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Nash implementation and the bargaining problem

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Abstract We study Nash implementation of the bargaining solutions in a cake sharing set up. We argue that the minimal Pareto optimal, *symmetric* and Nash implementable SBS is the one inducing all Pareto optimal and *midpoint-dominating* utility vectors in each state.

1 Introduction

Bargaining theory aims at specifying a reasonable compromize to each bargaining *problem*. The compromize is called a bargaining solution. Since Nash (1950), an extensive literature has explored bargaining solutions from the axiomatic basis (for a survey, see e.g. Thomson and Lensberg 1989). The so called *Nash Program* requires that a good solution can not only be motivated axiomatically but it also should be supported in a non-cooperative framework. *Implementation theory* studies the general question of what outcome functions can be implemented non-cooperatively. It is shown by Trockel (2002a) that any support result in Nash Program can be embedded into implementation theory (the "Embedding Principle") (see also Bergin and Duggan 1999, Serrano 1997, 2005; Trockel 2002b). Conversely, if a solution *cannot* be implemented under any meaningful structural assumptions, then it cannot be supported in Nash Program.

We study implementability of bargaining solutions in the canonical cake sharing scenario. Both the players and the planner are aware of the cake sharing structure but only the players know the payoff functions.

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Literature contains many permissive results on implementation of bargaining solutions in subgame perfect Nash equilibrium in a similar set up.[1] The general rule is that practically *any* bargaining solution can be implemented with complex enough extensive form mechanism in subgame perfect Nash equilibrium (see Miyagawa 2002; Vartiainen 2002). In the light of Nash Program, this is slightly frustrating: strategic constraint does not matter much, at least if there is no bound on how complex mechanisms are feasible. Hence, if the strategic constraint is to have a bite, one must assume some restrictions on feasible mechanisms.

In this paper, we focus on normal form mechanisms, and employ the Nash equilibrium as the solution concept. Normal form games are "simple" and do not require too much strategic sophistication from the players' part. We ask what two-person bargaining solutions—or solution correspondences—can be *Nash implemented*.

The set of states consists of all representations of increasing and continuous von Neumann–Morgenstern utility functions over the cake. Random mechanisms are permitted. Since any increasing and continuous utility functions are possible, and lotteries are feasible, the domain of the utility possibility sets consists of all compact, convex, and strictly comprehensive sets.

We prove that any monotonic, Pareto-optimal, and *symmetric* solution necessarily contains the Nash bargaining solution.[2] Since the Nash solution is also shown to be *non-monotonic* (in the Maskin sense), which is a necessary condition for Nash implementation, an impossibility result follows. There is no Pareto-optimal, symmetric, and Nash implementable bargaining solution in the cake sharing problem.[3]

The impossibility result can be avoided by allowing a solution to induce a *multi-valued* set of utility vectors in each state. By using Corollary 3 of Moore and Repullo (1990), it follows that (Maskin) monotonicity is *also* a sufficient condition for implementation of any *strictly individually rational* solution in our context. Hence, it follows that there are multi-valued Pareto-optimal and symmetric solutions which are implementable (e.g., the Pareto-correspondence).

From two set valued bargaining solutions, the smaller one (in the sense of set inclusion) is presumably more desirable since it gives a more accurate prediction of the eventual outcome. The question is whether one can rank multi-valued solutions based on their size. The main result of the paper is to develop a tight characterization of the *minimal* set valued solution.[4] We show that there is a unique minimal Pareto-optimal, symmetric and Nash implementable

[1] Howard (1992) (the Nash solution), Moulin (1984) (the Kalai–Smorodinsky solution), and Anbarci (1993) (the Area monotonic solution).

[2] Benoit and Ok (2005) and Bochet (2005) study Nash implementation via randomized mechanisms when the number of players is at least three.

[3] For another negative result in a different domain, see Serrano (1997). However, in other contexts positive results are feasible, see e.g. Trockel (2000, 2002b), Naeve (1999), and Vartiainen (2005).

[4] Bargaining solution is minimal in a class of solutions if it is contained (as a subset) by all other solutions in this class.

bargaining solution. Such solution consists of all Pareto-optimal and *midpoint dominating* outcomes.

2 Fundamentals

2.1 The set up

Let $\{1,2\}$ be the set of players, with typical elements i and j. There is a perfectly divisible cake of size 1, to be shared among the two players. The set of possible allocations is $A = \{(a_1, a_2) \in \mathbb{R}^2_+ : a_1 + a_2 \leq 1\}$, with a_i denoting a typical share of player i. The sets of players and outcomes are kept fixed throughout.

Let the index set Θ comprise *all* representations of continuous and strictly monotonic von Neumann–Morgenstern preferences on $[0,1]$. That is, each $\theta_i \in \Theta$ can be identified with a continuous and strictly increasing payoff function $v_i(\cdot, \theta_i) : [0,1] \to \mathbb{R}$, and for any continuous and strictly increasing function $w : [0,1] \to \mathbb{R}$ there is $\theta \in \Theta$ such that $w = v_i(\cdot, \theta_i)$. To simplify the exposition we shall focus on the subset Θ_0 of Θ that is defined by the property

$$\Theta_0 = \left\{\theta_i \in \Theta : v_i(0, \theta_i) = 0\right\}.$$

This normalization will not affect the results. The *state space* is the product $\Theta_0 \times \Theta_0$. The prevailing pair $\theta = (\theta_1, \theta_2) \in \Theta_0 \times \Theta_0$ of payoff functions is known by the players but not by outsiders. We customarily use the vector notation $\theta' = (\theta'_1, \theta'_2), \theta'' = (\theta''_1, \theta''_2)$, etc.

Denote by Δ the outcome set, i.e. the set of all probability measures on the Borel sigma algebra of A. Denote the typical elements of Δ by p and q. Abusing the notation, denote by a the degenerate measure in Δ that chooses $a \in A$ with probability one. For any $p \in \Delta$, the expected payoff of $i = 1,2$ under θ is

$$v_i[p, \theta_i] = \int_A v_i(a_i, \theta_i) \mathrm{d}p(a_1, a_2).$$

Write $v[p, \theta] = (v_1[p, \theta_1], v_2[p, \theta_2])$ and, for $B \subseteq \Delta$,

$$v[B, \theta] = \left\{v[p, \theta] \in \mathbb{R}^2 : p \in B\right\}.$$

Denote the *lower contour set* of i at $q \in \Delta$ under θ by

$$L_i(q, \theta_i) = \left\{p \in \Delta : v_i[q, \theta_i] \geq v_i[p, \theta_i]\right\},$$

and the *Pareto-optimal* set of lotteries by[5]

$$P(\theta) = \{p \in \Delta : \text{there is no } q \in \Delta \text{s.t.} v[q,\theta] \geq v[p,\theta]\}.$$

Since $v_i(\cdot,\theta_i)$ is strictly increasing, $p \in P(\theta)$ if and only if there is no $q \in \Delta$ such that $v[q,\theta] > v[p,\theta]$. Or, in other words, $p \in P(\theta)$ if and only if $L_1(p,\theta_1) \cup L_2(p,\theta_2) = \Delta$.

2.2 A bargaining solution and a set valued bargaining solution

Let us take the dispose-all outcome as the *disagreement* outcome $d = (0,0)$ and the utility vector $v[d,\theta] = (0,0)$ as the *status quo* point. By Lyapunov's Theorem [see e.g., Vind (1964), Lemma A], $v[\Delta,\theta]$ is a nonempty, convex, and compact set in $\mathbb{R}^2_+$. Moreover, since v_1 and v_2 are strictly increasing, $v_1(1,\theta) = v_2(1,\theta) = 1$ and $v_1(0,\theta) = v_2(0,\theta) = 0$, the set $v[\Delta,\theta]$ is strictly comprehensive.[6] Conversely, for any nonempty, convex, compact, and strictly comprehensive set $U \subset \mathbb{R}^2_+$, there is $\theta \in \Theta_0 \times \Theta_0$ such that $v[\Delta,\theta] = U$. Thus there is a surjective function from the family $\{v[\Delta,\theta]\}_{\theta\in\Theta_0\times\Theta_0}$ of utility possibility sets to a standard class of *bargaining problems* in $\mathbb{R}^2_+$ (see e.g., Thomson and Lensberg 1989).[7]

A *bargaining solution* is defined in the domain of bargaining problems, and specifies a utility vector for each problem. The solution is defined directly in terms of utilities. However, implementation forces us to concentrate on the underlying physical structure.

A set valued bargaining solution(SBS) is a correspondence $f : \Theta_0 \times \Theta_0 \Longrightarrow \Delta$ such that $f(\theta) = \{p \in \Delta : v[p,\theta] = v[q,\theta] \text{ for some } q \in f(\theta)\}$, for all θ. That is, f contains *all* outcomes that generate a desirable payoff vector. This restriction can be defended by the standard welfarist argument. If a particular outcome has desirable welfare properties, then another outcome generating the same payoffs should be considered equally desirable, and not precluded from the choice set.

However, we are also interested in a more specific form of set valued bargaining solutions. An SBS $f : \Theta_0 \times \Theta_0 \Longrightarrow \Delta$ is a bargaining solution (BS) if $v[f(\theta),\theta]$ is *single valued* for all θ. Thus while an SBS may generate *many* payoff vectors, a BS generates a *unique* payoff vector in each state. To simplify the notation, we customarily identify the set $v[f(\theta),\theta]$ of a BS f with the element that it contains.

We review some of the axioms on bargaining solutions.

Definition 1 (PO) An SBS f is Pareto-optimal if $f(\theta) \subseteq P(\theta)$, for all $\theta \in \Theta_0 \times \Theta_0$.

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[5] Vector inequalities: given $x,y \in \mathbb{R}^2$, $x \geqq y$ means $x_i \geqq y_i$ for $i = 1,2$, $x \geq y$ means $x \geqq y$ and $x \neq y$, and $x > y$ means $x_i > y_i$ for $i = 1,2$.

[6] Set $U \subset \mathbb{R}^2$ is strictly comprehensive if, for all $u,v \in U$, $u \geq v$ implies $w > v$ for some $w \in U$ (see Thomson and Lensberg (1989), p.7).

[7] Note that two utility function pairs can induce the same utility possibility set.

A bargaining problem under θ is *symmetric* if $(u_1, u_2) \in v[\Delta, \theta]$ if and only if $(u_2, u_1) \in v[\Delta, \theta]$.

Definition 2 (SYM) An SBS f is symmetric if there is $q \in f(\theta)$ such that $v_1[q, \theta_1] = v_2[q, \theta_1]$, whenever $\theta \in \Theta_0 \times \Theta_0$ is symmetric.

We want the solution to improve both players' payoffs relative to status quo.

Definition 3 (SIR) An SBS f is *strictly individually rational* if $v[q, \theta] > 0$ for all $q \in f(\theta)$, for all $\theta \in \Theta_0 \times \Theta_0$.

SBS f is *scale invariant* if it is invariant to representation of vNM preferences. To define this formally, denote the coordinatewise product $(\alpha_1 v_1(\cdot, \theta_1), \alpha_2 v_2(\cdot, \theta_2))$ by $\alpha v(\cdot, \theta)$ for any $\alpha = (\alpha_1, \alpha_2) \in \alpha \in \mathbb{R}^2$.

Definition 4 (INV) An SBS f is scale invariant if $\alpha v(\cdot, \theta) = v(\cdot, \theta')$ implies $f(\theta) = f(\theta')$, for all $\alpha \in \mathbb{R}^2_{++}$, for all $\theta, \theta' \in \Theta_0 \times \Theta_0$

Scale invariance can be equivalently stated by saying that f is sensitive only to preferences over *lotteries*. Formally, if, for any pair $\theta, \theta' \in \Theta_0 \times \Theta_0, v[p, \theta] \geqq v[q, \theta]$ implies $v[p, \theta'] \geqq v[q, \theta']$, for all pairs $p, q \in \Delta$, then $f(\theta) = f(\theta')$. To see the equivalence, note that if $\alpha v(\cdot, \theta) = v(\cdot, \theta')$, for some α, then $\alpha v[p, \theta] = v[p, \theta']$, for all p. Thus $v[p, \theta] \geqq v[q, \theta]$ if and only if $v[p, \theta'] \geqq v[q, \theta']$ for all p, q. To see the other direction, suppose that $\alpha v(\cdot, \theta) \neq v(\cdot, \theta')$ for *all* α. Then there must be $a, b \in A$, $i \in \{1, 2\}$, and $\lambda \in (0, 1)$ such that

$$\frac{v_i(a_i, \theta_i')}{v_i(b_i, \theta_i')} = \lambda \neq \frac{v_i(a_i, \theta_i)}{v_i(b_i, \theta_i)}.$$

Construct lottery $p = \lambda \cdot b + (1 - \lambda) \cdot d$. Then $v_i[p, \theta_i'] = v_i[a, \theta_i']$ but $v_i[p, \theta_i] \neq v_i[a, \theta_i]$.

Hence, if f is scale invariant, and there is no *preference reversal* when moving from θ to θ', then the solution should be the same under both preference profiles. Therefore, as equilibrium behavior within a mechanism only depends on the underlying preferences, and as the difference between two preferences manifests itself in a preference reversal, it follows that if a mechanism implements a choice rule, then the rule must be scale invariant. This fact is stated in Lemma 1, below.

2.3 Nash implementation

While it is common knowledge among the players that they know the true state θ (and that they are rational), the only thing that is known by the planner is Θ_0 (and the rationality of players). To implement f, a mechanism must be invoked which reconciles planners' goals with the individual incentives.

A *game form* $\Gamma = (M_1 \times M_2, g)$ consists of *strategy sets* M_1 and M_2, and an *outcome function* $g : M_1 \times M_2 \to \Delta$. Since players' preferences over lotteries, the outcome space, change when moving from one profile to another, a game must be defined by the state θ. Given θ, the pair (Γ, θ) defines a *normal form game,* where player i's real valued payoff function is defined on $M_1 \times M_2$, and satisfies $w_i(\cdot) = v_i[g(\cdot), \theta_i]$.

Let $NE(\Gamma, \theta)$ denote the set of Nash equilibria of (Γ, θ). Then $m \in NE(\Gamma, \theta)$ if and only if

$$w_i(m) \geq w_i(m_i', m_j), \quad \text{for all } m_i' \in M_i \text{ and for all } i \neq j.$$

That is, $m \in NE(\Gamma, \theta)$ if and only if

$$v_i[g(m), \theta_i] \geq v_i[g(m_i', m_j), \theta_i], \quad \text{for all } m_i' \in M_i \text{ and for all } i \neq j.$$

Mechanism Γ is said to *Nash implement* SBS f if, for all $\theta \in \Theta_0 \times \Theta_0$,

$$g(NE(\Gamma, \theta)) = f(\theta).$$

If there is a mechanism Γ which Nash implements SBS f, then f is Nash implementable.

3 Results

3.1 Nash implementation and the bargaining problem

Note that SBS and BS are special cases of more general classes of social choice correspondences. Therefore, general results on the implementability of choice correspondences apply here. Recall the condition by Maskin (1999).

Definition 5 (Maskin monotonicity) SBS f is *Maskin monotonic* if for all pairs $\theta, \theta' \in \Theta_0 \times \Theta_0$ it is true that $p \in f(\theta')$ and $L_i(p, \theta') \subseteq L_i(p, \theta)$, for $i = 1, 2$, implies $p \in f(\theta)$.

In words, if p is chosen by a social choice rule f under θ', and the ranking of p *does not* strictly decrease in either players' preferences when moving from θ' to θ, then f also chooses p under θ. Maskin showed that any Nash implementable social choice correspondence is Maskin monotonic. In our set-up, this condition turns out to be also sufficient.

We are interested in implementing strictly individually rational SBSses. Corollary 3 of Moore and Repullo (1990) provides an operationalizable sufficient condition for Nash implementation (see also Dutta and Sen 1991). It says that any choice rule that satisfies Maskin monotonicity and restricted veto power, and permits the existence of a “bad outcome” is Nash implementable.[8]

[8] Solution f satisfies restricted veto power if i does have the power to veto p that is maximal for $j \neq i$ if $v_i[q, \theta] > v_i[p, \theta]$ for all q in the range of f. A bad outcome is strictly worse for both agents that any outcome in the range of f.

Now if the SBS f is strictly individually rational both players consume a positive share of the cake. Then d can be chosen as the bad outcome and, given that the maximal payoff of player i is generated by he consuming the whole cake, the restricted veto power is also met.[9]

Proposition 1 *Let SBS f satisfy SIR. Then f can be Nash implemented if and only if it is Maskin monotonic.*

By Maskin monotonicity, there has to be a preference reversal if SBS changes from one state to another. Hence, any admissible SBS must be scale invariant.

Lemma 1 *Any Maskin monotonic SBS satisfies INV.*

Proof Suppose that an SBS f is Maskin monotonic. Given θ, take any $\alpha \in \mathbb{R}^2_{++}$, and let θ' satisfy

$$\alpha v(\cdot,\theta) = v(\cdot,\theta').$$

Now, for all $i \in \{1,2\}$,

$$\begin{aligned} L_i(p,\theta_i) &= \{q \in \Delta : v_i[p,\theta_i] \geq v_i[q,\theta_i]\} \\ &= \{q \in \Delta : \alpha_i v_i[p,\theta_i] \geq \alpha_i v_i[q,\theta_i]\} \\ &= \left\{q \in \Delta : v_i[p,\theta_i'] \geq v_i[q,\theta_i']\right\} \\ &= L_i(p,\theta_i'). \end{aligned}$$

By Maskin monotonicity, $p \in f(\theta)$ if and only if $p \in f(\theta')$. Thus f satisfies INV. □

It is clear that the proof applies to the case where the transformations are affine.[10] Since we are interested in Nash implementable solutions, this means that restricting the type space to Θ_0 is, indeed, without loss of generality.

Nash (1950) showed that PO, SYM, INV and the axiom of *independence of irrelevant alternatives* (IIA)[11] determine a unique bargaining solution, the *Nash solution*. Since $\{v[\Delta,\theta]\}_{\theta\in\Theta_0\times\Theta_0}$ coincides with the family of all nonempty convex, compact, and strictly comprehensive utility possibility sets, Nash's result extends to our case.[12]

We now argue that the concepts of Nash solution and Nash implementation are closely linked in the cake sharing context. There are two dimensions where the two concepts interact: first, Lemma 1 means that INV is a necessary condition for Nash implementation. Second, as the next theorem proves, Maskin

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[9] My thanks to a referee for pointing out that the result is implied by Moore and Repullo (1990).

[10] That is, $v(\cdot,\theta') = \alpha v(\cdot,\theta) + \beta$ implies $f(\theta) = f(\theta')$, for any $\theta,\theta' \in \Theta \times \Theta$, for any $\alpha \in \mathbb{R}^2_{++}$ and $\beta \in \mathbb{R}^2$.

[11] The independence of irrelevant alternatives axiom in our context: For all $\theta,\phi \in \Theta_0$, if $v[\Delta,\theta] \subseteq v[\Delta,\phi]$ and $v[f(\phi),\phi] \in v[\Delta,\theta]$, then $f(\theta) = \{p \in \Delta : v[p,\theta] = v[f(\phi),\phi]\}$.

[12] (See e.g., Thomson and Lensberg 1989 Chap. 2).

monotonicity implies IIA and hence also IIA is a necessary condition for Nash implementation. Therefore, by Nash's Theorem, any Nash implementable SBS that satisfies PO and SYM must contain the Nash solution.

More formally, denote by $f^N : \Theta_0 \times \Theta_0 \to \Delta$ the Nash solution where, for each θ,

$$f^N(\theta) = \arg\max_{p \in \Delta} v_1[p, \theta_1] v_2[p, \theta_2]. \tag{1}$$

Hence, f^N is a BS that always maximizes the *Nash product*.

Proposition 2 *Let $f : \Theta_0 \times \Theta_0 \to \Delta$ be a SBS meeting PO and SYM. If f can be Nash implemented, then $f^N(\theta) \subseteq f(\theta)$ for all $\theta \in \Theta_0 \times \Theta_0$.*

Proof Suppose that SBS f meeting PO and SYM can be Nash implemented. We show that this implies that $f^N(\theta) \subseteq f(\theta)$ for all $\theta \in \Theta_0 \times \Theta_0$. This is done by identifying θ' and θ'' such that θ is a linear transformation of θ' and such that since θ' is a Maskin monotonic transformation of θ'' with respect to $q \in f^N(\theta'') \subseteq f(\theta'')$. The latter implies $f^N(\theta') \subseteq f(\theta')$, and the former $f^N(\theta) \subseteq f(\theta)$.

Linear transformation $\theta' \to \theta$:

Let θ' satisfy $v(\cdot, \theta') = \alpha v(\cdot, \theta)$, for $\alpha_i = v_i[f^N(\theta), \theta_i]^{-1}$, and for $i = 1, 2$. Then $v[f^N(\theta'), \theta'] = (1, 1)$. By INV, $f^N(\theta) = f^N(\theta')$ and, by Lemma 1, $f(\theta) = f(\theta')$. Since f^N maximizes the Nash product, and $v[f^N(\theta'), \theta'] = (1, 1)$, set $v[\Delta, \theta']$ is supported by the tangent $u_1 + u_2 = 2$ at $(1, 1)$. To show that $f^N(\theta) \subseteq f(\theta)$, it suffices to find $q \in f^N(\theta')$ such that $q \in f(\theta')$.

Constructing θ'':

Our aim is to find $\theta'' \in \Theta_0 \times \Theta_0$ such that necessarily $q \in f(\theta'')$ and such that θ' is a Maskin monotonic transformation of θ''. Since $v[\Delta, \theta']$ is supported by tangent $u_1 + u_2 = 2$ at $(1, 1)$, there are allocations $a^1, a^2 \in \{(a_1, a_2) : v_1(a_1, \theta'_1) + v_2(a_2, \theta'_2) = 2\}$ such that $v_1(a^1_1, \theta'_1) \geq 1$ and $v_2(a^2_2, \theta'_2) \geq 1$ (see Fig. 1).[13] Construct $\theta''_i \in \Theta_0$ such that, for any $(a_1, a_2) \in A$ and $i \neq j$,

$$v_i(a_i, \theta''_i) = \begin{cases} 2 - v_j(1 - a_i, \theta'_j), & \text{if } a_i \geq a^i_i, \\ v_i(a_i, \theta'_i), & \text{if } a_i < a^i_i. \end{cases} \tag{2}$$

To guarantee that θ''_i is in Θ_0, we have to verify that $v_i(\cdot, \theta''_i)$ meeting (2) is an increasing and continuous function. By construction, $v_i(\cdot, \theta''_i)$ is increasing and continuous on $[0, a^i_i)$ since $v_i(a_i, \theta_i)$ is, and on $(a^i_i, 1]$ since $v_j(1 - a_i, \theta'_j)$ is. To check the continuity of $v_i(\cdot, \theta''_i)$ at a^i_i, note that, by construction, $v_1(a^1_1, \theta'_1) + v_2(a^2_2, \theta'_2) = 2$ and $a^i \in P(\theta)$. Thus $2 - v_j(1 - a^i_i, \theta'_j) = v_i(a^i_i, \theta'_i)$ and $v_i(a^i_i, \theta'_i) = v_i(a^i_i, \theta''_i)$, implying that $v_i(\cdot, \theta''_i)$ is continuous at a^i_i.

SYM and PO:

Identify a lottery q^* with support $\{a^1, a^2\}$ such that $v_i(a^1_i, \theta'_i) q^*(a^1) + v_i(a^2_i, \theta'_i)$ $q^*(a^2) = 1$, for all $i = 1, 2$. By construction, then

$$v_i[q^*, \theta'_i] = v_i[q^*, \theta''_i] = 1. \tag{3}$$

[13] If $v[\Delta, \theta']$ (i.e. $v[\Delta, \theta]$) is strictly convex, then $a^1 = a^2 = a^*$ such that $v(a^*, \theta') = (1, 1)$.

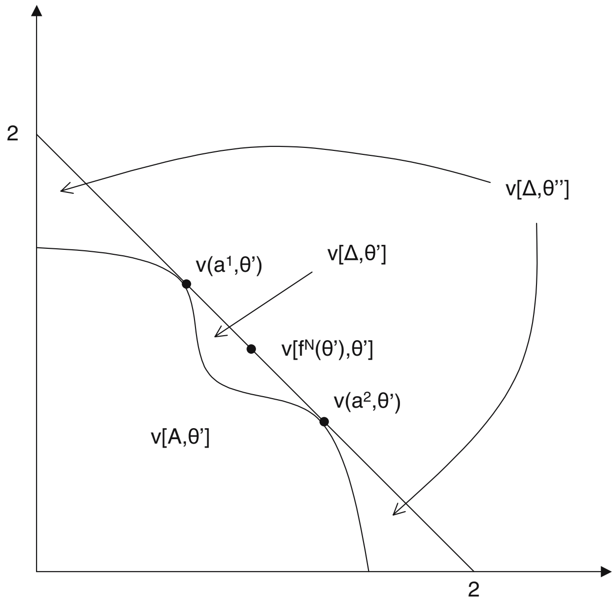

Fig. 1

Since f^N is a SBS, and since $v[f^N(\theta'), \theta'] = (1, 1)$, we have $q^* \in f^N(\theta')$. Moreover, since $v[\Delta, \theta'']$ is the convex hull of points $(0, 2)$, $(2, 0)$ and $(0, 0)$, and since SBS f meets SYM and PO, it follows that also $q^* \in f(\theta'')$, as desired.

Monotonicity of transformation $\theta'' \to \theta'$:

We now show that θ' is a Maskin monotonic transformation of θ''. Let $q^* \in f(\theta')$. First, for any $a_i \in [0, 1]$,

$$v_i(a_i, \theta_i'') - v_i(a_i, \theta_i') = \begin{cases} 2 - v_j(1 - a_i, \theta_j') - v_i(a_i, \theta_i'), & \text{if } a_i \geq a_i^i, \\ 0, & \text{if } a_i < a_i^i. \end{cases} \tag{4}$$

By the convexity of $v[\Delta, \theta']$ and the fact that $v[\Delta, \theta']$ is supported by tangent $u_1 + u_2 = 2$ at $(1, 1)$, it must be that $v_j(1 - a_i, \theta_j') + v_i(a_i, \theta_i') \leq 2$. Thus (4) is non-negative, or $v_i(a_i, \theta_i'') \geq v_i(a_i, \theta_i')$, for all $(a_1, a_2) \in A$. Since p puts non-negative weight on elements of A, also $v_i[p, \theta_i''] \geq v_i[p, \theta_i']$ for any $p \in \Delta$. By (3), then

$$v_i[p, \theta_i''] - v_i[q^*, \theta_i''] \geq v_i[p, \theta_i'] - v_i[q^*, \theta_i'], \qquad \text{for all } p \in \Delta, \text{ for } i = 1, 2. \tag{5}$$

If the left hand side of (5) is non-positive, so is the right hand side. That is,

$$L_i(q^*, \theta_i'') \subseteq L_i(q^*, \theta_i'), \qquad \text{for } i = 1, 2.$$

By Maskin monotonicity, then, $q^* \in f(\theta'')$ implies $q^* \in f(\theta')$. Since also $q^* \in f^N(\theta')$ the proof is completed. □

The proof is based on the following simple - and standard - geometric argument. Take $\theta \in \Theta_0 \times \Theta_0$, and scale it to θ' such that $v[f^N(\theta'), \theta'] = (1,1)$ as depicted in Fig. 1.

Now there are outcomes a^1 and a^2 (not necessarily distinct) in the Pareto frontier such that a lottery over them induces a $(1,1)$-payoff. Then one finds θ_1'' which expands 1's payoffs for all $a_1 \geq a_1^1$, and θ_2'' that expands 2's payoffs for all $a_2 \geq a_2^2$ such that the resulting $v[\Delta, \theta'']$ coincides with the convex hull of points $(0,2)$, $(2,0)$, and $(0,0)$. Pareto-optimality and symmetry then require that outcomes generating $(1,1)$ belong to f under θ'', hence also $f^N(\theta'')$ is contained by $f(\theta)$. Since payoffs are increased when moving from from θ to θ'', it follows by Maskin monotonicity that $f(\theta'')$ must be contained by $f(\theta')$. Thus $f^N(\theta'')$ is contained by $f(\theta')$. A fortiori, since $f^N(\theta'')$ and $f^N(\theta')$ have a common element, $f^N(\theta')$ is contained by $f(\theta')$. Finally, by scale invariance, $f^N(\theta)$ is contained by $f(\theta)$.

Thus, any Pareto-optimal, symmetric, and Maskin monotonic SBS must generate the Nash bargaining payoffs. Is, then, the Nash solution Nash implementable? The next example shows it is not.

Example 1 Take θ_1, θ_2 and θ_2' such that $v_1(a_1, \theta_1) = a_1$ and $v_2(a_2, \theta_2) = a_2$ and $v_2(a_2, \theta_2') = \sqrt{a_2/2}$. Then $(1/2, 1/2) \in f^N(\theta)$ while $\{(2/3, 1/3)\} = f^N(\theta_1, \theta_2')$. Note that, $v_2(1/2, \theta_2) = v_2(1/2, \theta_2') = 1/2$. Moreover,

$$v_2[p, \theta_2] = \int_A a_2 \, \mathrm{d}p(a).$$

By Jensen's inequality,

$$\begin{aligned} v_2[p, \theta_2'] &= \int_A \sqrt{\frac{a_2}{2}} \mathrm{d}p(a) \\ &\leq \sqrt{\frac{\int_A a_2 \mathrm{d}p(a)}{2}} \\ &= \sqrt{\frac{v_2[p, \theta_2]}{2}}. \end{aligned}$$

Hence, if $v_2[p, \theta_2] \leq 1/2$ then $v_2[p, \theta_2'] \leq 1/2$. This implies that if $p \in L_2(1/2, 1/2, \theta_2)$ then $p \in L_2(1/2, 1/2, \theta_2')$. However, $(1/2, 1/2) \in f^N(\theta) \backslash f^N(\theta_1, \theta_2')$. Thus f^N violates Maskin monotonicity.

Thus, the Nash bargaining solution *is not* Nash implementable on the domain $\Theta_0 \times \Theta_0$.[14] Maskin monotonicity is too restrictive when players' risk attitudes

[14] For example, see Howard (1992) and Serrano (1997).

vary (cf. Kihlström et al. 1981). In fact *any* risk sensitive BS fails Maskin monotonicity.[15] By Proposition 2 and Example 1, we have the following negative result.

Corollary 1 *There is no Nash implementable BS meeting PO and SYM on* $\Theta_0 \times \Theta_0$.

3.2 Minimal set valued bargaining solution

Ideally, a solution induces a unique payoff vector, i.e. is a BS. One way to go around the impossibility result in Corollary 1 is to allow the solution to be multi-valued SBS. It is clear that there are large enough symmetric and Pareto-optimal SBSs that are Nash implementable (take the Pareto correspondence $f(\cdot) = P(\cdot)$). We now look for the SBSs that are Nash implementable and contain the *least* amount of indeterminacy. The minimality property below captures our desideratum.

Definition 6 **(**Minimal SBS) Let $\mathcal{F}$ be a family of SBSs. Then $f \in \mathcal{F}$ is *minimal* in $\mathcal{F}$ if $f(\theta) \subseteq g(\theta)$ for all $g \in \mathcal{F}$, for all $\theta \in \Theta_0 \times \Theta_0$.

Minimality weakens in a natural way a desire to have a single valued payoff vector in each state. If a set valued SBS exists having desired properties, and this solution is minimal in the class of all SBSs having these properties, then the solution would be a natural choice by an ambiguity averse planner: a minimal solution, if it exists, contains the least amount of indeterminacy.

The appealing properties that we want a SBS to have are Pareto-optimality, symmetry and Nash implementability, given the domain Θ_0 of utility functions. Define

$$\mathcal{F}^* = \left\{f : \Theta_0 \times \Theta_0 \Rightarrow \Delta : f \text{ is PO, SYM and Nash implementable}\right\}.$$

Since the Pareto correspondence is Nash implementable, symmetric, and Pareto-optimal, $\mathcal{F}^*$ is nonempty.

A *midpoint* gives the winner of a coin toss the right to choose an outcome. The property of *midpoint domination* requires that the solution should never be dominated by the midpoint. More formally, denote by e the lottery that awards the cake to 1 with probability 1/2 and to 2 with probability 1/2. That is,

$$e = \frac{1}{2} \cdot \delta_{(1,0)} + \frac{1}{2} \cdot \delta_{(0,1)},$$

where δ_a is the Dirac measure on $a \in A$.

[15] Risk sensitive solution gives a *smaller* share to a player whose payoff function becomes *more concave*.

Definition 7 **(**Midpoint domination) An SBS f satisfies *midpoint domination* if, for all $\theta \in \Theta_0 \times \Theta_0$,

$$f(\theta) \subseteq M(\theta) = \left\{p \in \Delta : v[p,\theta] \geqq v[e,\theta]\right\}.$$

An intuitive motivation for condition of the midpoint domination is that a player may justifiably object an outcome by demanding the midpoint instead. Salonen (85) (see also Sobel 1981) studies bargaining solutions with the condition. He shows that most of the solutions proposed in the literature satisfy it. In particular, in the class of convex, compact and comprehensive problems, the Nash solution meets the midpoint dominance.

Proposition 3 *An SBS f is minimal in $\mathcal{F}^*$ if and only if $f(\theta) = P(\theta) \cap M(\theta)$, for all $\theta \in \Theta_0 \times \Theta_0$.*

That $P \cap M$ belongs to $\mathcal{F}^*$ follows from Proposition 1. The minimality of $M \cap P$ in $\mathcal{F}^*$ is based on fact that whichever midpoint dominating, Pareto-optimal outcome p^* we choose under θ, we can find another state θ'' that spans a symmetric utility possibility set such that p^* belongs by PO and SYM to the SBS, and such that we can monotonically transform θ'' to θ' and linearly θ' to θ. It then follows that p^* must belong to the choice set of the Nash implementable SBS that symmetric, and Pareto-optimal SBS.

Figure 2a, b, c clarify the situation. Let $v_i(1,\theta) = 2$ for all i. Choose any $p^* \in P(\theta) \cap M(\theta)$. Then find θ', the scaled version of θ, such that $v[p^*,\theta'] = (1,1)$. That is, $v[p^*,\theta'] = v[e,\theta]$. The new utility set is supported at $(1,1)$ by a tangent T with slope $-c$, for $c \in [0,1]$ (depending on the choice of axis, see Fig. 2a). Then we find, θ_2'' that expands, say, 2's payoffs such that all divisions $(1-a_2, a_2)$ in the range $a_2 \geq 1 - b_1$ induce payoffs in the line $((0,2),(2,0))$ (see Fig. 2b). By construction $L_2(p^*,\theta_2'') \subseteq L_2(p^*,\theta_2')$. Finally, we find θ_1'' that changes 1's payoffs such that all $(a_1, 1-a_1)$ in the range $a_1 \geq b_1$ also induce payoffs in the line $((0,2),(2,0))$ (see Fig. 2c). Since $v_1[p^*,\theta_1'] = v_1[p^*,\theta_1'']$ it now follows that $L_1(p^*,\theta_1'') \subseteq L_1(p^*,\theta_1')$. This follows from the fact that $v[\cdot,(\theta_1',\theta_2'')]$ lies below the tangent T which crosses the line $((0,2),(2,0))$ at $(1,1)$. Since $p^* \in f(\theta'')$ by Proposition 2, the desired result is now implied by Maskin monotonicity (first when moving from θ'' to (θ_1',θ_2'') and then from (θ_1',θ_2'') to θ'), and scale invariance (when moving from θ' to θ).

Proof First we prove that if $f(\theta) = P(\theta) \cap M(\theta)$ for all θ, then f is Nash implementable and, hence, $f \in \mathcal{F}^*$. By construction, $f(\theta)$ satisfies SIR. By Proposition 1, it suffices that f is Maskin monotonic.

Take any θ, θ' and $q \in f(\theta)$. Suppose that $L_i(q,\theta_i) \subseteq L_i(q,\theta_i')$ for $i = 1,2$. We show that $q \in f(\theta')$. First, since $q \in P(\theta)$ it follows that $L_1(q,\theta_1) \cup L_2(q,\theta_2) = \Delta$. Thus also $L_1(q,\theta_1') \cup L_2(q,\theta_2') = \Delta$ and, consequently, $q \in P(\theta')$. Second, since $q \in M(\theta)$ it follows that $e \in L_1(q,\theta_1) \cap L_2(q,\theta_2)$. Thus also $e \in L_1(q,\theta_1') \cap L_2(q,\theta_2')$ and, consequently $q \in \left\{p \in \Delta : v[p,\theta'] \geqq v[e,\theta']\right\} = M(\theta')$. Collecting the results, $q \in P(\theta') \cap M(\theta')$, as desired.

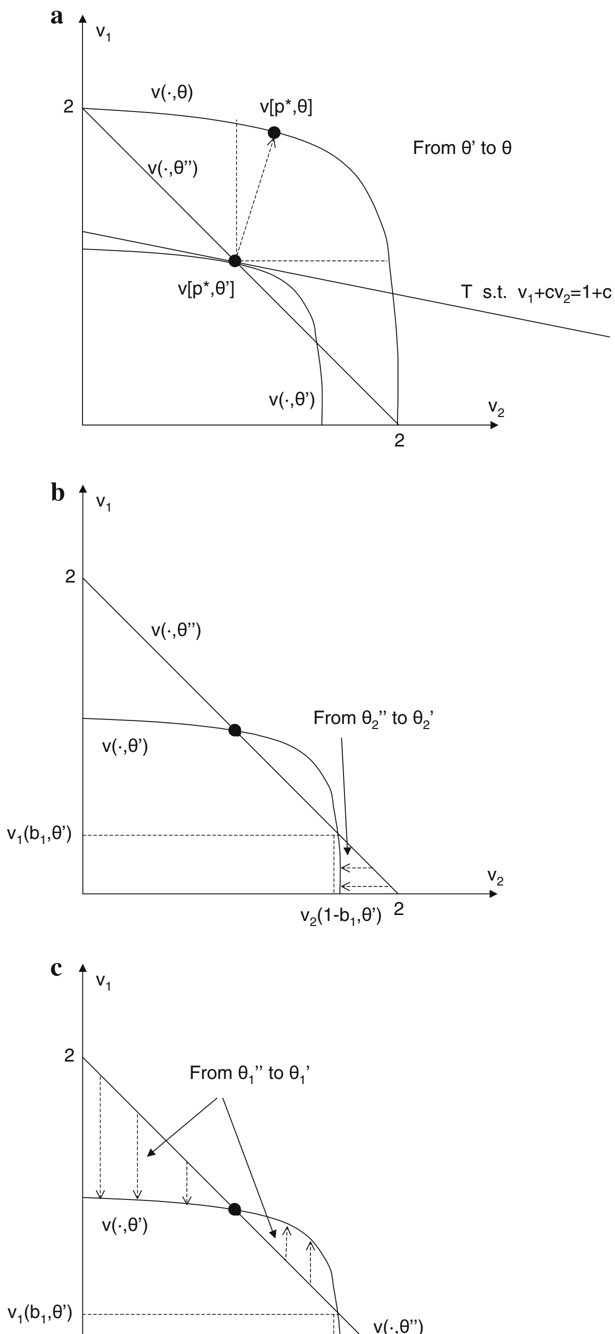
a
v1
v(·,θ)
2
v[p*,θ]
From θ' to θ
v(·,θ'')
v[p*,θ']
T s.t. v1+cv2=1+c
v(·,θ')
v2
2
b
v1
2
v(·,θ'')
From θ2'' to θ2'
v(·,θ')
v1(b1,θ')
v2
v2(1-b1,θ')
2
c
v1
2
From θ1'' to θ1'
v(·,θ')
v1(b1,θ')
v(·,θ'')
v2
v2(1-b1,θ')
2

Fig. 2

Now we prove that if f is Nash implementable, and satisfies SYM and PO, then $P(\theta) \cap M(\theta) \subseteq f(\theta)$ for all θ, and hence f is minimal in $\mathcal{F}^*$. Take any θ and $p^* \in P(\theta) \cap M(\theta)$. We find θ'' such that necessarily $p^* \in f(\theta'')$. Then we conduct two successive Maskin monotonic transformations of payoffs to θ'. Finally we make a linear transformation of payoffs from θ' to θ. To keep track on transformations, they are done in the reversed order.

Linear transformation $\theta' \to \theta$:

Without loss of generality, assume $v_i(1, \theta_i) = 2$ for $i = 1, 2$. By construction, there is $\alpha_i \in (0, 1]$ and $\theta_i' \in \Theta_0$ such that $\alpha_i v_i[\cdot, \theta_i] = v_i[\cdot, \theta_i']$, and $v_i[p^*, \theta_i'] = 1$, for all i. Then

$$v_i(1, \theta_i') = 2\alpha_i \leq 2, \quad \text{for all } i = 1, 2. \tag{6}$$

Constructing θ_2'' :

Since $p^* \in P(\theta')$ and $v[p^*, \theta'] = (1, 1)$. $v[\Delta, \theta']$ is supported by a tangent $u_i + cu_j = 1 + c$, for some $i \neq j$, and some $c \in [0, 1]$. Assume, without loss of generality, that $i = 1$ and $j = 2$. Then,

$$v_1(a_1, \theta_1') + cv_2(1 - a_1, \theta_2') \leq 1 + c, \quad \text{for all } a_1 \in [0, 1]. \tag{7}$$

Find a $b_1 \in [0, 1]$ such that (see Fig. 2a)

$$b_1 = \min_b \{b : v_1(b, \theta') + v_2(1 - b, \theta') = 2\}.$$

Since $v_1(\cdot, \theta_1')$ and $v_2(\cdot, \theta_1')$ are continuous functions, and $v_1[p^*, \theta'] + v_2[p^*, \theta'] = 2$, such b_1 exists. Since $v[\Delta, \theta']$ is a convex set, and (7) supports $v[\Delta, \theta']$ at $(1, 1)$, we have

$$v_1(b_1, \theta') \leq 1 \leq v_2(1 - b_1, \theta'). \tag{8}$$

Find $\theta_2'' \in \Theta_0$ such that, for any $a_1 \in [0, 1]$,

$$v_2(1 - a_1, \theta_1'') = \begin{cases} 2 - v_1(a_1, \theta_1'), & \text{if } a_1 \leq b_1, \\ v_2(1 - a_1, \theta_2'), & \text{if } a_1 > b_1. \end{cases}$$

Function $v_2(\cdot, \theta_2'')$ is increasing and continuous since $v_1(\cdot, \theta_1')$ and $v_2(\cdot, \theta_2')$ are, and since $v_1(b_1, \theta') = 2 - v_2(1 - b_1, \theta')$.

Monotonicity of the transformation $\theta_2'' \to \theta_2'$:

Suppose $p \in L_2(p^*, \theta_2'')$. Since $v_2(a_2, \theta_2'') \geq v_2(a_2, \theta_2')$ for all a_2, also $v_2[p, \theta_2''] \geq v_2[p, \theta_2']$, for any $p \in \Delta$. Hence,

$$1 - v_2[p, \theta_2'] \geq 1 - v_2[p, \theta_2''], \qquad \text{for all } p \in \Delta. \tag{9}$$

From (9) it is clear that $1 - v_2[p, \theta_2''] \geq 0$ implies $1 - v_2[p, \theta_2'] \geq 0$. Since necessarily $v_2[p^*, \theta_2''] = v_2[p^*, \theta_2'] = 1$ (see Fig. 2b), it follows that $p \in L_1(p^*, \theta_1'')$ implies $p \in L_1(p^*, \theta_1')$, for any p. That is,

$$L_2(p^*, \theta_2'') \subseteq L_2(p^*, \theta_2').$$

Constructing θ_1'':
Construct $\theta_1'' \in \Theta_0$ such that

$$v_1(a_1,\theta_1'') = \begin{cases} 2 - v_2(1-a_1,\theta_2'), & \text{if } a_1 \geq b_1, \\ v_1(a_1,\theta_1'), & \text{if } a_1 < b_1. \end{cases} \tag{10}$$

Function $v_1(\cdot,\theta_1'')$ is increasing and continuous since $v_1(\cdot,\theta_1')$ and $v_2(\cdot,\theta_2')$ are, and since $2 - v_2(1-b_1,\theta_2') = v_1(b_1,\theta_1')$. By (10),

$$\begin{aligned} v_1[p,\theta_1''] &= \int_{\{a:a_i\geq b_1\}} v_1(a_1,\theta_1'')\mathrm{d}p(a) + \int_{\{a:a_i<b_1\}} v_1(a_1,\theta_1'')\mathrm{d}p(a) \\ &= \int_{\{a:a_i<b_1\}} v_1(a_1,\theta_1')\mathrm{d}p(a) + \int_{\{a:a_i\geq b_1\}} [2 - v_2(1-a_1,\theta_2')]\mathrm{d}p(a). \end{aligned}$$

Thus,

$$1-v_1[p,\theta_1''] = \int_{\{a:a_i<b_1\}} [1-v_1(a_1,\theta_1')]\mathrm{d}p(a) - \int_{\{a:a_i\geq b_1\}} [1-v_2(1-a_1,\theta_2')]\mathrm{d}p(a). \tag{11}$$

By (7),

$$\begin{aligned} v_1[p,\theta_1'] &= \int_{\{a:a_i<b_1\}} v_1(a_1,\theta_1')\mathrm{d}p(a) + \int_{\{a:a_i\geq b_1\}} v_1(a_1,\theta_1')\mathrm{d}p(a) \\ &\leq \int_{\{a:a_i<b_1\}} v_1(a_1,\theta_1')\mathrm{d}p(a) + \int_{\{a:a_i\geq b_1\}} [1 + c - cv_2(1-a_1,\theta_2')]\mathrm{d}p(a). \end{aligned}$$

After some manipulation,

$$1 - v_1[p,\theta_1'] \geq \int_{\{a:a_i<b_1\}} [1 - v_1(a_1,\theta_1')]\mathrm{d}p(a) - c \int_{\{a:a_i\geq b_1\}} [1 - v_2(1-a_1,\theta_2')]\mathrm{d}p(a). \tag{12}$$

Lemma 3.1 $v_1[p^*,\theta_1'] = v_1[p^*,\theta_1''] = 1$.
For proof, see the appendix.
Monotonicity of the transformation $\theta_1'' \to \theta_1'$:
Suppose that $p \in L_1(p^*,\theta_1'')$, for any p. By Lemma 3.1,

$$1 - v_1[p,\theta_1''] \geq 0. \tag{13}$$

To obtain $p \in L_1(p^*,\theta_1')$ it suffices, by Lemma 3.1, that (12) is nonnegative. By (8), $1 \geq v_1(a_1,\theta_1')$ for all $a_1 < b_1$ which means that the first term in the right-hand-side of (12) is nonnegative. There are two subcases. If

$$\int_{\{a:a_i \geq b_1\}} [1 - v_2(1 - a_1, \theta_2')]\mathrm{d}p(a) \leq 0,$$

then both terms in (12) are nonnegative, and hence

$$1 - v_1[p, \theta_1'] \geq 0.$$

Conversely, if

$$\int_{\{a:a_i \geq b_1\}} [1 - v_2(1 - a_1, \theta_2')]\mathrm{d}p(a) > 0,$$

then (13), (11), and (12) imply, since $c \in [0, 1]$, that

$$\begin{aligned} 1 - v_1[p, \theta_1'] &\geq \int_{\{a:a_i < b_1\}} [1 - v_1(a_1, \theta_1')]\mathrm{d}p(a) - \int_{\{a:a_i \geq b_1\}} [1 - v_2(1 - a_1, \theta_2')]\mathrm{d}p(a) \\ &= 1 - v_1[p, \theta_1''] \\ &\geq 0. \end{aligned}$$

Thus $1 - v_1[p, \theta_1''] \geq 0$ implies $1 - v_1[p, \theta_1'] \geq 0$. Since p is arbitrarily chosen, we have

$$L_1(p^*, \theta_1'') \subseteq L_1(p^*, \theta_1').$$

Maskin monotonicity:

Since transformation $\theta_1'' \to \theta_1'$ is monotonic, if $p^* \in f(\theta'')$, then $p^* \in f(\theta_1', \theta_2'')$. Since transformation $\theta_2'' \to \theta_2'$ is monotonic, if $p^* \in f(\theta_1', \theta_2'')$, then $p^* \in f(\theta')$. Thus if $p^* \in f(\theta'')$, then $p^* \in f(\theta')$.

Nash solution:

Since $v[\Delta, \theta'']$ is the convex hull of points $(0, 2)$, $(2, 0)$ and $(0, 0)$, and $v[p^*, \theta''] = (1, 1)$, we have $p^* \in f^N(\theta'')$. By Proposition 2, $p^* \in f(\theta'')$. By Maskin monotonicity, $p^* \in f(\theta')$. Since transformation $\theta' \to \theta$ is linear, by Lemma 1 and INV, $p^* \in f(\theta)$. □

If a planner does not know players' payoff functions, then the best she can do is to choose from the Pareto-optimal and midpoint dominating set of lotteries. This is to be contrasted against the well known impossibility results in the implementation literature.[16] What is novel is the explicit expression of the minimal implementable, Pareto-optimal and symmetric choice rule.

[16] E.g. Hurwicz and Schmeidler (1978) prove the impossibility of Nash implementing two-player Pareto-optimal social choice *function*.

4 Conclusion

We focus on a cake sharing problem, and argue that Maskin monotonicity is the main obstacle in Nash implementing interesting bargaining solutions. In particular, there are no Pareto-optimal and symmetric, single valued bargaining solutions that can be Nash implemented. A natural way to Nash implement a Pareto-optimal and symmetric solution is to *extend* the solution, to allow it contain many payoff vectors in a given state. The main contribution of the paper is to identify *the* minimal set valued bargaining Pareto-optimal, symmetric, and Nash implementable solution in the domain of all increasing and continuous utility functions. Such solution implements the set of Pareto-optimal and mid-point dominating outcomes in each state.

The impossibility of implementing a single valued solution crucially depends on the generality of the domain of preferences, i.e. that all combinations of continuously increasing vNM utility function over the cake are possible. In some scenarios, however, one may be able to rule out some combinations. In such case, more permissive results become possible. Trockel (2002b) (see also van Damme 1986) shows that the Nash solution can be implemented in a meta game where the players bargain over solutions (see also Naeve 1999). Trockel (2000) implements the Nash solution in "Walrasian equilibrium".

The analysis here can be extended beyond the cake sharing structure. The obvious extension is towards the general "economic environment", of which the current, one good scenario is a special case. In the general case *also* preferences over pure outcomes may reverse. This means that, since more preference reversals is potentially available, at least as many solutions are Nash implementable. However, since Leontief preferences over many goods are isomorphic to preferences over one good, not more solutions can be implemented if the domain allows all convex, continuous preferences over bundles. Thus, all our results would hold unchanged.

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A Appendix

Proof Lemma 3.1 By construction, $1 = v_1[p^*, \theta_1']$. If p^* is degenerate, then, also by construction, $1 = v_1[p^*, \theta_1'']$. If p^* is a non-degenerate measure, then, since p^* is in $P(\theta')$, and tangent $T = \{(u_1, u_2) : u_1 + cu_1 = 1 + c\}$ supports $v[\Delta, \theta']$ at $(1, 1)$, the support of p^* must be in T. Thus

$$
\begin{aligned}
0 &= 1 - v_1[p^*, \theta_1'] \\
&= \int\limits_{\{a:v_1(a_1,\theta_1')\geq 1\}} [1 - v_1(a_1, \theta_1')]\mathrm{d}p^*(a) + \int\limits_{\{a:v_1(a_1,\theta_1')\leq 1\}} [1 - v_1(a_1, \theta_1')]\mathrm{d}p^*(a) \\
&= c \int\limits_{\{a:v_1(a_1,\theta_1')\geq 1\}} [1 - v_2(1 - a_1, \theta_2')]\mathrm{d}p^*(a) \\
&\quad + c \int\limits_{\{a:v_1(a_1,\theta_1')\leq 1\}} [1 - v_2(1 - a_1, \theta_2')]\mathrm{d}p^*(a),
\end{aligned}
$$

where the third equality follows from the definition of T. Thus, dividing by c,

$$
\begin{aligned}
0 &= \int\limits_{\{a:v_1(a_1,\theta_1')\geq 1\}} [1 - v_2(1 - a_1, \theta_2')]\mathrm{d}p^*(a) + \int\limits_{\{a:v_1(a_1,\theta_1')\leq 1\}} [1 - v_2(1 - a_1, \theta_2')]\mathrm{d}p^*(a) \\
&= \int\limits_{\{a:v_1(a_1,\theta_1')\geq 1\}} [1 - v_1(a_1, \theta_1'')]\mathrm{d}p^*(a) + \int\limits_{\{a:v_1(a_1,\theta_1')\leq 1\}} [1 - v_1(a_1, \theta_1'')]\mathrm{d}p^*(a) \\
&= 1 - v_1[p^*, \theta_1''],
\end{aligned}
$$

where the second equality from (10), and the definition of b_1. □

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