ who neither ranks a bottom under R'_i nor top under R_i, whenever $a \in f(R') \setminus f(R)$.

In recent papers, Benoit and Ok (2005) and Bochet (2005) study Nash implementation in a similar framework (but allow more general preferences). They show that when randomization is permitted off-the-equilibrium path, Maskin monotonicity alone characterizes the implementable choice rules. Hence the characterization of Nash implementable rules simplifies remarkably. Similarly, this paper establishes a simple characterization of a class of rules that can be SPE implemented by a randomized mechanism. Interestingly, the simplification concerns the structure of the AS characterization itself. This is useful since the AS condition is difficult to check in its original form.

Through our characterization SPE implementability of voting rules can easily be studied. We study voting rules that are neutral and, hence, onto on a rich domain of preferences. Thus the voting rule is SPE implementable by a randomized mechanism if and only if it meets the our condition. We show that *no generalized scoring*[2] rule meets the condition, and is hence not SPE implementable by any randomized mechanisms. However, the *top-cycle* and the *uncovered* rule as well as the *plurality with runoff* rule meet the condition, and can therefore be SPE implemented by some mechanism. In addition, *any strongly Condorcet consistent*[3] can be implemented. However, we also show that many interesting Condorcet consistent but not strongly Condorcet consistent rules, such as the *Copeland* rule, the *Kramer* rule and the *Simpson* rule, *cannot* be SPE implemented.

By and large, these observations add to the long list of results (see e.g. Sjöström 1993 or May 1952) that favor majority based voting rules over scoring rules for implementability reasons. Taken together, our results closely parallel to those obtained by Herrero and Srivastava (1992) and, in particular, Dutta and Sen (1993) in the context of implementation of voting rules via backwards induction. This suggests, rather surprisingly, that integer games and alike devices may not be fundamentally important in the voting context.

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1 See e.g. Abreu and Matsushima (1990), Abreu and Sen (1991), Glazer and Perry (1996), and Sjöström (1993).

2 Allowing scores be contingent on the number of voters.

3 We use a strong version of Condorcet consistency that prevents selecting a strong loser.

2 Notation and the basic set up

There is set $N = \{1, 2, \ldots, n\}$ players, with generic elements i, j, and a finite set A of feasible pure social alternatives with three or more elements. Denote by $\mathcal{R}_A$ the set of all linear orders on A, i.e. $a\ R_i\ b$ and $b\ R_i\ a$ imply $a = b$ for all $R_i \in \mathcal{R}_A$. Assume that each $R_i \in \mathcal{R}_A$ has an extension over lotteries Δ on A, also denoted R_i, which satisfies the vNM axioms. Denote by P_i the asymmetric part of R_i.

Recall the continuity and independence properties of vNM preferences $R_i \in \mathcal{R}_A$. First, take $\mu \in [0, 1]$, and define a lottery $p_\mu : \Delta \times \Delta \to \Delta$ as follows

$$p_\mu(a, b) = a\mu + b(1 - \mu), \quad \text{for } a, b \in \Delta.$$

Take $a, b \in \Delta$ such that $a\ P_i\ b$. By independence,

$$p_\varepsilon(a, c)\ P_i\ p_\varepsilon(b, c), \text{ for any } \varepsilon \in (0, 1) \quad \text{and for any } c \in \Delta. \tag{1}$$

By continuity, there is $\lambda > 0$ such that

$$a\ R_i\ p_\lambda(b, c). \tag{2}$$

To complete the description of the environment, there is a set $\mathcal{R} \subseteq \mathcal{R}_A^n$ of admissible preference profiles. This implies that for any distinct $R = (R_1, \ldots, R_n)$ and $R' = (R'_1, \ldots, R'_n)$ in $\mathcal{R}$ there is i and $a, b \in A$ such that i experiences a preference reversal over a and b when switching from R to R'. We assume that $R \in \mathcal{R}$ is observed by all $i \in N$ but not by the others.

Denote by $\overline{M}_i(R, B)$ and $\underline{M}_i(R, B)$ the sets of i's *R-maximal* and *R-minimal* alternatives, respectively, in $B \subseteq \Delta$ under R. Formally, $\overline{M}_i(R, B) = \{a \in B : a\ R_i\ b \text{for all } b \in B\}$ and $\underline{M}_i(R, B) = \{a \in B : b\ R_i\ a \text{ for all } b \in B\}$. For simplicity, write $\overline{M}_i(R) = \overline{M}_i(R, A)$ and $\underline{M}_i(R) = \underline{M}_i(R, A)$. Since R_i's are linear on A, also $\overline{M}_i(R) = \overline{M}_i(R, D)$ and $\underline{M}_i(R) = \underline{M}_i(R, D)$ for all $A \subseteq D \subseteq \Delta$. Denote the *lower contour set* of i at $a \in \Delta$ under R by $L_i(a, R) = \{b \in \Delta : a\ R_i\ b\}$.

We say that i experiences a *preference reversal over a and b* when switching from R to R' if $a\ R_i\ b$ and $b\ P'_i\ a$. Preference reversals are a fundamental requirement for implementation.

3 SPE implementation and a voting rule

We allow mechanisms in the class of *extensive game forms with simultaneous moves* (for precise definitions, see Appendix A). We allow mechanism Γ to contain randomization and call it *randomized*. Denote by $\mathrm{SPE}(\Gamma, R)$ the set of outcomes in Δ that are implemented under SPE in state $R \in \mathcal{R}$ via a randomized mechanism Γ.

A *social choice rule* is a correspondence $f : \mathcal{R} \to A$, defining a non-empty set of "socially desirable" pure outcomes in each state. Denote $f(\mathcal{R}) = \cup_{R \in \mathcal{R}} f(R)$.

We will focus on *deterministic* and *ordinal* choice rules.[4] That is, $f(R) \subseteq A$ for all $R \in \mathcal{R}$ and if, for any $R, R' \in \mathcal{R}$, $R_i \cap (A \times A) = R'_i \cap (A \times A)$ for all i, then $f(R) = f(R')$.

Given that f is ordinal, it would be without loss of generality to replace the strong assumption $\mathcal{R} \subseteq \mathcal{R}^n_A$ with a weaker assumption that $\mathcal{R}$ is *compact* in the domain of all vNM preferences that are linear orders on A.[5] Namely, in the latter case there is $\bar{\lambda} > 0$ such that for any $R \in \mathcal{R}$, and for all pure outcomes a and b, $a\ P_i\ b$ implies $a\ P_i\ p_\lambda(q, b)$ for all lotteries q, and for all λ in $(0, \bar{\lambda}]$. This is sufficient for the results below (in particular, Lemma 8) to hold.

We are interested in *full* implementation where the set of SPE outcomes of the implementing mechanism coincides with the desired choice rule in all states.

Definition 1 *Choice rule f is SPE implemented by a randomized mechanism* Γ *if* $\mathrm{SPE}(\Gamma, R) = f(R)$, *for all* $R \in \mathcal{R}$.

We say that if there is a Γ that SPE implements choice rule f, then f is SPE *implementable.*

4 Random mechanisms

To express the AS characterization of SPE implementable choice rules, define the following pair of sequences.[6] Given a triple $(R, R', a) \in \mathcal{R} \times \mathcal{R} \times A$ and a set of $D \subseteq \Delta$ of outcomes, a pair of sequences $h(0), \ldots, h(K)$ in N, and $d(0), \ldots, d(K+1)$ in D satisfy $a = d(0)$ and

α(i) $d(k)\ R'_{h(k)}\ d(k+1)$, for $k = 0, \ldots, K$,
α(ii) $d(K+1)\ P_{h(K)}\ d(K)$,
α(iii) $d(k) \notin \overline{M}_{h(k)}(R, D)$, for $k = 0, \ldots, K$,
α(iv) $d(K+1) \in \cap_{i \neq h(K)} \overline{M}_i(R, D)$ implies that $K = 0$ or $h(K-1) \neq h(K)$.

Given triple (R, R', a), denote a typical sequence meeting α (i–iv) by $(h, d)(R, R', a)$. In general, there may exist great many such sequences.

Definition 2 (Condition α) *Choice rule f satisfies* Condition α *with respect to* $D \supseteq f(\mathcal{R})$ *if there exists a sequence* $(h, d)(R, R', a)$ *meeting* α*(i–iv) whenever* $a \in f(R') \setminus f(R)$, *for all* $R, R' \in \mathcal{R}$.

Parts α(i) and α(ii) were derived by MR, AS introduced α(iii) and α(iv). Note that α(i) and α(ii) imply that there is player $h(K)$ who experiences a preference reversal between $d(K)$, $d(K+1)$ when switching from R' to R. This is a much weaker condition than Maskin monotonicity, which requires that there is a preference reversal between a choice *in f* and some other alternative.

4 With vNM preferences and lotteries a choice rule could be responsive to *risk attitudes*.

5 That is, in the domain $\{(x_a)_{a \in A} \in \mathbb{R}^{|A|} : x_a \neq x_b, \text{ for all } a \neq b\}$.

6 To be precise, the necessary condition of AS concerns the situation where the set of possible outcomes of the mechanism is a subset of A. However, nothing in their construction requires the outcomes to be deterministic. Hence the necessary condition holds also when the set of possible outcomes is a subset of Δ.

Lemma 3 (Abreu and Sen 1990) *Choice rule f is* SPE *implementable only if it satisfies Condition α with respect to some $D \supseteq f(\mathcal{R})$.*[7]

AS also show that without the linearity restriction of preferences Condition α and no-veto power (NVP) are sufficient for SPE implementation.[8] However, Vartiainen (2005) proves that when preferences *are* linear on A, as is assumed throughout this paper, one can replace NVP with *unanimity*.[9] Since Condition α and unanimity are also necessary conditions for SPE implementation of a choice rule, it follows that they completely characterize SPE implementable choice rules.[10]

When the choice rule f is *onto,* i.e. $f(\mathcal{R}) = A$, the characterization simplifies even further: since any outcome is selected in some state, Condition α implies unanimity. Thus Condition α is also sufficient to SPE implement this choice rule.

Lemma 4 (Vartiainen 2005) *An ordinal and onto choice rule f defined on $\mathcal{R}$ is* SPE *implementable if it satisfies Condition α with respect to $D = A$.*

This result uses a deterministic mechanism.

Lemma 5 *If a choice rule f defined on $\mathcal{R}$ satisfies Condition α with respect to some $D \supseteq A$, then it satisfies Condition α with respect to $D = \Delta$.*

Proof Given triple (R, R', a), let sequence $\underline{(h, d)}(R, R', \underline{a})$ satisfy α(i–iv) with respect to $D \supseteq A$. By the linearity of R_i's, $\overline{M}_i(R, D) = \overline{M}_i(R, A) = \overline{M}_i(R)$, for all i. Since $D \subseteq \Delta$, sequence $(h, d)(R, R', a)$ satisfies α(i – iv) with respect to Δ . Equivalently, if f meets Condition α with respect to D, then f meets Condition α with respect to Δ. □

Combining Lemma 4 with Lemmata 3 and 5 we now have a tight characterization.

Theorem 6 *A choice rule f that is ordinal and onto on $\mathcal{R}$ is* SPE *implementable by a randomized mechanism if and only if it satisfies Condition α with respect to Δ.*

Condition α is difficult to use in practice. The number of α-sequences grows fast as the number of players and alternatives increases. Thus, for practical purposes, the condition needs to be simplified. When f is ordinal and onto, randomization helps.

Definition 7 (Condition α^{Δ}) *Choice rule f defined on $\mathcal{R}$ satisfies* Condition α^{Δ} *if $a \in f(R')$ and $a \in \underline{M}_i(R') \cup \overline{M}_i(R)$ for all $i \in N$ imply $a \in f(R)$, for all $R, R' \in \mathcal{R}$.*

[7] Abreu and Sen (1990) use deterministic mechanisms and assume $D \subseteq A$.

[8] Choice rule f satisfies NVP with respect to D if $a \in \cap_{i \neq j} \overline{M}^i(\theta, D)$ implies $a \in f(\theta)$.

[9] Choice rule f satisfies *unanimity* with respect to D if $a \in \cap_{i \in N} \overline{M}^i(\theta, D)$ implies $a \in f(\theta)$.

[10] Vartiainen also shows that with strict preferences α(iv) of Condition α becomes superfluous.

That is, if a subset H of players consider choice $a \in f(R')$ as R'-minimal, and all players not in H as R-maximal, then f must pick a also under R. It is essential that H may be empty or equal to N.

Note that if $a \in f(R')$ and $\{a\} = \underline{M}_i(R')$ or $\{a\} = \overline{M}_i(R)$ for all i, then there cannot be $b \neq a$ such that $a\, R'_i\, b$ and $b\, P_i\, a$, for any i. Thus, Maskin monotonicity implies $a \in f(R)$. Thus Maskin monotonicity binds whenever Condition α^{Δ} binds.

Condition α and Condition α^{Δ} are equivalent when randomization is allowed and $D = \Delta$.

Lemma 8 *An ordinal choice rule f on $\mathcal{R}$ satisfies Condition α with respect to Δ if and only if f satisfies Condition α^{Δ}.*

Proof It suffices to show that, for any distinct $R, R' \in \mathcal{R}_A^n$ and $a \in A$, there does not exist $(h,d)(R,R',a)$ that meets α(i–iv) if and only if $a \in \underline{M}_i(R') \cup \overline{M}_i(R)$, for all $i \in N$.

"If": Suppose that $a \in \underline{M}_i(R') \cup \overline{M}_i(R)$, for all $i \in N$. We show that there does not exist sequence $(h,d)(R,R',a)$ that meets α (i–iv). Suppose, to the contrary, that sequence $(h,d)(R,R',a)$ of length K exists. By construction $d(0) = a$. Let $d(k) = a$ for any $k \in \{0,\ldots,K\}$. Since $d(k) \notin \overline{M}_{h(k)}(R)$, α(iii) implies $d(k) \in \underline{M}_{h(k)}(R')$. By α(i), $d(k+1) \in L_{h(k)}(R', d(k))$. Since $\{d(k)\} = \underline{M}_{h(k)}(R') = L_{h(k)}(R', d(k))$, necessarily $d(k) = d(k+1)$. Since this is true for $k = 0$, it must be true for any $k = 1, \ldots$. Thus there is no K such that α(ii) is satisfied.

"Only if": Suppose that $a \notin \overline{M}_i(R') \cup \underline{M}_i(R)$, for some i We show that there exists sequence $(h,d)(R,R',a)$ that meets α (i–iv). In proving this we use the continuity (2) and independence (1) properties of vNM preferences.

Since $R \neq R'$, there are $b, c \in A$ and $j \in N$ such that $c\, R'_j\, b$ and $b\, P_j\, c$. Construct lotteries[11]

$$q = \frac{1}{\#A} \sum_{e \in A} e,$$

$$q_{bc} = \frac{1}{\#A} \left\{ \sum_{e \in A \setminus \{c\}} e + \frac{b}{2} + \frac{c}{2} \right\}.$$

That is, q_{bc} shifts c's probability mass in q on b. Let $a \notin \overline{M}_i(R) \cup \underline{M}_i(R')$. By (2) there is $\lambda > 0$ such that, for $e \in \underline{M}_i(R')$, $a\, R'_i\, p_\lambda(q,e)$. From (1) it follows that

$$p_\lambda(q,e)\; R'_j\; p_\lambda(q_{bc},e) \text{ and } p_\lambda(q_{bc},e)\; P_j\; p_\lambda(q,e), \text{ for all } j \in N.$$

[11] Degenerate lotteries that put probability one to pure outcomes are denoted by $a, b, \ldots$.

Construct sequences $h(R,R',a)$ and $d(R,R',a)$ such that

$$\begin{aligned} &h(0)=i, \qquad && d(0)=a,\\ &h(1)=j, && d(1)=p_\lambda(q,e),\\ & && d(2)=p_\lambda(q_{bc},e). \end{aligned}$$

As q's and q_{bc}'s support is A, and $\lambda > 0$, also the grand lotteries $p_\lambda(q,e)$ and $p_\lambda(q_{bc},e)$ have full support. Since preferences over A are linear it follows that $d(1)$ and $d(2)$ are not R-maximal for any player. By supposition, $d(0)=a \notin \overline{M}_i(R)$. Finally $d(k)\ R'_{h(k)}\ d(k+1)$ for $k \in \{0,1\}$, and $d(2)\ P_{h(1)}\ d(1)$. Checking that the constructed $(h,d)(R,R',a)$ meets α(i–iv) is routine. □

The central problem is to construct $(h,d)(R,R',a)$ sequence whenever $a \notin \underline{M}_i(R') \cup \overline{M}_i(R)$, for some $i \in N$. Our construction uses lotteries that are not top nor bottom ranked for any player.

By Theorem 6 and Lemma 8 we now get an operationalizable characterization of ordinal and onto SPE implementable choice rules.

Theorem 9 *A choice rule f that is ordinal and onto on $\mathcal{R}$ is SPE implementable by a randomized mechanism if and only if it satisfies Condition α^Δ.*

4.1 SPE implementation of voting rules

This section studies SPE implementability of *voting rules,* a class of choice rules that are ordinal, neutral, and deterministic.[12] We demand that a voting rule v is implementable under universal preference domain and under any number of players.

Definition 10 *A voting rule v is SPE implementable by a randomized mechanism if, for any $n = 3, 4, \ldots$, there is a randomized mechanism that SPE implements it under universal domain $\mathcal{R}^n_A$ of preferences.*

The voting rules we shall focus are neutral, and hence they are onto on the universal domain. By Theorem 9, a necessary and sufficient condition for SPE implementability will be Condition α^Δ.

First we study *scoring rules*.[13] The question we impose is whether there is a way to associate an implementable system of scores to any voting set up.

Definition 11 (Generalized scoring rule) *A generalized scoring rule v^s specifies, for each n, a system of scores $s^n_1 \geq s^n_2 \geq \cdots \geq s^n_{\#A}$ such that $s^n_1 \neq s^n_{\#A}$. For given n, a preference profile R associates score s^n_1 to i's top ranked alternative in R_i, s^n_2 to i's second ranked alternative in R_i, and so forth, for all $i \in N$. Scoring rule $v^s(R,n)$ selects an alternative with the highest sum of scores.*

12 Neutrality means that the names of the social alternatives do not matter.

13 For voting methods, see e.g. Moulin (1988, Ch. 9).

Thus a generalized scoring rule associates a scoring system to any voting scenario. For a fixed number of voters the scoring system is fixed, and specifies a winner for all preference profiles. The scoring system may, however, be contingent on the number of voters.

One example of a scoring rule is the *plurality rule* where each player casts a vote for his favorite candidate and the candidate who is named most often is then elected. Another example is the *Borda rule* where $s_k = k - 1$, for all $k = 1, \ldots, \#A$. The general class of scoring rules contains infinitely many rules.

Proposition 12 *There is no generalized scoring rule that is SPE implementable by a randomized mechanism.*

Proof Let $A = \{a, b, c\}$.[14] It suffices to focus on the $n = 12$ case. We show that there is no system of scales $s_1 \geq s_2 \geq s_3$, $s_1 > s_3$ that allows to implement $v^s(\cdot, 12)$. To obtain a contradiction, suppose that there is a system of scores (s_1, s_2, s_3) such that $v^s(\cdot, 12)$ can be SPE implemented. Construct six preference profiles $R, R', \bar{R}, \bar{R}', \tilde{R}, \tilde{R}' \in \mathcal{R}_A^{12}$ as follows.

Case 1 Let profile R' contain three different linear orderings on A:[15]

	$4 \times R'_1$	$4 \times R'_2$	$4 \times R'_3$
s_1	a	c	c
s_2	b	a	b
s_3	c	b	a.

Given any system of scores $s_1 \geq s_2 \geq s_3$, we have that $c \in v^s(R')$. Let players of type 3 experience a preference reversal over a and b, to get profile R :

	$4 \times R_1$	$4 \times R_2$	$4 \times R_3$
s_1	a	c	c
s_2	b	a	a
s_3	c	b	b.

By Theorem 9, and since v^s meets Condition α^{Δ}, $c \in v^s(R)$. Thus also

$$s_1 + s_3 \geq 2s_2. \tag{3}$$

Case 2 Let profile $\bar{R}'$ generate the following linear orderings on A:

	$3 \times \bar{R}'_1$	$3 \times \bar{R}'_2$	$3 \times \bar{R}'_3$	$3 \times \bar{R}'_3$
s_1	a	a	c	c
s_2	b	b	b	b
s_3	c	c	a	a.

[14] As pointed out by a referee, the proof would go through with A containing any number ≥ 3 of elements by assuming that any new e is ranked below a, b, c in all the used profiles.

[15] The table reads: there are four players with preferences R'_1. Call them type 1 players. Type 2 and 3 players' preferences are defined similarly. Score s_1 is associated to an alternative that is top ranked by a player, s_2 to the second, and so on.

From (3) it follows that $a, c \in v^s(\bar{R}')$. Let type 3 and 4 players experience a preference reversal with respect to a and b to get profile $\bar{R}$:

	$3 \times \bar{R}_1$	$3 \times \bar{R}_2$	$3 \times \bar{R}_3$	$3 \times \bar{R}_4$
s_1	a	a	c	c
s_2	b	b	a	a
s_3	c	c	b	b.

By Theorem 9, and since v^s meets Condition α^{Δ}, $c \in v^s(\bar{R})$. Thus also

$$s_1 > s_2 = s_3. \tag{4}$$

Case 3 Let profile $\tilde{R}'$ generate the following linear orderings on A:

	$4 \times \tilde{R}'_1$	$4 \times \tilde{R}'_2$	$4 \times \tilde{R}'_3$
s_1	a	b	c
s_2	b	a	a
s_3	c	c	b.

From (4) it follows that $a, b, c \in v^s(\tilde{R}')$. Let type 2 players experience a preference reversal with respect to a and b to get profile $\tilde{R}$:

	$4 \times \tilde{R}_1$	$4 \times \tilde{R}_2$	$4 \times \tilde{R}_3$
s_1	a	a	c
s_2	b	b	a
s_3	c	c	b.

By Theorem 4, and since v^s meets Condition α^{Δ}, $c \in v^s(\tilde{R})$. But, by (4), $v^s(\tilde{R}) = \{a\}$. A contradiction. □

The proof relies on there being 12 voters. Whether this is the smallest domain of voters where a scoring rule cannot be implemented remains an open question.

Scoring rules and *majority based* voting rules constitute two main approaches to voting. Given R, denote by $SD(R)$ and $D(R)$ the *strict majority* and *majority* preference relations on A : for any $a, b \in A$

$$\begin{aligned} a\, SD(R)\, b \quad &\text{if and only if} \quad \#\{i : a\, P_i\, b\} > \#\{i : b\, P_i\, a\}, \\ a\, D(R)\, b \quad &\text{if and only if} \quad \#\{i : a\, P_i\, b\} \geq \#\{i : b\, P_i\, a\}. \end{aligned}$$

Given R, alternative a is the *Condorcet winner* if $a\, SD(R)\, b$ for all $b \neq a$, and a is a *strong loser* if a is bottom ranked by a majority. *Condorcet consistency* implies that an alternative that the Condorcet winner, if it exists, is chosen by the voting rule. We will slightly strengthen this condition.

Definition 13 (Strong Condorcet consistency) *Voting rule v^c is* strongly Condorcet consistent *if (i) the Condorcet winner under R, if it exists, belongs to $v^c(R)$, and (ii) a strong loser under R, if it exists, does not belong to $v^c(R)$.*

This definition of Condorcet consistency is stronger than the usual one since we require that a strong loser is never selected. It is clear a strong loser cannot ever be the Condorcet winner. Thus property (ii) binds only when there is no winner. With $\#A \geq 3$, there always exists an outcome that is not a strong loser.

Proposition 14 Any *strongly Condorcet consistent voting rule* can *be SPE implemented by a randomized mechanism.*

Proof Let v^c be a strongly Condorcet consistent rule. We show that Condition α^{Δ} is met by v^c. Suppose not. Then there are R, R' such that $a \in v(R') \setminus v(R)$, and such that $\underline{M}_i(R') = \{a\}$ for all $i \in H$ and $\overline{M}_i(R) = \{a\}$ for all $i \in N \setminus H$, for some $H \subseteq N \cup \emptyset$. Since v^c does not select strong losers, and $a \in v^c(R')$, it cannot be the case that a is bottom ranked by the majority. Thus $2\#H < \#N$. By construction, $\#H + \#N \setminus H = \#N$, thus $2\#N \setminus H > \#N$ implying that $a\ SD(R)\ b$ for all $b \neq a$. But Condorcet consistency implies that $a \in v^c(R)$, a contradiction. □

This is consistent with the fact that *no* scoring rule is Condorcet consistent (e.g. Moulin 1988, Theorem 9.1). Condorcet consistency is a property, it does not define a choice rule. Many well known voting rules satisfy the property, e.g. the *top-cycle set* (Condorcet set) and the *uncovered set*.

Definition 15 (Top-cycle set) *Given R, the* top-cycle set $v^{\text{tc}}(R)$ *satisfies*

$$v^{\text{tc}}(R) = \cap\{B \subseteq A : b \in B, a \in A \setminus B \textit{ implies } b\ SD(R)\ a\}.$$

To define the uncovered set, we say that *a covers b* if $aSD(R)b$ and $b\ SD(R)\ c$ implies $a\ SD(R)\ c$ for all $c \in A$.

Definition 16 (Uncovered set) *Given R, the* uncovered set $v^u(R)$ *satisfies*

$$v^u(R) = \{a \in A :\ a \textit{ is not covered by any } b \in A \textit{ under } R\}$$

The top-cycle set is the smallest subset of A with the property that nothing outside the set is preferred by a strict majority anything in the set. The uncovered set is a subset of the top-cycle set and contains only Pareto optimal allocations. In particular, if a is the Condorcet winner under R, then $a = v^c(R) = v^u(R) = v^{\text{tc}}(R)$.

One commonly used voting rule is *plurality with runoff.*[16] In principal, the rule seeks to select two candidates that favored by most of the players, and then compare these candidates against each other. To define the plurality with runoff correspondence also in the indeterminate situations where one cannot find exactly two mostly preferred candidate, we use the notion of top cycle set to identify the candidates that are strong in pairwise comparisons. Define $n(a, R) = \#\{i \in N : \{a\} = \overline{M}_i(R)\}$ and $A(R) = \{a \in A$:there is $c \in A$ s.t. $n(a, R) \geq n(b, R)$ for all $b \in A \setminus \{c\}\}$.

[16] Applied widely, e.g. in France and in Finland.

Definition 17 (Plurality with runoff) *Given* R, *the* plurality with runoff *rule* $v^{\mathrm{pr}}(R)$ *satisfies*

$$v^{\mathrm{tc}}(R) = \cap\{B \subseteq A(R) : b \in B, a \in A(R) \setminus B \textit{ implies } b\ SD(R)\ a\}.$$

If there are two candidates that are top ranked by more players than the rest of the candidates, then $A(R)$ consists of these two candidates. Only in the case of a three or more players' tie $A(R)$ contains more than two elements. If $A(R)$ consists of two candidates, then the one which wins the pairwise majority comparison is selected by $v^{\mathrm{pr}}(R)$. Finally, if there is any candidate which is top ranked by a majority, then v^{pr} selects this alternative.

Proposition 18 *The top-cycle correspondence, the uncovered correspondence, and plurality with runoff correspondence* can *be SPE implemented by a randomized mechanism.*

Proof Denote the top-cycle correspondence, the plurality with runoff correspondence, and the uncovered correspondence by $v^{\mathrm{tc}}, v^{\mathrm{pr}}$, and v^{u}, respectively. We show that the correspondences satisfy Condition α^{Δ}. Suppose that $v \in \{v^{\mathrm{tc}}, v^{\mathrm{pr}}, v^{u}\}$ does not. Then there are R, R' and $a \in A$ such that $a \in v(R') \setminus v(R)$, and such that $\underline{M}_i(R') = \{a\}$ for all $i \in H$ and $\overline{M}_i(R) = \{a\}$ for all $i \in N \setminus H$, for some $H \in N \cup \emptyset$.

Case $v = v^{\mathrm{tc}}$: Suppose that $2\#H > \#N$. Then $b\ SD(R')\ a$, for all $b \neq a$. But then, by definition, $v^{\mathrm{tc}}(R') \subseteq A \setminus \{a\}$, a contradiction. Hence $2\#H \leq \#N$ and, therefore, $2\#(N \setminus H) \geq \#N$. Then it must also be true that $a\ D(R)\ b$, for all $b \neq a$. But then it *cannot* be the case that $b\ SD(R)\ a$, for any $b \neq a$. Thus, by definition, $a \in v^{\mathrm{tc}}(R)$, a contradiction.

Case $v = v^{\mathrm{pr}}$: By definition, $a \in A(R')$. Suppose that $2\#H > \#N$. Then $b\ SD(R')\ a$, for all $b \in A(R') \setminus \{a\}$. But then, by definition, $v^{\mathrm{pr}}(R') \subseteq A(R') \setminus \{a\}$, a contradiction. Hence $2\#N \setminus H \geq \#N$. Then it must also be true that $n(a, R) \geq n(b, R)$, for all $b \in A$. Thus $a \in A(R)$. Moreover, it *cannot* be the case that $bSD(R)a$, for any $b \in A(R) \setminus \{a\}$. Thus, by definition, $a \in v^{\mathrm{pr}}(R)$, a contradiction.

Case $v = v^{u}$: Suppose that $2\#H > \#N$. Then $b\ SD(R')\ a$, for all $b \neq a$. But then every $b \neq a$ covers a, a contradiction. Hence $2\#H \leq \#N$ and, therefore, $2\#N \setminus H \geq \#N$. Then it must also be true that $a\ D(R)\ b$, for all $b \neq a$. But then it *cannot* be the case that $b\ SD(R)\ a$, for any $b \neq a$. Thus, by definition, $a \in v^{u}(R)$, a contradiction. □

Our results are similar to Sjöström (1993) who studies implementation in *trembling hand perfect equilibrium*. He founds out that strongly Condorcet consistent voting rules and the top-cycle correspondence are implementable whereas the Borda rule is not. However, these results are strikingly different from those obtained by Jackson et al. (1994) in the context of implementation in *undominated Nash equilibrium with bounded mechanism*. They showed that the top-cycle set cannot be implemented by using their solution concept whereas the plurality correspondence can be implemented.[17,18]

To identify other prominent Condorcet consistent voting rules, define the following. For given R, the *Copeland score* of an outcome a is $c(a, R) = \#\{b \in A : a\, SD(R)\, b\}$, the *Kramer score* of an outcome a is $k(a, R) = \max_{b \neq a} \#\{i \in N : a\, P_i\, b\}$, and the *Simpson score* of an outcome a is $s(a, R) = \min_{b \neq a} \#\{i \in N : a\, P_i\, b\}$.

Definition 19 –*The* Copeland rule:

$$v^C(R) = \{a \in A : c(a, R) \geq c(b, R),\ \textit{for all } b \in A\},\ \textit{for all } R \in \mathcal{R}_A^n,$$

–*The* Kramer rule:

$$v^K(R) = \{a \in A : k(a, R) \geq k(b, R),\ \textit{for all } b \in A\},\ \textit{for all } R \in \mathcal{R}_A^n,$$

–*The* Simpson rule:

$$v^S(R) = \{a \in A : s(a, R) \geq s(b, R),\ \textit{for all } b \in A\},\ \textit{for all } R \in \mathcal{R}_A^n.$$

Proposition 20 *The Copeland rule, the Kramer rule or the Simpson rule* cannot *be* SPE *implemented by a randomized mechanism.*

Proof The Copeland rule v^C and the Kramer rule v^K : let $A = \{a, b, c\}$ and $n = 4$, and let profile R' contain two kinds of orderings:

$2 \times R_1'$	$2 \times R_2'$
a	c
b	b
c	$a.$

Then $v^C(R') = v^K(R') = \{a, b, c\}$. To get profile R, let type 2 players experience a preference reversal over a and b,

$2 \times R_1$	$2 \times R_2$
a	c
b	a
c	$b.$

[17] However, Jackson et al. (1994) also show that general scoring rules, the Borda rule in particular, cannot be boundedly implemented in undominated Nash equilibrium.

[18] Palfrey and Srivastava (1991) argue that SPE implementation is a weak implementation procedure relative to undominated Nash implementation by constructing examples where attractive Pareto optimal or Condorcet consistent rules (Examples 1 and 3) cannot be implemented by using the former even if they can be by using the latter. However, this observation holds true only for deterministic mechanisms: both examples satisfy Condition α^{Δ} and can therefore be SPE implemented by a random mechanism.

Then $v^C(R) = v^K(R) = \{a\}$. However, Condition α^Δ implies $c \in v^C(R)$ and $c \in v^K(R)$.

Simpson rule v^S : Let $A = \{a,b,c,e\}$ and $n = 9$ and let R' generate six different preference orderings:

$$\begin{array}{cccccc} \multicolumn{3}{c}{\overbrace{\hspace{6em}}^{2\times}} & & & \\ R'_1 & R'_2 & R'_3 & R'_4 & R'_5 & R'_6 \\ a & b & c & e & e & e \\ b & c & a & a & b & c \\ c & a & b & b & c & a \\ e & e & e & c & a & b. \end{array}$$

Then $v^S(R') = \{a,b,c,e\}$. Shift a upwards to get profile R

$$\begin{array}{cccccc} \multicolumn{3}{c}{\overbrace{\hspace{6em}}^{2\times}} & & & \\ R_1 & R_2 & R_3 & R_4 & R_5 & R_6 \\ a & a & a & e & e & e \\ b & b & c & a & a & a \\ c & c & b & b & b & c \\ e & e & e & c & c & b. \end{array}$$

Then $v^S(R) = \{a\}$ $(s(a,R) = 6, s(e,R) = 3)$. However, Condition α^Δ implies $e \in v^S(R)$. □

While the proof of the result uses even number of voters, it does not rely on that assumption (examples available from the author).

The three rules are not strongly Condorcet consistent as they may choose a strong loser. This is interesting since Dutta and Sen (1993) show that selections from the uncovered set *can* be implemented (see also Herrero and Srivastava 1992) whereas any selection from the Kramer nor the Copeland correspondence *cannot* be implemented by using mechanisms that are solvable via *backwards induction*. Thus, in general, this suggests that allowing a mechanism to have simultaneous moves *does not* dramatically increase the number of interesting implementable voting rules. This is desirable since implementation via backwards induction does not rely on integer games or alike unattractive constructions.

5 Conclusion

We show that under linear preferences, randomization can be used to simplify the complex characterization of SPE implementable voting rules. We assume that the used mechanisms randomize only off-the-equilibrium path. It is shown that *no* genralized scoring rule is SPE implementable even with randomized mechanisms. However, the top-cycle rule, the uncovered rule as well as the plurality with runoff rule can be SPE implemented, as well as any strongly Condorcet consistent voting rule (that always selects the Condorcet winner and never selects and a strong loser). However, not all condorcet consistent rules can be implemented. We show that the Copeland rule, the Kramer rule or the Simpson rule cannot be SPE implemented.

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A Appendix

A mechanism is an array[19] $\Gamma = \langle Y, S, g \rangle$ where Y is a *set of histories* y, $S = S_1 \times \cdots \times S_n$ and $S_i = \times_{y \in Y} S_i^y$ for all i. An element of $S^y = S_1^y \times \cdots \times S_n^y$, say $s^y = (s_1^y, \ldots, s_n^y)$, is a *message vector* while s_i^y is i's *message* at y. Histories and messages are tied together by the property that $S^y = \{s^y : (y, s^y) \in Y\}$. An element of S_i, say s_i, is i's (pure) *strategy*[20], specifying i's choices at each nonterminal history, and an element of S, say s, is a (pure) *strategy profile.* There is an *initial history* $y^0 \in Y$ and *each* history $y^k \in Y$ is represented by a *finite* sequence $(y^0, s^1, \ldots, s^{k-1}) = y^k$.[21] If $y^{k+1} = (y^k, s^k)$, then history y^{k+1} *proceeds* history y^k. As Γ contains finitely many stages, there is a set of *terminal histories* $\overline{Y}$ of Y such that $\overline{Y} = \{y \in Y :$ there is no $y' \in Y$ proceeding $y\}$. Any strategy profile $s \in S$ defines a unique terminal history given the initial history. Denote this dependence by $\bar{y}(s : y) \in \overline{Y}$. Thus, if $y^k = (y^0, s^1, \ldots, s^{k-1}) \in Y \setminus \overline{Y}$, then there is a chain $\bar{y}(s : y^k) = (y^0, s^1, \ldots, s^{k-1}, \ldots, s^K) \in \overline{Y}$ for some $K \geq k$. Sometimes terminal history $\bar{y}(s : y^k)$ is called a *path,* given s and y^k. The *outcome function* $g(\cdot : y) : S \to \Delta$ specifies an outcome for each strategy profile with a property that if $\bar{y}(s : y) = \bar{y}(s' : y)$ for any $s, s' \in S$, $y \in Y \setminus \overline{Y}$, then $g(s : y) = g(s' : y)$. For short, if players obey strategy profile s and start from y^0, denote the outcome of g by $g(s) = g(s : y^0)$. Given state $R \in \mathcal{R}$, the pair (Γ, R) constitutes an extensive form game with simultaneous moves.

By the construction of Γ, every $y \in Y \setminus \overline{Y}$ identifies a *subgame* $\Gamma(y)$ of Γ, as follows: y is an initial history of the game $\Gamma(y) = \langle Y(y), S(y), g \rangle$ where $Y(y) = \{y' \in Y : y'$ proceeds $y\}$ and $S(y) = \times_{y' \in Y(y)} S^{y'}$. Hence $\Gamma = \Gamma(y^0)$. Denote by $D(y, s_{-i})$ the set of outcomes player i can reach by varying his own strategy $s_i \in S_i$ given that history y is reached, and all $j \neq i$ adopt strategy $s_j \in S_j$. Formally, $D(y, s_{-i}) = \{a \in A : g(s : y) = a, s_i \in S_i(y)\}$. Denote by $\overline{\text{SPE}}(\Gamma, R)$ the set of SPE *strategies* of a game Γ given R .

[19] For analogous definition see Osborne and Rubinstein (1994).

[20] For simplicity, we confine our attention to pure strategies. This restriction does *not* affect the results.

[21] Thus, like MR, we confine our attention to games having *finitely* many stages. This assumption is made for sake of simplicity. AS showed that allowing infinitely many stages should not affect the conclusions.

Definition 21 *Take* $\Gamma = \langle Y, S, g\rangle$ *and* $R \in \mathcal{R}$. *Then* $s \in S$ *is an element of* $\overline{\text{SPE}}(\Gamma, R)$ *if and only if* $g(s : y) \in \overline{M}_i(R, D(y, s_{-i}))$ *for all* $y \in Y$ *and for all* $i \in N$.

By construction, $\overline{\text{SPE}}(\Gamma(y), R)$ refers to the equilibria of the subgame starting at history y. An important and natural additional restriction on any applicable mechanism $\Gamma = \langle Y, S, g\rangle$ is that $\overline{\text{SPE}}(\Gamma(y), R)$ is nonempty for all $y \in Y$ and for all $R \in \mathcal{R}$. Thus a mechanism must be well defined independently of the history of the play.

Denote

Definition 22 *Take* $\Gamma = \langle Y, S, g\rangle$ *and* $R \in \mathcal{R}$. *Then* $a \in \Delta$ *is an element of* $\text{SPE}(\Gamma, R)$ *if and only if* $g(s) = a$ *for some* $s \in \overline{\text{SPE}}(\Gamma, R)$.

Note that $\text{SPE}(\Gamma(y), R)$ refers to outcomes induced by SPE strategies after history y is reached. Of course, $\text{SPE}(\Gamma(y), R)$ and $\text{SPE}(\Gamma, R)$ do not typically coincide if y does not belong to the *equilibrium path*.

References

Abreu D, Matsushima H (1990) Virtual implementation in iteratively undominated strategies: complete information. Econometrica 60:993–1008

Abreu D, Sen A (1990) Subgame perfect implementation: a necessary and almost sufficient condition. J Econ Theory 50:285–99

Abreu D, Sen A (1991) Virtual implementation in nash equilibrium. Econometrica 59:997–1021

Benoit J-P, Ok E (2005) Nash implementation without No-Veto power, manuscript, NUY

Bochet O(2005) Nash implementation via lottery mechanism. Soc Choice Welf (forthcomming)

Dutta B, Sen A (1993) Implementing generalized condorcet social choice functions via backwards induction. Soc Choice Welf 11:148–60

Glazer J, Perry M (1996) Virtual implementation in backwards induction. Games Econ Behav 15:27–32

Herrero M, Srivastava S (1992) Implementation via backward induction. J Econ Theory 56:70–88

Jackson M, Palfrey T, Srivastava S (1994) Undominated Nash implementation in bounded mechanisms. Games Econ Behav 13:474–501

May K (1952) A set of independent necessary and sufficient conditons for for simple majority decision. Econometrica 46:317–30

Moore J, Repullo R (1988) Subgame perfect implementation. Econometrica 56:1191–1220

Moulin H (1988) Axioms of cooperative decision making. Cambridge University Press, New York, NY

Osborne M, Rubinstein A (1994) A course in game theory. MIT, Cambridge, MA

Palfrey T, Srivastava S (1991) Nash implementation using undominated strategies. Econometrica 59:479–501

Sjöström T (1993) Implementation in perfect equilibria. Soc Choice Welf 10:97–106

Vartiainen H (2005) Subgame perfect implementation: a full characterization. J Econ Theory (forthcoming)