# On the number of pure Nash equilibria in random $n$-person games*

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**Abstract**

We study pure Nash equilibria in random $n-$player games. Unlike previous studies, we do not restrict the best replies to be single-valued. When all best reply correpondences are equally likely, the probability of at least one pure Nash equilibrium approaches one and the expected number of pure Nash equilibria approaches infinity, when the size of the game becomes large. When the utilities of the players are drawn from a finite set of utility indices, the limit distribution of pure Nash equilibria depends on how fast the set of utility indeces grows relative to the the size of the game. We show that the limit distribution is Poisson with mean that depends on this factor.

Keywords: random games, pure Nash equilibria, $n$ players
JEL: C62, C72

## 1 Introduction

It is of importance to understand how Nash equilibrium behaves as a solution concept, on the "average". A vast literature has analyzed pure Nash equilibria (PNE) in games whose payoffs are drawn randomly from a maximum entropy distribution (see e.g. Stanford 1995a,b, 1996; Powers 1990, Goldberg et al. 1968, Dresher 1970). Of particular interest is the asymptotic distribution of pure Nash equilibria when the size of the game becomes large.

A standard assumption in the literature has been that the payoffs are drawn randomly from a set that is much larger than the finite choice set,

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typically from a continuum. This assumption guarantees that players are never indifferent, and that best responses are always unique. This simplifies the analysis.

However, we feel that the possibility of multiple best responses should be accounted for. First, it can be argued that best reply correspondences are sufficient descriptions of games, at least as far as one is interested in solutions that depend only on best replies. If all best response correspondences are equally likely, then multiple best responses cannot be ruled out (as is observed by looking at examples is standard game theory texts). Second, even if there are aspects in strategic interaction that are not captured by best reply correspondences alone, it is (arguably) not natural to think that one could really choose payoffs from an infinite set of possible payoffs - especially in a model where everything else is finite. This again means that to multiple best responses occur with positive likelihood.

We evaluate the likelihood pure Nash equilibria in random games where multiple best responses are allowed. We study "large" $n$-player matrix games where the size of each player's choice set $K$ approaches infinity. We first show that, when all best reply correspondences are equally likely, the probability of *at least* one pure Nash equilibrium approaches one, and the expected number of pure Nash equilibria approaches infinity.

The situation is more vexed, however, when randomness concerns the underlying utilities. To model this, we let the payoffs of the players be drawn independently from a finite set of utility indices. Letting the cardinality of the set of utilities of player $i$, $f_i(K)$, depend on the size of the game $K$ in such a way that $f_i(K)/K$ approaches some real number $r_i$ as $K$ becomes large, the probability of multiple best responses does not vanish even in the limit.

The standard result when the utility indices are drawn from a continuum is that the distribution of pure Nash equilibria converges to the Poisson distribution with mean 1 as $K$ becomes large. This, however, does not hold when utilities are drawn from a set of $f(K)$ distinct indices. Our main finding is that the limit distribution of pure Nash equilibria converges to Poisson with mean

$$\prod_{i=1}^{n} \left( \frac{1/r_i}{1 - e^{-1/r_i}} \right).$$

Since this number converges to 1 as $r_i$ tends to infinity, our result can be taken as a generalization of the previous findings.

The key challenge is to take into account the probabilistic dependency between Nash equilibria: existence of a Nash equilibrium in a row affects the probability - but does not rule out the possibility - that there is another one in the same row. Tackling this phenomenon requires a combinatorial argument. We show that, in the limit, the dependencies between pure Nash equilibria vanish (with probability one) and they can be treated "as if" they are independent.

It is important to note that our limit results cover also the case where the sizes of the choice sets of the players, say $K_1, ..., K_n$, increase with different speed. So long as the limit of the ratios $f_i(K_i)/K_i$ are well defined for all players, they can be used as the basis of $(r_1, ..., r_n)$ in the above characterization of the limit distribution of pure Nash equilibria. Alternatively, we could draw the utilities from a same set and vary the sizes of the strategy sets, without affecting the qualitative nature of the results.

There are important omissions. Our focus is restricted to pure strategies, finite action sets and independently drawn payoffs. McLennan (1997) allows mixed strategies and Bade et al. (2007) infinite action sets. Rinott and Scarsini (2000) study the case where players' payoffs are dependent.

The paper is organized as follows. The notation is given in Section 2. In the rest of the paper, we characterize the distribution of the number of equilibria and give the expected number of pure Nash equilibria, as the number of pure strategies goes to infinity.

## 2 Preliminaries

There are players $1, ..., n$, playing a $K^n$ matrix game. We assume that player $i$s payoffs are drawn uniformly from the set $\{1/f_i(K), ..., 1\}$, where $f_i(K)$ is a natural number. A *payoff matrix* of player $i$ defines a utility index for all action pairs $(a_1, ..., a_n) \in \{1, ..., K\}^n$

$$U_i = [u_i(a)]_{a \in \{1,...,K\}^n} \,.$$

Given a payoff matrix $U_i$, denote the induced *best response matrix* by

$$B(U_i) = [b(a : U_i)]_{a \in \{1,...,K\}^n}$$

where

$$b(a : U_i) = \begin{cases} 1, & \text{if } u_i(a) \geq u_i(a_i', a_{-i}), \quad \text{for all } a_i' \in \{1, ..., K\}, \\ 0, & \text{otherwise.} \end{cases}$$

Note that at least one element in a row of a best response matrix must be equal to one. If $U_i$ is random, then $B(U_i)$ is random.

Given the $n$ payoff matrices $(U_1, ..., U_n)$, an action profile $a \in \{1, ..., K\}^n$ forms a *pure Nash equilibrium* (PNE) if and only if

$$\prod_{i=1}^{n} b(a : U_i) = 1.$$

Denote by $E_x g(x)$ the expected value of the function $g$ that is dependent on the random variable $x$.

## 3 Random best reply matrices

Two games with the same players and the same strategy sets are *best reply equivalent* if they induce the same best response matrices. Note that all games in the same equivalence class have the same PNE. Given an equivalence class of games, we may take the corresponding best reply matrices (one for each player) as representing this class, since best reply matrices can of course be interpreted as payoff matrices. In this section we draw equivalence classes from a uniform distribution, for each number $K$ of actions. We determine the limit probability that the chosen equivalence class of games has $k$ PNEs as $K$ goes to infinity.

**Proposition 1** *Suppose that a best reply equivalence class is drawn from the uniform distribution over all equivalence classes, given $K$. The probability that a game in this class possesses exactly $k$ PNEs, for $k = 0, 1, ...,$ goes to zero as $K$ goes to infinity.*

**Proof.** Assume first that $f_i(K) = 2$ for all $K$ and for all $i$. Let us call this situation as a $0-1$ game. First we claim that the payoff matrices of a $0-1$ game coincides with probability one with the best response matrices as $K$ becomes large. For this it suffices to show that, when $K$ becomes large, for each action of a player $i$'s best response matrix there is a combination of the other players' actions that induces $i$ a payoff 1, *i.e.,* there is no player $i$ with an action that induces $i$ zero payoff against all actions profiles of the the players. This materializes with probability

$$\left(1 - \frac{1}{2^K}\right)^K.$$

This number goes to 1 as $K$ becomes large. To see this, observe that

$$\left(1 - \frac{1}{2^K}\right)^K = \left[\left(1 - \frac{1}{2^K}\right)^{2^K}\right]^{K/2^K},$$

and that

$$\ln\left(1 - \frac{1}{2^K}\right)^K = \frac{K}{2^K}\ln\left(1 - \frac{1}{2^K}\right)^{2^K}$$

goes to zero as $K$ grows to infinity. This proves the claim.

Now let $f_i(K) \geq 2$ for all $K$ and $i$. By the claim made in the previous paragraph, a game with random best response matrices has $k$ PNEs with the same probability as a game of random $0-1$ payoffs has $k$ PNEs with payoffs $(1, ..., 1)$. Since the utility indices of a $0-1$ game are drawn independently from the set $\{0, 1\}$, an event "strategy $(x, y)$ induces payoffs $(1, ..., 1)$" is a Bernoulli trial that is independent of $x$ and $y$, and has success probability

$1/2^n$. The probability that there exist exactly $k = 0, 1, ...$ such PNEs is binomially distributed, and equals

$$\binom{K^n}{k}\left(1-\frac{1}{2^n}\right)^{K^n-k}\left(\frac{1}{2^n}\right)^k = \frac{K^n!}{(K^n-k)!k!}\left(1-\frac{1}{2^n}\right)^{K^n-k}\left(\frac{1}{2^n}\right)^k.$$

This number goes to zero as $K$ goes to infinity. ■

We end this section by stating two immediate corollaries of the previous result. Suppose that a best reply equivalence class is drawn from the uniform distribution over all equivalence classes, given $K$. Then:

1. The probability that a game in this class possesses at least one PNE goes to one as $K$ goes to infinity.

2. The expected number of PNEs goes to infinity as $K$ goes to infinity.

# 4 Random payoff matrices

In this section, we let the utility indices be the primitive of the model. We assume that, for each $i = 1, ..., n$, there is a nonnegative real number $r_i$ such that

$$\lim_{K\to\infty} \frac{f_i(K)}{K} = r_i.$$

First we observe the following lower bound on the number of pure PNEs in the limit game. When payoffs for agent $i$ are taken from the set $\{1/f_i(K), \ldots, 1\}$, the best possible PNE is the one with payoffs $(1, ..., 1)$. We firstr observe that the distribution of number of such best equilibria is approximately Poisson with mean $1/\Pi_i r_i$ as $K$ becomes large.

**Remark 2** *The number of PNE with payoffs* $(1, ..., 1)$ *is Poisson distributed with mean* $1/\Pi_i r_i$ *as* $K$ *goes to infinity.*

**Proof.** The probability that an action profile $(x, y)$ results in payoffs $(1, ..., 1)$ gets arbitrarily close to $K^{-n}\Pi_i r_i$ as $K$ grows. The probability of payoffs $(1, ..., 1)$ for a given action profile is independent of the realization of the payoffs for other action profiles. Thus the number of action profiles with payoffs $(1, ..., 1)$ is binomially distributed with success probability $K^{-n}/\Pi_i r_i$. The number of trials is $K^n$ and so the mean of this distribution is $K^n \cdot (K^{-n}/\Pi_i r_i) = 1/\Pi_i r_i$. By the well-known approximation result, the limit distribution is Poisson with mean $1/\Pi_i r_i$. ■

As a corollary of the previous remark it follows that the probability of at least one PNE with payoffs $(1, 1)$ converges to $1 - e^{-1/\Pi_i r_i}$ as $K$ becomes large. However, for all $K$ there is also a positive probability that a PNE materializes with payoffs strictly lower than 1. As long as $r_i > 0$, this probability does not vanish when $K$ becomes large, and it needs to be taken into account when evaluating the distribution of PNEs.

Define

$$\bar{\mu}_i := \frac{1/r_i}{1 - e^{-1/r_i}}.$$

Now we can state our main result.

**Proposition 3** *The number of PNEs is Poisson distributed with mean* $\Pi_{i=1}^{n} \bar{\mu}_i$ *as* $K$ *goes to infinity.*

We prove the result via a series of subresults. The proof is by induction on $n$. Let $U_i$ be a $K^n$ payoff matrix of player $i$ in a $K-$game. Denote the proportion of the $K^{n-1}$ action profiles of the other players against which $i$ has $k = 0, ..., K$ distinct best responses by

$$\alpha^K(k : U_i) = \frac{\sum_{a_{-i}} I\left\{\sum_{a_i} b(a_i, a_{-i} : U_i) = k\right\}}{K^{n-1}}, \quad \text{for all } k = 1, ..., K^{n-1}.$$

Similarly, denote the proportion of the $K$ actions of $i$ that are best responses to $k = 0, ..., K$ distinct action profiles of the other players

$$\beta^K(k : U_i) = \frac{\sum_{a_i} I\left\{\sum_{a_{-i}} b(a_i, a_{-i} : U_i) = k\right\}}{K}, \quad \text{for all } k = 0, ..., K.$$

Since it is immaterial whether one counts the total number of best responses on the basis of rows or columns, the average number of $i$'s best responses $\mu_i^K(U_i)$ is satisfies the parity

$$\mu_i^K(U_i) := \sum_{m=1}^{K^{n-1}} \alpha^K(k : U_i) k = \frac{\sum_{k=1}^{K} \beta^K(k : U_i) k}{K^{n-2}}. \tag{1}$$

**Lemma 4** *For any* $i$, $\mu_i^K(U_i)$ *converges to* $\bar{\mu}_i$ *as* $K$ *goes to infinity.*

**Proof.** The probability that the number of player profiles $a_{-i}$ against which $i$ has $m$ best responses is the probability that $m$ actions generate the same payoff $v$ times the probability that all other actions generate lower payoffs, given $v$. Since the distribution over the set $\{1/f(K), 2/f(K), ..., 1\}$ is uniform, we have, under given $K$,

$$\begin{aligned} E_{U_i} \alpha_i^K(m : U_i) &= \sum_{v=1}^{f_i(K)} \binom{K}{m} \left(\frac{1}{f_i(K)}\right)^m \left(\frac{v-1}{f_i(K)}\right)^{K-m} \\ &= \binom{K}{m} \left(\frac{1}{f_i(K)}\right)^m \sum_{x=1}^{f_i(K)} \left(1 - \frac{v}{f_i(K)}\right)^{K-m}, \end{aligned}$$

where the second equality follows by reversing the order of summation. Letting $K$ become large,

$$\begin{aligned}\lim_K E_U \alpha_i^K(m:U_i) &= \lim_K \frac{1}{m!}\left(\frac{K}{f_i(K)}\right)^m \sum_{v=1}^{f_i(K)}\left(1-\frac{v}{f_i(K)}\right)^{K-m} \\ &= \frac{\sum_{v=1}^{\infty} e^{-v/r_i}}{r_i^m m!} \\ &= \frac{e^{-1/r_i}}{(1-e^{-1/r_i})r_i^m m!},\end{aligned}$$

where the second equality follows by taking a component wise limit of the summation. Since best responses of player $i$ against $a_{-i}$ are independently distributed, it follows, by the law of large numbers, that

$$\lim_K \alpha_i^K(m:U_i) = \lim_K E_{U_i}\alpha_i^K(m:U_i), \quad \text{for all } m,$$

almost surely. Thus

$$\begin{aligned}\lim_K \mu_i^K(U_i) &= \lim_K \sum_{m=1}^{f_i(K)} n\alpha_i^K(m:U_i) \\ &= \sum_{m=1}^{\infty} \frac{ne^{-1/r_i}}{(1-e^{-1/r_i})r_i^m m!} \\ &= \frac{e^{-1/r_i}/r_i}{1-e^{-1/r_i}}\sum_{m=1}^{\infty}\frac{1}{r_i^{m-1}(m-1)!} \\ &= \frac{1/r_i}{1-e^{-1/r_i}},\end{aligned}$$

where the final equality follows from noting that $\sum_{n=1}^{\infty}\left[r_i^{n-1}(n-1)!\right]^{-1}$ is a Taylor expansion of $e^{1/r_i}$. ■

Based on Lemma 4, the content of our Proposition 3 is that, in the limit, the expected of number of PNEs is simply Denote the number of PNEs in a $K-$game given payoff matrices $U_1, ..., U_n$ by $\pi_n^K(U_1, ..., U_n)$. Denote

$$\lim E_{U_1,...,U_n}\pi_n^K(U_1, ..., U_n) = \bar{\pi}_n,$$

whenever the limit is well defined.

**Lemma 5** *Assume that the limit $\bar{\pi}_{n-1}$ is well defined. Denote by $\theta^K(k)$ the prior probability that action $a_n$ of player $n$ is a component of $k$ PNEs in the $K$-game. Then*

$$\lim_K K\cdot\theta^K(k) = \begin{cases}\bar{\pi}_{n-1}\bar{\mu}_n, & \text{if } k=1, \\ 0, & \text{if } k>1.\end{cases} \tag{2}$$

**Proof.** For any $K$, fix an arbitrary action $a_n$ of player $n$. Let there be $\ell$ distinct profiles $a_{-n}$ against which $a_n$ is a best response of player $n$. Further, let there be $m$ PNEs in the $n-1$ -player game restricted to the player set $\{1,...,n-1\}$ when player $n$ chooses action $a_n$. Since each allocation of the given $\ell$ and $m$ in the in the set $\{(a_n, a_{-n}) : a_{-n} \in \{1,...,K\}^{n-1}\}$ is equally likely, the number $k \leq \max\{\ell, m\}$ of PNEs in this set is hypergeometrically distributed. The probability $\eta^K(k : \ell, m)$ of there being $k$ PNEs, given $m$ and $\ell$, is

$$\begin{aligned} \eta^K(k : \ell, m) &= \frac{\begin{pmatrix} m \\ k \end{pmatrix}\begin{pmatrix} K^{n-1} - m \\ \ell - k \end{pmatrix}}{\begin{pmatrix} K^{n-1} \\ \ell \end{pmatrix}} \\ &= \frac{m!\ell!}{(m-k)!(\ell-k)!} \frac{(K^{n-1}-\ell)!(K^{n-1}-m)!}{K^{n-1}!(K^{n-1}-m-\ell+k)!}. \end{aligned}$$

It then follows that

$$\lim_K K^{n-1} \cdot \eta^K(k : \ell, m) = \begin{cases} \ell m, & \text{if } k = 1, \\ 0, & \text{if } k > 1. \end{cases} \tag{3}$$

By definition,

$$\theta^K(k) = E_{m,\ell}\eta^K(k : \ell, m). \tag{4}$$

Since the best responses of player $n$ and the PNEs of the game played by players $1,...,n-1$ are independently distributed, $\ell$ and $m$ are independent random variables. The expected value of $m$ is

$$E_{U_1,...,U_{n-1}}\pi^K_{n-1}(U_1,...,U_{n-1}).$$

The share of player $n$'s choices $a_n$ that are best responses against $\ell$ distinct profiles $a_{-n}$ is $\beta^K(\ell : U_n)$. The expected value of $\ell$ is then, by (1),

$$E_{U_n}\sum_{\ell=1}^{K}\beta^K(\ell : U_n)\ell = K^{n-2} \cdot E_{U_i}\mu^K_n(U_n).$$

Since $m$ and $\ell$ are independently distributed, and since the limit $\bar{\pi}_{n-1}$ is well defined, Lemma 4, (4), and (3) imply

$$\begin{aligned} \lim_K K \cdot \theta^K(k) &= \lim_K \frac{K^{n-1} \cdot E_{m,\ell}\eta^K(k : \ell, m)}{K^{n-2}} \\ &= \begin{cases} \lim_K E_{U_1,...,U_{n-1}}\pi^K_{n-1}(U_1,...,U_{n-1}) \cdot E_{U_n}\mu^K_n(U_n), & \text{if } k = 1 \\ 0, & \text{if } k > 1 \end{cases} \\ &= \begin{cases} \bar{\pi}_{n-1}\bar{\mu}_n, & \text{if } k = 1 \\ 0, & \text{if } k > 1 \end{cases} \end{aligned}$$

■

**Lemma 6** *Assume that the limit $\bar{\pi}_{n-1}$ is well defined. The number of PNEs is Poisson distributed with mean $\bar{\pi}_{n-1}\bar{\mu}_n$ as $K$ goes to infinity.*

**Proof.** Recall that $\beta^K(k : U_n)$ is the proportion of actions of player $n$ under $U_n$ that are best responses to $k = 0, 1, ...$ distinct strategy profiles $a_{-n}$ of the other players when the size of the game is $K$. Let $K$ be large so that, by the law of large numbers, $\beta^K(\cdot : U_n)$ is approximately equal to the empirical distribution of a $K$-sequence of independent trials drawn from the distribution $\beta^K(\cdot : U_n)$ itself. This means that the preconditions of Lemma 5 are met by each trial in this $K-$sequence: the probability that *any* action $a_n$ of player $n$ is a component of $k$ PNEs is $\theta^K(k)$. Because of the independency of the trials, the number of choices of player $n$ that are components of $k$ PNEs is binomially distributed with mean $K \cdot \theta^K(k)$. By Lemma 5, this mean is approximately zero for all $k > 1$. That is, for large $K$, the number of PNEs is approximately binomially distributed with mean $K \cdot \theta^K(1)$. By the standard approximation result, the claim now follows from Lemma 5. ■

Finally, we argue by induction that $\bar{\pi}_n = \Pi_{i=1}^n \bar{\mu}_i$, for all $n$. By Lemma 6, this also proves Proposition 2.

**Lemma 7** $\bar{\pi}_n = \Pi_{i=1}^n \bar{\mu}_i$, *for all $n = 1, ...$ .*

**Proof.** The initial step: By the definition of PNE and Lemma 4, the statement of the lemma holds for $n = 1$.

The inductive step: Let, for any $n = 2, ...,$ $\bar{\pi}_{n-1} = \Pi_{i=1}^{n-1} \bar{\mu}_i$. By definition, $\bar{\pi}_n$ is the expected number of PNEs in an $n-$player limit game. Thus, by Lemma 6, $\bar{\pi}_n = \Pi_{i=1}^n \bar{\mu}_i$. ■

Thus, in the limit, the expected number of PNE is

$$\prod_{i=1}^{n} \bar{\mu}_i = \prod_{i=1}^{n} \left( \frac{1/r_i}{1 - e^{-1/r_i}} \right).$$

Since $(1 - e^{-1/r})/r$ is monotonous in $r$, and

$$\begin{aligned} \lim_{r \to 0} \frac{1/r}{1 - e^{-1/r}} &= \infty, \text{ and} \\ \lim_{r \to \infty} \frac{1/r}{1 - e^{-1/r}} &= 1, \end{aligned}$$

it follows that:

1. When payoffs are drawn from a set that is much (infinitely times) larger than the set of choices, the number of pure PNE is Poisson distributed with mean 1 as the set of choices becomes large (cf. Goldberg et al., 1968; Drescher, 1970; Powers, 1990; Stanford, 1995a,b).

2. When payoffs are drawn from a set that is small relative to the size of the game the expected number of PNEs approaches infinity and the probability of at least one PNE approaches one, a result parallel to Remark 2.

3. Adding a new player $n+1$ with $r_{n+1} \in (0, \infty)$ increases the expected number of PNEs. If the parameters $r_1, ..., r_n$ are drawn from a bounded set, then the number of expected PNEs grows exponentially in $n$.

Note also that the ratio between Poisson mean $1/\Pi_i r_i$ in Remark 2 - the lower bound of the expected number of equilibria - and the Poisson mean $\Pi_i \bar{\mu}_i$ in Proposition 3, i.e. $1/\Pi_i(1 - e^{-1/r_i})$ tends to one when all $r_i$s tend to 0, reflecting the fact that when the set of utility indices is small relative the size of the game, most of the PNE are with maximal payoffs.

# 5 A note on the limit game

The natural limit game when $K$ becomes large is the one in which all players have $\mathbb{N} = \{0, 1, \ldots\}$ as their strategy sets. If $f_i(K)$ increases without limit as well, for all $i$, then the uniform distribution over $\{1/f_i(K), \ldots, 1\}$ weakly converges to the uniform distribution over $[0, 1]$. Assume indeed that the strategy sets are $\mathbb{N}$ and payoffs to both players and to each strategy pair are *i.i.d.* draws from the uniform distribution over $[0, 1]$. In this game there are *no* pure Nash equilibria with probability 1. To see this, note that player $i = 1, ..., n$ gets utility strictly less than 1 from every strategy pair with probability 1. Hence a Nash equilibrium $(a_1, ..., a_n)$ should be such that player, say, $i$ gets equilibrium payoff $y < 1$. But with probability one he gets payoff $x > y$ from some other action $a' \neq a_i$. This is one reason why the limit results are of interest: if there were pure Nash equilibria in the limit game, then such an equilibrium might qualify as an approximate solution to a large but finite matrix game.

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