FDPE Decision Theory
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Problem set 2

1. Assume de Finetti's framework. Suppose that $\succsim$ is a weak order and that there is no Dutch book. Suppose further that any x has a certanty equivalent c - a prospect whose outcome is independent of the state - such that $x \sim c$. Prove that $\succsim$ satisfies monotonicity and additivity.

   A: A Dutch book: $(x_i, y_i)_{i=1}^k$ such that $x_i \succsim y_i$ and such that $\sum_{i=1}^k x_i(s) < \sum_{i=1}^k y_i(s)$ for all s.

   For monotonicity, suppose that there are x and y such that $x(s) < y(s)$ for all s but $x \succsim y$. But then (x, y) is a Dutch book.

   For additivity, we first argue that $c(x)$, the certainty equivalent of x, satisfies $c(x + y) = c(x) + c(y)$. Since $c(x + y) \sim x + y$, $c(x) \sim x$ and $c(y) \sim y$, if $c(x + y) < c(x) + c(y)$ then one can construct a Dutch book from $(c(x+y), x+y)$, $(x, c(x))$, and $(y, c(y))$. To the other direction, replace the order of the elements in the binary sets. Thus $c(x + y) = c(x) + c(y)$.

   Now assume $x \succsim y$. Then by monotonicity $c(x) \geq c(y)$ and $c(x)+c(z) \geq c(y) + c(z)$. By the previous paragraph, $c(x + z) \geq c(y + z)$ and hence $x + z \succsim y + z$.

2. Assume we find in experiment that $1_x \sim 100 \cdot \alpha_x + 0 \cdot (1 - \alpha_x)$ for all $x \in [0, 100]$. Let $\alpha_x = \sqrt{x/100}$.

   (a) Assume expected utility maximization. Identify the underlying Bernoulli utility function u on the interval $[0, 100]$.

   A: Assume w.l.o.g. that $u(100) = 1$ and $u(0) = 0$. Then utility function then satisfies $u(x) = \sqrt{x/100}$.

   (b) Assume that the indiffernece is obtained via distortion of the probabilities by $\alpha \mapsto w(\alpha)$. Identify w on the interval $[0, 1]$.

   A: Because of the indifference, $x = w(\sqrt{x/100})100$, or

$$\frac{x}{100} = w\left(\sqrt{\frac{x}{100}}\right) = \sqrt{\frac{x}{100}}^2$$

Thus $w(\alpha) = \alpha^2$.

(c) Show that the latter way of evaluating the lotteries violates the first-order stochastic dominance

A: The value of the prospect that gives 50 with probability 1 is $1^2 \cdot 50 = 50$. The value of a first order dominating prospect that gives 51 with prob. 0.5 and 50 with prob. 0.5 is $(0.5)^2 \cdot 51 + (0.5)^2 \cdot 50 = 25.25$

3. Rank dependent utility satisfies the first-order stochastic dominance -criterion

A: Let p first-order stochastically dominate p'. Then

$$r_i = \sum_{j=i}^{n} p_j \geq \sum_{j=i}^{n} p'_j = r'_i, \quad \text{for all } i = 0, ..., n.$$

Recall that $r_{i+1} = r_i - p_i$ and $r_0 = 1$, $r_n = 0$. Since w is an increasing function,

$$\begin{aligned}
V(p) - V(p') &= \sum_{i=0}^{n} [\pi_i(p) - \pi_i(p')] x_i \\
&= \sum_{i=0}^{n} [(w(r_i) - w(r_{i+1})) - (w(r'_i) - w(r'_{i+1}))] x_i \\
&= \sum_{i=0}^{n} [w(r_i) - w(r'_i)] x_i - \sum_{i=1}^{n} [w(r_i) - w(r'_i)] x_{i-1} \\
&= \sum_{i=1}^{n} [w(r_i) - w(r'_i)](x_i - x_{i-1}) - [w(r_n) - w(r'_n)] x_n + [w(r_0) - w(r'_0)] x_0 \\
&= \sum_{i=1}^{n} [w(r_i) - w(r'_i)](x_i - x_{i-1}) \\
&\geq 0
\end{aligned}$$

4. Expected utility, rank dependent utility, and the sure thing principle - discuss

A: The sure thing principle demands that preferences over lotteries (prospects) be independent of the common outcomes. Under expected utility this follows by the additivity of the probability measure. Under rank dependent utility, this follows if probabilities are replaced with ranks.

5. Let time preferences $\succsim$ satisfy the Fishburn-Rubinstein axioms, and be represented by function $\delta^t u(x)$. Show that if u is concave, then the loss of delay $x - f(x)$ is increasing in x, where $(f(x), 0) \sim (x, 1)$ for all x. Interpret this property in terms of uncertainty.

   A: Assume f and u are continuous. The function f satisfies

$$u(f(x)) = u(x)\delta, \quad \text{for all } x.$$

   Concavity of u implies that $u'(x)/u(x)$ is a monotonically decreasing, strictly positive function under all $x > 0$. Since $f(x) < x$,

$$\begin{aligned} f'(x) &= \frac{u'(x)\delta}{u'(f(x))} \\ &= \frac{u'(x)/u(x)}{u'(f(x))/u(f(x))} \\ &\in (0, 1). \end{aligned}$$

   Risk aversion thus implies increasing loss of delay.

6. Prove that a binary relation $\succsim$ on $2^X \backslash \emptyset$ is rationalizable under uncertainty if and only if the following conditions hold true:

   (a) if $A \subseteq B \subseteq X$, then $B \succsim A$

   (b) if $A \subseteq B \subseteq X$, and $A \sim B$, then $A \cup C \sim B \cup C$, for all $C \subseteq X$

   A: Let $\succsim$ be rationalizable under uncertainty. Then there are k utility functions $u_1, ..., u_k$ on X such that $B \succsim A$ if and only if

$$\sum_{i=1}^{k} \max_{x \in A} u_i(x) \leq \sum_{i=1}^{k} \max_{x \in B} u_i(x) \tag{1}$$

   For (a), if $A \subseteq B$, then $\max_A u_i(x) \leq \max_B u_i(x)$ for all i, and hence $B \succsim A$ by (1). For (b), if $A \subseteq B$ then $\max_A u_i(x) \leq \max_B u_i(x)$ for all i. If, moreover,

$$\sum_{i=1}^{k} \max_{x \in A} u_i(x) = \sum_{i=1}^{k} \max_{x \in B} u_i(x)$$

then $\max_A u_i(x) = \max_B u_i(x)$ for all i. Then also $\max_{A\cup C} u_i(x) = \max_{B\cup C} u_i(x)$ for all i, which implies

$$\sum_{i=1}^{k} \max_{x\in A\cup C} u_i(x) = \sum_{i=1}^{k} \max_{x\in B\cup C} u_i(x),$$

as desired.

Let $\succsim$ satisfy (a) and (b). We we show it is rationalizable by *any* single utility function u on X. If $A \subseteq B$, then $\max_A u(x) \leq \max_B u(x)$ as desired. If $A \subseteq B$ and $\max_A u(x) = \max_B u(x)$, then $\max_{A\cup C} u(x) = \max_{B\cup C} u(x)$ as desired.