Int J Game Theory
DOI 10.1007/s00182-009-0212-3

ORIGINAL PAPER

Multilateral non-cooperative bargaining in a general utility space

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Accepted: 11 November 2009

Abstract We consider an n-player bargaining problem where the utility possibility set is compact, convex, and stricly comprehensive. We show that a stationary subgame perfect Nash equilibrium exists, and that, if the Pareto surface is differentiable, all such equilibria converge to the Nash bargaining solution as the length of a time period between offers goes to zero. Without the differentiability assumption, convergence need not hold.

Keywords Multilateral · Bargaining · General utility set

JEL Classification C7 · D7

1 Introduction

One of the most celebrated observations in the bargaining literature is by Binmore et al. (1986), who show that the unique subgame perfect equilibrium outcome of the two-player alternating offers bargaining game à la Rubinstein (1982) converges to the

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Published online: 11 December 2009

Nash bargaining solution (Nash 1950) when frictions vanish, i.e., when the length of time period that it takes to make offers and counter offers goes to zero.

We study how far these results extend to a multiplayer setting. To do this, we employ the natural multiplayer version of the alternating offers bargaining game by Binmore (1985); Herrero (1985), and Shaked (as reported by Osborne and Rubinstein 1990).[1]

Vastness of equilibria is a well known problem of multiplayer bargaining games. To circumvent the problem, we shall focus on equilibria in *stationary* strategies. It is known that under stationarity restriction convergence does hold (see Sutton 1986) when bargaining concerns linear, one dimensional cake.[2] Our question is whether convergence holds if we relax the restrictions on the underlying physical structure.

While in the two-player case the underlying physical structure is immaterial, in the multiplayer case it is not. In fact, when there are at least three players, the one-cake restriction makes many of the interesting trade-offs between players immaterial. We prove via an example that convergence of stationary equilibria to the Nash solution need not hold when the utility domain is only asked to meet the more liberal assumptions made in the cooperative bargaining literature, i.e. that the payoffs are drawn from a compact, convex and comprehensive utility possibility set.[3] Such more general utility set can be generated e.g. by multiplicity of goods or consumption externalities. Our aim is to delineate conditions under which convergence does take place in such a utility set.

We start by showing that a stationary subgame perfect equilibrium exists in any compact, convex, and comprehensive utility possibility set. To our knowledge, this is the most general existence result concerning stationary bargaining equilibria in the literature.[4] Our main result is that by slightly restricting the feasible set, the convergence result can be saved: if the Pareto surface is *differentiable*, then all stationary equilibria converge to the Nash bargaining solution.[5]

A distinctive feature of our analysis is—since no specific correspondence need be determined between payoffs and the physical environment—that a closed form representations of strategies and payoffs cannot be obtained. Instead, the results have to be derived via indirect means. At the heart of the proof of the convergence is a novel dimensionality argument.[6]

[1] Players are ordered into a circle. A player (say 1) proposes an outcome. If all other players accepts the proposal, then it is implemented. Otherwise, the next player (say 2) in the order makes an offer. The play continues this way until an offer is accepted.

[2] Chatterjee and Sabourian (2000) show that the stationarity restriction is without loss of generality if players are complexity averse.

[3] Thomson and Lensberg (1989) is an authorative reference.

[4] Recently, and independently of this study, Herings and Predtetchinski (2007) have obtained the same convergence result when the recognition of an offering player follows a Markov process.

[5] We thank an associate editor for suggesting this weakening of our original restriction.

[6] Kultti and Vartiainen (2007) develop a similar argument in a different context.

There are many papers on n-player non-cooperative bargaining games. Herrero (1985) and Sutton (1986) demonstrate the difficulties in extending the analysis of two-player bargaining to the general $n-$player case. The convergence to the Nash bargaining solution typically requires that the bargaining game has a unique subgame perfect equilibrium. Various examples of such games are available in Chae and Yang (1988, 1994); Huang (2002); Krishna and Serrano (1996), and Suh and Wen (2006).

2 The set up

There is a set $\{1, \ldots, n\}$ of players and a nonempty, compact, convex and strictly comprehensive utility possibility set $U \subset \mathbb{R}^n_+$.[7,8] A vector of utilities is denoted by $u = (u_1, \ldots, u_n)$, or $u = (u_i, u_{-i})$. The (weak and strong) *Pareto frontier* of U is then defined by $P \coloneqq \{u \in U : v \not> u, \text{ for all } v \in U\}$.

Delay is costly: The present value of player i's next period utility u_i is $\delta^{\Delta} u_i$, where $0 < \delta < 1$ is the common discount factor, and $\Delta > 0$ is the length between two stages.

We study a class of unanimity bargaining games, defined as follows.

- At stage $t = 0, 1 \ldots$ player $i = (t+1) \bmod n$ makes an offer $v \in U$ and the players $j \neq i$ accept or reject the offer in the ascending order of their index.
- If all $j \neq i$ accept, then v is implemented. Otherwise the game moves to stage $t + 1$.

We concentrate on the *stationary* (subgame perfect Nash) equilibria of the game, where:

1. Each i's proposal is dependent only on the continuation game.
2. Each i's acceptance decision in period t depends only on the offer on the table and the continuation game.

Define

$$\rho(i, j) = \begin{cases} i - j, & \text{if } i \geq j \\ i - j + n, & \text{if } i < j. \end{cases}$$

Given the stationarity assumption, player j always offers u^j when it is his turn to make an offer. Since player j's offer $v = (v_1, \ldots, v_n)$ is accepted if

$$v_i \geq \delta^{\Delta\rho(i,j)} u^i_i, \text{ for all} i \neq j,$$

the equilibrium offer u^j satisfies,

$$u^j_i = \delta^{\Delta\rho(i,j)} u^i_i, \text{ for all } i, j. \tag{1}$$

[7] Vector notation: $x \geqq y$ if $x_i \geq y_i$ for all i, $x \geq y$ iff $x \geqq y$ and not $x_i = y_i$ for all i, and $x > y$ iff $x_i > y_i$ for all i.

[8] $X \subset \mathbb{R}^k_+$ is comprehensive if $x \in X$ and $x \geq y \geq 0$ imply $y \in X$. It is strictly comprehensive if $x \not> y$ for all $x \in X - \{y\}$ implies $x \not\geq y$ for all $x \in X - \{y\}$.

Thus an equilibrium is characterized by a profile $(u^1, \ldots, u^n) \in U^n$, where each $u^i = (u_1^i, \ldots, u_n^i) \in U$ specifies all players' payoffs when it is i's turn to make an offer. We conclude that $(u^1, \ldots, u^n)$ is an equilibrium profile if and only if it meets (1) for all $i, j \in \{1, \ldots, n\}$.

Theorem 1 *A stationary equilibrium exists.*

Proof Assume $\Delta = 1$. Let U^n be the n-copy of the utility set U. Denote a typical element of U^n by $(u^1, \ldots, u^n)$. For any i, j, define function $g_j^i : U \to \mathbb{R}_+$ such that

$$g_j^i(u^j) := \delta^{\rho(j,i)} \max\{u_j : (u_j, (u_k^j)_{k \neq j}) \in U\}. \tag{2}$$

By the compactness of U, g_j^i is well defined and, by the convexity of U, it is continuous. For each i, let $g^i(\cdot) := g_j^i(\cdot))_{j=1}^n : U^n \to \mathbb{R}_+^n$. Define function $\xi^i : U^n \to \mathbb{R}_+$ such that

$$\xi^i(u^1, \ldots, u^n) := \max\{x \in \mathbb{R} : x g^i(u^1, \ldots, u^n) \in U\},$$
$$\text{for all } (u^1, \ldots, u^n) \in U^n.$$

By the compactness of U^n, also ξ^i is well defined, and by the convexity of U^n, it is continuous. Construct a function $h^i : U^n \to \mathbb{R}_+^n$ such that

$$h^i(u^1, \ldots, u^n) := g^i(u^1, \ldots, u^n) \min\{\xi^i(u^1, \ldots, u^n), 1\},$$
$$\text{for all } (u^1, \ldots, u^n) \in U^n.$$

Let $h(u^1, \ldots, u^n) = (h^1(u^1, \ldots, u^n), \ldots, h^n(u^1, \ldots, u^n))$. Then

$$h(u^1, \ldots, u^n) : U^n \to U^n.$$

By continuity of $\left(g^1, \ldots, g^n\right)$ and $\left(\xi^1, \ldots, \xi^n\right)$, h is continuous. By Brouwer's Theorem, there is a $\left(v^1, \ldots, v^n\right)$ in U^n such that

$$h(v^1, \ldots, v^n) = (v^1, \ldots, v^n). \tag{3}$$

If also

$$g(v^1, \ldots, v^n) \in U^n, \tag{4}$$

then $g(v^1, \ldots, v^n) = (v^1, \ldots, v^n)$, i.e., by (2),

$$v_i^i = g_i^i(v^i) = \max\{u_i : (u_i, (v_k^i)_{k \neq i}) \in U\}, \text{ for all } i, \tag{5}$$

$$v_j^i = g_j^i(v^i) = \delta^{\rho(j,i)} \max\{u_j : (u_j, (v_k^j)_{k \neq j}) \in U\}, \text{ for all} i \neq j, \tag{6}$$

or, by plugging (5) into (6),

$$v_j^i = \delta^{\rho(j,i)} v_j^j, \text{ for all } i, j.$$

Thus (4) is sufficient for $(v^1, \ldots, v^n)$ to satisfy (1).

Suppose (4) does not hold. Then there is i such that

$$\xi^i(v^1, \ldots, v^n) < 1. \tag{7}$$

By (3),

$$v^i = g^i(v^1, \ldots, v^n)\xi^i(v^1, \ldots, v^n).$$

Hence

$$v_j^i < g_j^i(v^1, \ldots, v^n), \text{ for all } j. \tag{8}$$

By construction,

$$v^i = g^i(v^1, \ldots, v^n)\xi^i(v^1, \ldots, v^n) \in P.$$

Thus, by (8),

$$(g_i^i(v^i), v_{-i}^i) \notin U. \tag{9}$$

But, by (2), $g_i^i(v^i) = \max\{u_i : (u_i, v_{-i}^i) \in U\}$, i.e. $(g_i^i(v^i), v_{-i}^i)$ is the element *in* U that maximizes i's payoff given that the other players get at least v_{-i}^i, a contradiction to (9). □

3 Relationship with the Nash solution

Denote the *Nash solution* of U by

$$u^N \coloneqq \arg\max_{u \in U} \prod_{i=1}^n u_i. \tag{10}$$

Also denote by

$$H(u) \coloneqq \left\{ (v_1, \ldots, v_n) \in \mathbb{R}^n : \prod_{i=1}^n v_i = \prod_{i=1}^n u_i \right\},$$

the hyperbola that contains u. Note that, by construction, u^N is the unique point at which U is supported by a hyperbola, this time by $H(u^N)$.

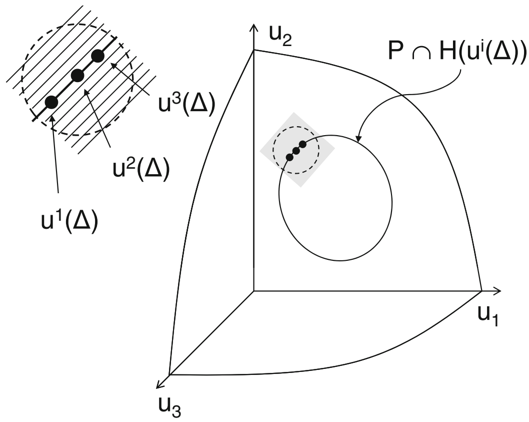


Fig. 1 Intersection of a hyperbola and the Pareto frontier

It is not difficult to see that equilibrium offers $u^1(\Delta), \ldots, u^n(\Delta)$ under $\Delta > 0$ lie in the same hyperbola: for any j,

$$\prod_{i=1}^{n} u_i^j(\Delta) = \prod_{i=1}^{n} \delta^{\Delta\rho(j,i)} u_i^i(\Delta) = \delta^{\Delta n(n-1)/2} \prod_{i=1}^{n} u_i^i(\Delta). \tag{11}$$

The last expression is independent of the proposer index j.

Theorem 2 *Let P be differentiable. Then all stationary equilibrium outcomes converge to u^N as Δ tends to 0.*

The proof can be summarized as follows. Consider the three-player case. Think of the surface P of U as a chart of 1-dimensional curves, each reflecting an intersection of P and a hyperbola. Identify the equilibrium offers $u^1(\Delta), u^3(\Delta), u^3(\Delta)$ under Δ. As Δ becomes small, the maximum distance between vectors $u^1(\Delta), u^2(\Delta), u^3(\Delta)$ becomes small. Since, by (11), they all lie in the same hyperbola, they must either converge to the Nash solution, or, in the limit, the vectors are contained by a 1 -dimensional subspace (see Fig. 1). We show that $u^1(\Delta), u^2(\Delta), u^3(\Delta)$ are always linearly independent, and hence cannot be embedded into a 1−dimensional subspace. Thus $u^1(\Delta), u^2(\Delta), u^3(\Delta)$ cannot converge anywhere but to the Nash solution.

Proof of Theorem 2 Denote by $u^j(\Delta) = (u_i^j(\Delta))_{i=1}^n$ player j's equilibrium offer when the period length is $\Delta > 0$. Note that, since U is bounded, the difference $(\delta^{-\Delta\rho(i,j)} - 1)u_i^j(\Delta)$ tends to 0 as Δ becomes negligible for all i, j. Hence so does the difference $u_i^i(\Delta) - u_i^j(\Delta)$. This implies that $\|u^j(\Delta) - u^i(\Delta)\|$ approaches 0 as Δ tends to 0, for all i, j.

Let $\{\Delta\}$ be a subsequence under which $u^1(\Delta), \ldots, u^n(\Delta)$ converge to u^*. Since U is bounded, it suffices for us to show that $u^* = u^N$. All the limits below are taken with respect to the sequence $\{\Delta\}$.

For expositional reasons, denote the logarithmized variables by

$$\begin{aligned}
\bar{u} &= \ln u = (\ln u_1, \dots, \ln u_n), \text{ for any } u \in U, \\
\bar{S} &= \{\bar{u} : u \in S\}, \text{ for any } S \subseteq \mathbb{R}^n_{++}, \\
\bar{\delta} &= \ln \delta.
\end{aligned}$$

Then, by (1),

$$\bar{u}_i^j(\Delta) = \Delta\bar{\delta}\rho(i, j) + \bar{u}_i^i(\Delta), \text{ for all } i, j. \tag{12}$$

Note that logarithmization transforms hyperbolas into hyperplanes without affecting their local nature.[9] Logarithmization also preserves the convexity of U and the differentiability of P. Thus $\bar{P}$ is an $n-1$ -manifold supporting, at each of its point, a unique $n-1$ dimensional hyperplane. In particular, it supports a unique hyperplane at $\bar{u}^*$.

Player j's equilibrium offer under $\Delta > 0$ is $\bar{u}^j(\Delta) = (\bar{u}_i^j(\Delta))_{i=1}^n$.[10] Then, by (12),

$$\begin{aligned}
\ln \prod_{i=1}^{n} u_i^j(\Delta) &= \sum_{i=1}^{n} \bar{u}_i^j(\Delta) \\
&= \Delta\bar{\delta} \sum_{i=1}^{n} \rho(i, j) + \sum_{i=1}^{n} \bar{u}_i^i(\Delta) \\
&= \frac{n(n-1)\Delta\bar{\delta}}{2} + \sum_{i=1}^{n} \bar{u}_i^i(\Delta).
\end{aligned} \tag{13}$$

Construct the $n-1$ -dimensional hyperplane L^{Δ} such that,

$$L^{\Delta} = \left\{ x \in \mathbb{R}^n : \sum_{i=1}^{n} x_i = \frac{n(n-1)\Delta\bar{\delta}}{2} + \sum_{i=1}^{n} \bar{u}_i^i(\Delta) \right\}.$$

Since (13) is independent of the index j, it follows that

$$\bar{u}^1(\Delta), \dots, \bar{u}^n(\Delta) \in L^{\Delta}. \tag{14}$$

[9] Since it is easier to verify whether an object can be embedded into a hyperplane than into a hyperbola.

[10] To be precise, logarithmic scales require that all equilibrium offers are bounded away from zero for all i and all $\Delta > 0$. This fact is easy to verify from the equilibrium conditions.

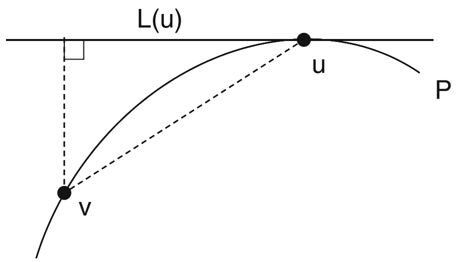


Fig. 2 Differentiability and the limit

Identify

$$C = \left\{ \begin{pmatrix} 0 \\ 1 \\ 2 \\ \vdots \\ n-1 \end{pmatrix}, \begin{pmatrix} n-1 \\ 0 \\ 1 \\ \vdots \\ n-2 \end{pmatrix}, \ldots, \begin{pmatrix} 1 \\ 2 \\ \vdots \\ n-1 \\ 0 \end{pmatrix} \right\},$$

a collection of $n-1$ linearly independent vectors. By (12), the collection of equilibrium offers can now be written compactly

$$\{\bar{u}^1(\Delta), \ldots, \bar{u}_i^n(\Delta)\} = (\bar{u}_i^i(\Delta))_{i=1}^n + \Delta\bar{\delta}C. \tag{15}$$

Thus, by (14),

$$(\bar{u}_i^i(\Delta))_{i=1}^n + \Delta\bar{\delta}C \subset L^{\Delta}, \quad \text{for all } \Delta.$$

By making an affine transformation of both sides,[11]

$$C \subset \frac{1}{\Delta\bar{\delta}} L^{\Delta} - \frac{1}{\Delta\bar{\delta}} (\bar{u}_i^i(\Delta))_{i=1}^n, \quad \text{for all } \Delta. \tag{16}$$

Let $L(\bar{u})$ be the hyperplane that supports $\bar{U}$ at $\bar{u}$.[12] Since P is compact and differentiable, there is a function $\sigma > 0$ such that $\sigma(\varepsilon) \to 0$ as $\varepsilon \to 0$, and such that (see Fig. 2)

$$\sigma(\|\bar{u} - \bar{v}\|) \geq \frac{\min\{\|\bar{v} - \bar{v}'\| : \bar{v}' \in L(\bar{u})\}}{\|\bar{u} - \bar{v}\|}, \quad \text{for all } \bar{u}, \bar{v} \in \bar{P}. \tag{17}$$

For a formal proof of that such function exists, see the working paper version.

[11] By an affine transformation of a set $X \subset \mathbb{R}^m$ we mean a map $X \mapsto aX + b$, for $a \in \mathbb{R}_{++}$ and $b \in \mathbb{R}^m$, where $aX + b = \{ax + b : x \in X\}$.

[12] That is, $L(\bar{u}) = \{x \in \mathbb{R}^n : px = k\}$, where $p \in \mathbb{R}^n$ and $k \in \mathbb{R}$ such that $p\bar{v} \geq k$, for all $\bar{v} \in \bar{U}$, and $p\bar{u} = k$.

By (17),

$$\sigma(\|\bar{u}^i(\Delta)-\bar{u}^1(\Delta)\|) \geq \frac{\min\{\left\|\bar{u}^i(\Delta)-\bar{v}\right\| : \bar{v} \in L(\bar{u}^1(\Delta))\}}{\left\|\bar{u}^i(\Delta)-\bar{u}^1(\Delta)\right\|}. \quad (18)$$

Denote by

$$\sigma^{\Delta} = \max_{i=1,\ldots,n} \sigma(\|\bar{u}^i(\Delta)-\bar{u}^1(\Delta)\|).$$

Since $\left\|\bar{u}^i(\Delta)-\bar{u}^1(\Delta)\right\| \to 0$ for all i it follows that $\sigma^{\Delta} \to 0$ as $\Delta \to 0$.

Let $\|X\|$ be the sup-norm of the set X, i.e. $\|X\| = \sup_{x,y\in X}\|x-y\|$. By (15),

$$\left\|\bar{u}^i(\Delta)-\bar{u}^1(\Delta)\right\| \leq \Delta\bar{\delta}\,\|C\|, \quad \text{for all } i. \quad (19)$$

By (18) and (19),

$$\min\{\|\bar{u}^i(\Delta)-\bar{v}\| : \bar{v} \in L(\bar{u}^1(\Delta))\} \leq \sigma^{\Delta}\Delta\bar{\delta}\,\|C\|, \quad \text{for all } i. \quad (20)$$

From (20) we have that

$$(\bar{u}_i^i(\Delta))_{i=1}^n + \Delta\bar{\delta}C \subset \left\{\bar{u} : \|\bar{u}-\bar{v}\| < \sigma^{\Delta}\Delta\bar{\delta}\,\|C\| \text{ and } \bar{v} \in L(\bar{u}^1(\Delta))\right\}.$$

Or, by making an affine transformation of both sides,

$$\begin{aligned} C &\subset \frac{1}{\bar{\delta}\Delta}\left[\left\{\bar{u} : \|\bar{u}-\bar{v}\| < \sigma^{\Delta}\Delta\bar{\delta}\,\|C\| \text{ and } \bar{v} \in L(\bar{u}^1(\Delta))\right\} - (\bar{u}_i^i(\Delta))_{i=1}^n\right] \\ &= \left\{\bar{u} : \|\bar{u}-\bar{v}\| < \sigma^{\Delta}\,\|C\| \text{ and } \bar{v} \in \frac{1}{\bar{\delta}\Delta}L(\bar{u}^1(\Delta))\right\} - \frac{1}{\bar{\delta}\Delta}(\bar{u}_i^i(\Delta))_{i=1}^n. \quad (21) \end{aligned}$$

That is, C is contained in the $\sigma^{\Delta}\,\|C\|$ -neighborhood of an $n-1$ -dimensional hyperplane $(\bar{\delta}\Delta)^{-1}L(\bar{u}^1(\Delta)) - (\bar{\delta}\Delta)^{-1}(\bar{u}_i^i(\Delta))_{i=1}^n$.

Define

$$L^* = \left\{x \in \mathbb{R}^n : \sum_{i=1}^n x_i = \sum_{i=1}^n \bar{u}_i^*\right\}.$$

By construction, $\bar{u}^* \in L^*$. Suppose that $L(\bar{u}^*) \neq L^*$, i.e. the hyperplane that supports $\bar{U}$ at $\bar{u}^*$ does not coincide with the hyperplane L^*. Since both $L(\bar{u}^*)$ and L^* are $n-1$ dimensional hyperplanes, the hyperplane $L(\bar{u}^*) \cap L^*$ is only $n-2$ dimensional. Since C contains $n-1$ independent vectors, and $\sigma^{\Delta} \to 0$, there is $\Delta' > 0$ such that for all $\Delta < \Delta'$,

$$C \not\subset \left\{\bar{u} : \|\bar{u}-\bar{v}\| < \sigma^{\Delta}\,\|C\| \text{ and } \bar{v} \in \frac{1}{\bar{\delta}\Delta}L(\bar{u}^*)\right\} \cap \frac{1}{\bar{\delta}\Delta}L^* - \frac{1}{\bar{\delta}\Delta}\bar{u}^*.$$

Since $\bar{P}$ is differentiable and $\bar{u}^1(\Delta) \to \bar{u}^*$, it follows that $L^\Delta \to L^*$ and $L(\bar{u}^1(\Delta)) \to L(\bar{u}^*)$.[13] Thus there is $\Delta'' < \Delta'$ such that for all $\Delta < \Delta''$,

$$C \not\subset \left\{ \bar{u} : \|\bar{u} - \bar{v}\| < \sigma^\Delta \|C\| \text{ and } \bar{v} \in \frac{1}{\delta\Delta} L(\bar{u}^1(\Delta)) \right\} \cap \frac{1}{\delta\Delta} L^\Delta - \frac{1}{\delta\Delta} (\bar{u}_i^i(\Delta))_{i=1}^n,$$

which contradicts (16) and (21). Thus $L(\bar{u}^*) = L^*$.

Since $L(\bar{u}^*) = L^*, L^*$ supports $\bar{U}$ at $\bar{u}^*$. Since, by definition, L^* is the log-transformed hyperbola $H(u^*)$, and $\bar{U}$ is the log-transformed utility set U, this must mean that the U is supported by the hyperbola $H(u^*)$ at u^*. Thus $u^* = u^N$. □

4 Necessity of the differentiability of *P*

The result of the previous section, which is the main finding of the paper, is based on the local properties of the Pareto frontier, i.e. it shows that the Nash program works in any environment where the Pareto frontier is locally isomorphic to a hyperplane. However, when this is not the case, the result need not hold. Indeed, to this end we argue that the convergence result is sensitive to the differentiability assumption. We give an example of a scenario where stationary equilibria do not converge to the Nash bargaining solution.

Let

$$U = \{u \in \mathbb{R}_+^3 : u_1 + u_2 \leq 1, u_3 \leq 1\}.$$

Here 1 and 2 bargain over a linear cake and 3 is a "dummy" player, without strategic significance.

Equilibrium condition (1) implies that the stationary equilibrium offers u^1, u^2, and u^3 satisfy

$$\begin{aligned} \delta^{2\Delta} u_1^1 &= \delta^\Delta u_1^3 = u_1^2, \\ \delta^{2\Delta} u_2^2 &= \delta^\Delta u_2^1 = u_2^3, \\ \delta^{2\Delta} u_3^3 &= \delta^\Delta u_3^2 = u_3^1. \end{aligned} \tag{22}$$

In equilibrium, players do not waste their own consumption possibilities when making offers. This means

$$\begin{aligned} u_1^1 &= 1 - u_2^1, \\ u_2^2 &= 1 - u_1^2, \\ u_3^3 &= 1. \end{aligned} \tag{23}$$

[13] Where the limits are defined with respect to the normals of the hyperplanes.

Combining (23) with (22) gives the equilibrium offers for players 1, 2 and 3,

$$u^1 = \left(\frac{1-\delta^{\Delta}}{1-\delta^{3\Delta}}, \frac{\delta^{\Delta}-\delta^{3\Delta}}{1-\delta^{3\Delta}}, \delta^{2\Delta} \right),$$
$$u^2 = \left(\frac{\delta^{2\Delta}-\delta^{3\Delta}}{1-\delta^{3\Delta}}, \frac{1-\delta^{2\Delta}}{1-\delta^{3\Delta}}, \delta^{\Delta} \right),$$
$$u^3 = \left(\frac{\delta^{\Delta}-\delta^{2\Delta}}{1-\delta^{3\Delta}}, \frac{\delta^{2\Delta}-\delta^{4\Delta}}{1-\delta^{3\Delta}}, 1 \right).$$

By taking the limit $\Delta \to 0$, we obtain the common convergence point of u^1, u^2, and u^3

$$u^* = \left(\frac{1}{3}, \frac{2}{3}, 1 \right).$$

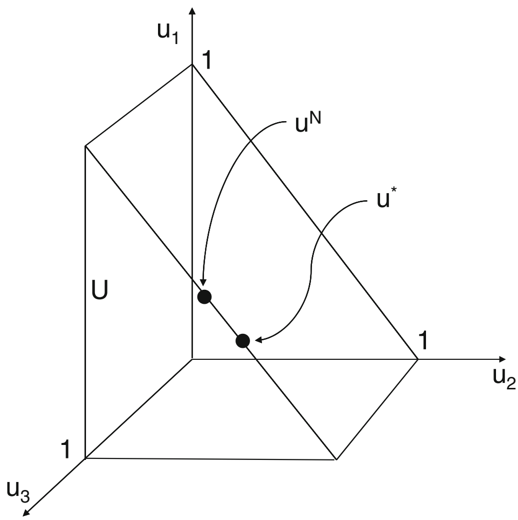
However, by symmetry, the Nash solution of the problem is

$$u^N = \left(\frac{1}{2}, \frac{1}{2}, 1 \right).$$

Thus the convergence point u^* of the stationary equilibrium does not coincide with the Nash solution u^N (see Fig. 3). Hence the differentiability of P is crucial for the convergence result (the seminal idea is by Lensberg and Thomson 1988).

However, our equilibrium is not the only stationary equilibrium that U entertains. This is due to the fact that U is comprehensive but not strictly comprehensive, as demanded by our characterization (1). However, using the above reasoning, it is clear that U can be approximated by a strictly comprehensive problem whose unique

Fig. 3 Non-convergence to the Nash solution



stationary equilibrium is close to the equilibrium we characterize.[14] Thus the no-convergence result does hold also in the class of convex, compact, and strictly comprehensive problems.

Discussion To understand why differentiability of the Pareto frontier is essential for the convergence to the Nash solution, note that nondifferentiability may allow a subgroup of players to agree on the desirable point in the Pareto frontier. In the above example, player 3 effectively groups with 2 to bargain against player 1. Because of the sequentiality of the offers, 3's role is to delay 1's offer once 2 has rejected his offer. This gives 2 more bargaining power even in the limit, with large discount factors. When the Pareto frontier is smooth, this scenario is not possible. The role of differentiability is to guarantee that a small portion of payoff of one player can be distributed to others in arbitrary way. When the discount factor is large, and hence the benefit from being a proposer vanishes, this means that all players bargain against one another in a symmetric way. Such property is met only at the Nash solution.

In the two player context, however, there is no problem with convergence even in nondifferentiable problems since each player only faces one opponent. Hence there is automatically a symmetric trade-off between the players. Technically, this is mirrored by the fact that the intersection of the Pareto surface and a hyperbola is zero dimensional but contains two points. Hence the intersection cannot be connected and, because of this, the shrinking set of players equilibrium offers cannot converge anywhere but to the point of intersection of the highest hyperbola and the Pareto surface.

To interpret this observation in terms of the so called "Nash program" which aims at reconciling the strategic approach and the axiomatic one, note that a nondifferentiable utility space can always be approximated by differentiable ones. Since the Nash solution is continuous with respect to small changes in the Pareto surface, it must be the case that the noncooperative equilibrium correspondence is discontinuous with respect to the underlying payoff parameters. Thus the "problem" seems to be with the noncooperative approach; (the limit) outcome of noncooperative bargaining can be sensitive to the fine details of the underlying physical model.

On the other hand, the non-convergence result may be interpreted as a manifestation of the fact that the Nash IIA axiom does not adequately take into account categorial changes in players' relationships, e.g. that one player becomes competely independent in terms of payoffs. Indeed, replacing IIA with the multilateral stability condition of Lensberg and Thomson (1988) (see also Thomson and Lensberg 1989, ch 8), whose target is precisely that, is known to characterize the Nash solution only in the domain of smooth problems.

Acknowledgements We want to thank an associate editor and the editor, William Thomson, for helpful comments. We also thank Geir Asheim and Hannu Salonen for useful discussions.

[14] Think of the problem $U^{\varepsilon} = U(1-\varepsilon) + V\varepsilon$, where $V = \{u \in \mathbb{R}^3_+ : 2u_1 + 2u_2 + u_3 \leq 3\}$, and $\varepsilon > 0$. Now U^{ε} is strictly comprehensive and for small ε, the unique stationary equilibrium converges to a point close to u^*.

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