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Implementing a surplus division rule

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Abstract

We show that in the class of *surplus division* problems, where the size of the surplus may be dependent on how it is shared, the *Nash bargaining solution* is the unique *symmetric* and *Nash implementable* division rule.

Keywords: Nash implementation; Bargaining; Surplus division

JEL classification: C78; D71; D78

1. Introduction

Let there be two players in a need to share a surplus from the joint project. The size of the surplus may be dependent on how the surplus is divided.[1] Thus the division affects players' payoffs not only directly but also indirectly via its effect on the size of the surplus.

A *surplus function* depicts the relationship between the division and the size of the surplus. Let the domain of surplus functions consist of all bounded functions under which all pure divisions are not Pareto-dominated by lotteries over divisions (allow randomization).

The players need to agree on the *surplus division rule* which specifies a division for each surplus function. The problem is that the prevailing surplus function is known only by the players and not by outsiders. Hence, to agree on a surplus division rule one must induce players to reveal the true surplus function. We ask which surplus division rules can be agreed upon, i.e. *Nash implemented*.

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[1] Due to, say, a hold-up problem (see e.g. Hart and Moore, 1990; Maskin and Tirole, 1999b).

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Our result is that the *Nash bargaining solution* is the *unique* symmetric and Nash implementable surplus division rule in this domain. What delivers the result is that Maskin's *monotonicity* condition implies Nash's independent irrelevant alternatives axiom in this domain.

2. Fundamentals

2.1. Payoffs

Let $\{1,2\}$ be the set of players, with typical elements i, j. Let $a \in [0,1]$ and $1-a \in [0,1]$ be the relative shares of surplus received by player 1 and 2, respectively, and d the disagreement outcome. Denoted by Δ the set of lotteries on $[0,1] \cup \{d\}$. The amount of shareable surplus depends on which element in $[0,1] \cup \{d\}$ is chosen. Let *surplus function* π: $[0,1] \cup \{d\} \to \mathbb{R}_+$ reflect this relationship.

Players' 1 and 2 vNM preferences are represented by the following linear payoff functions: for any π and a,

$$v_1[a,\pi] = a\pi(a),$$

$$v_2[a,\pi] = (1-a)\pi(a).$$

The expected payoff of player $i=1$, 2 under lottery $p \in \Delta$ is then

$$v_i[p,\pi] = \int_0^1 p(a) v_i[a,\pi] da.$$

For any $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}_{++} \times \mathbb{R}_{++}$, write $\lambda v[p,\pi] = (\lambda_1 v_1[p,\pi], \ \lambda_1 v_2[p,\pi])$ and $\lambda v[B,\pi] = \{\lambda v[p,\pi] : p \in B\}$, for any $B \subseteq \Delta$. Note that $\lambda v[\cdot,\pi]$ and $v[\cdot,\pi]$ represent the same preferences over Δ, and that these preferences are dependent on π.

Denote the lower contour set of i at $q \in \Delta$ by

$$L_i(q,\pi) = \{p \in \Delta : v_i[q,\pi] \geq v_i[p,\pi]\},$$

and the *Pareto-optimal* set of lotteries by[2]

$$P(\pi) = \{p \in \Delta : \text{there is no } q \text{ s.t. } v[q,\pi] \geq v[p,\pi]\}.$$

Let the *state space* Π comprise *all* bounded surplus functions π such that (i) $[0,1] \subseteq P(\pi)$, (ii) $\pi(a) > \pi(d) = 0$, for all $a \in [0,1]$. Assumption (i) guarantees that a deterministic division rule is never socially suboptimal.[3] Assumption (ii) means that any deterministic division is always strictly preferred over disagreement. We assume that π is observed by players but not by outsiders.

Note that the family $\{v[\Delta,\pi] : \pi \in \Pi\}$ of utility sets comprises all *comprehensive bargaining problems* in $\mathbb{R}^2_+$, i.e. $v[\Delta,\pi]$ is nonempty, convex, compact, and i's maximal element in $v[\Delta,\pi]$ coincides with a minimal element of j, $i \neq j$. It is well known (see e.g. Thomson and Lensberg, 1989, ch 2) that the standard Nash axioms characterize the Nash bargaining solution in this domain.

[2] Vector inequalities: given $x, y \in \mathbb{R}^2$, $x \geqq y$ means $x_i \geqq y_i$ for $i=1$, 2, $x \geqq y$ means $x \geqq y$ and $x \neq y$, and $x > y$ means $x_i > y_i$ for $i=1$, 2.

[3] Assumption (i) effectively implies that π is a continuous function.

2.2. Division rule

A *surplus division rule* is denoted by a function F: $\Pi \rightarrow [0,1]$. Note that F is *scale invariant*, i.e. only the underlying preferences affect the rule. We say that a surplus sharing function π is *symmetric* if there is $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}_{++} \times \mathbb{R}_{++}$ such that $(u_1, u_2) \in \lambda v[\Delta, \pi]$ implies $(u_2, u_1) \in \lambda v[\Delta, \pi]$, for all $(u_1, u_2) \in \mathbb{R}_{++} \times \mathbb{R}_{++}$.

Definition 1. F is symmetric if $v_1[F(\pi), \pi] = v_2[F(\pi), \pi]$ whenever π is *symmetric*.

Since F is a division rule, $F(\pi) \in P(\pi)$ for all $\pi \in \Pi$. Thus F is Pareto-optimal.

2.3. Nash implementation

While the state is common knowledge among the players, the planner only knows the state space Π. Thus, a mechanism must be invoked to implement F.

A *game form* $\Gamma = (M, g)$ consists of the *strategy space* $M = M_1 \times M_2$ and an *outcome function* g: $M \rightarrow \Delta$. Pair (Γ, π) now constitutes a *normal form game*. Let NE (Γ, π) denote the set of Nash equilibria of game (Γ, π). Then $m \in \mathrm{NE}(\Gamma, \pi)$ if and only if

$$v_i[g(m), \pi] \geq v_i\left[g\left(m_i', m_j\right), \pi\right], \text{for all } m_i' \in M_i \text{ and for all } i \neq j.$$

Mechanism Γ *Nash implements* F if and only if

$$v[g(\mathrm{NE}(\Gamma, \pi,), \pi] = v[F(\pi), \pi], \text{for all } \pi \in \Pi.$$

If there is a mechanism Γ which Nash implements F, then F is Nash implementable. Any Nash implementable F is clearly scale invariant.

3. The result

Recall the condition by Maskin (1999) [working paper 1977]: F is (Maskin) *monotonic* if $L_i(F(\pi'), \pi') \subseteq L_i(F(\pi'), \pi)$ for $i = 1$, 2, implies $F(\pi') = F(\pi)$, for all $\pi, \pi' \in \Pi$. Maskin showed that any Nash implementable choice rule is monotonic. Using Moore and Repullo (1990), monotonicity is also sufficient in our context.[4]

Theorem 1. *F can be Nash implemented if and only if it is monotonic.*

Given state π, the Nash product on $[0,1]$ is defined by

$$\prod_{i=1,2} v_i[a, \pi] = a(1-a)\pi(a)^2.$$

[4] Use d as a "bad outcome".

The *Nash bargaining rule* to the surplus sharing problem is

$$N(\pi) = \arg\max_{a\in[0,1]} \prod_{i=1,2} v_i[a,\pi]. \tag{1}$$

Nash (1950) showed that N is the unique rule that satisfies symmetry, Pareto-optimality, scale invariance, and the axiom of independence of *irrelevant alternatives*.

Which symmetric choice rules can be Nash implemented? In general, monotonicity is a too demanding condition to implement any bargaining solution (see Vartiainen, 2004). However, the surplus sharing restriction on the domain allows us to avoid the impossibility result. The following lemma is the key consequence of our structural assumption.

Lemma 1. *Fix $a\in[0,1]$. Then, for any $\pi,\pi'\in\Pi$, there is $\bar{c}=(c,c)\in\mathbb{R}^2_{++}$ such that $v[a,\pi]=\bar{c}v[a,\pi']$.*

Proof. Take any $\pi,\pi'\in\Pi$. Choose $c=\pi(a)/\pi'(a)$. Then

$$\begin{aligned} v[a,\pi] &= (a\pi(a),(1-a)\pi(a)) \\ &= (ac\pi'(a),(1-a)c\pi'(a)) \\ &= \bar{c}v[a,\pi']. \end{aligned} \tag{2}$$

□

That is, either $v[a,\pi]\geqq v[a,\pi']$ or $v[a,\pi']\geqq v[a,\pi]$. In Fig. 1, the three dashed lines represent utility vectors that can be induced by surplus functions π,π' and π'' for given a, a' or a''. It is clear

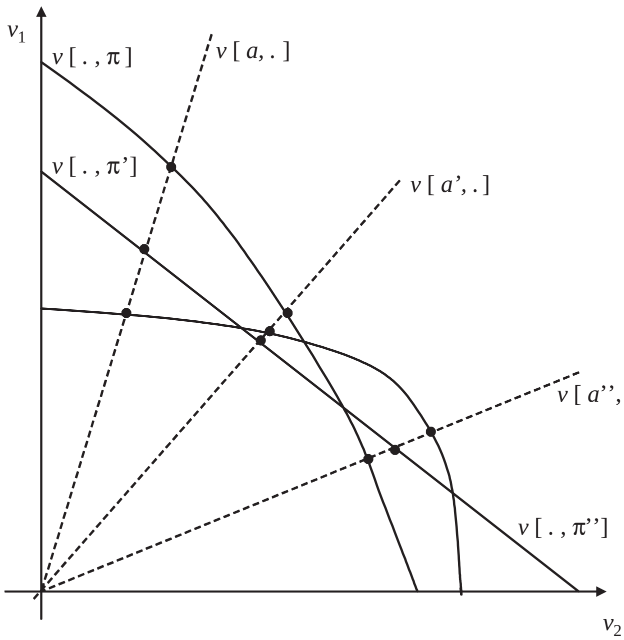

Fig. 1. Utility frontiers induced by three surplus functions.

that *any* convex, closed, and comprehesive utility possibility set can be induced by some surplus function in Π.

Note that F satisfies Pareto-optimality by assumption, and scale invariance by construction. We now show that independence of irrelevant alternatives is implied by the (Maskin) monotonicity condition. Thus Nash implementability and symmetry alone characterize the Nash solution in this domain.

Theorem 2. *F is symmetric and Nash implementable if and only if F = N.*

Proof. *Sufficiency*: N is symmetric. By Theorem 1, it suffices to show that F is monotonic on Π. Take any $\pi \in \Pi$, and identify $N(\pi)$. Suppose that, for some $\pi' \in \Pi$,

$$L_i(N(\pi), \pi') \subseteq L_i(N(\pi), \pi), \qquad \text{for } i = 1, 2. \tag{3}$$

We show that then $N(\pi) = N(\pi')$, too.

By Lemma 1, there is $\bar{c} = (c, c) \in \mathbb{R}^2_{++}$ such that

$$\bar{c}v[N(\pi), \pi'] = v[N(\pi), \pi]. \tag{4}$$

Suppose, ad absurdum, that for some $a \in [0, 1]$,

$$cv_i[a, \pi'] > v_i[a, \pi]. \tag{5}$$

Then, by Lemma 2,

$$cv_j[a, \pi'] > v_j[a, \pi], \qquad \text{for } j \neq i.$$

Since $N(\pi) \in [0, 1]$ and $a \in (0, 1)$, there is $k \in \{1, 2\}$ such that

$$cv_k[a, \pi'] > v_k[N(\pi), \pi].$$

Construct lottery $p = a \cdot p(a) + d \cdot (1 - p(a))$, where $p(a) \in (0, 1)$ satisfies

$$cv_k[p, \pi'] = cv_k[a, \pi']p(a) = cv_k[N(\pi), \pi']. \tag{6}$$

Then $p \in L_k(a^*, \pi')$. However, by Eqs. (4), (5), and (6),

$$\begin{aligned} v_k[p, \pi] &= v_k[a, \pi]p(b) \\ &> cv_k[a, \pi']p(b) \\ &= cv_k[N(\pi), \pi'] \\ &= v_k[N(\pi), \pi]. \end{aligned}$$

Hence, $p \notin L_k(N(\pi), \pi)$. But this violates Eq. (3). Hence Eq. (5) must be false. This means that

$$\bar{c}v[a, \pi'] \geqq v[a, \pi], \qquad \text{for all } a \in [0, 1].$$

But then also

$$\bar{c}v[p, \pi'] \geqq v[p, \pi], \qquad \text{for all } p \in \Delta,$$

or,

$$\bar{c}v[\Delta, \pi'] \subseteq v[\Delta, \pi]. \tag{7}$$

By Eq. (4) and $N(\pi)\in[0,1]$, this implies that

$$\max_{a\in[0,1]}\prod_{i=1,2} cv_i[a,\pi'] = \max_{a\in[0,1]}\prod_{i=1,2} v_i[a,\pi] = \prod_{i=1,2} v_i[N(\pi),\pi] = \prod_{i=1,2} cv_i[N(\pi),\pi'].$$

Thus $N(\pi)=N(\pi')$, as desired.

Necessity: to see that N is the *only* Nash implementable Pareto-optimal and symmetric rule, fix $\pi\in\Pi$. Define

$$v_1[N(\pi),\pi] = N(\pi)\pi(N(\pi)) = \lambda_1^{-1},$$

$$v_2[N(\pi),\pi] = (1-N(\pi))\pi(N(\pi)) = \lambda_2^{-1}.$$

Choose scale λ. Then $\lambda v[N(\pi),\pi]=(1,1)$.

Let Nash implementable F be symmetric. Construct function $\pi'\colon [0,1]\to\mathbb{R}_+$ such that

$$\pi'(a) = \frac{2}{a\lambda_1+(1-a)\lambda_2}, \qquad \text{for all } a\in[0,1].$$

Then

$$a\pi'(a)\lambda_1 + (1-a)\pi'(a)\lambda_2 = \lambda_1 v_1[a,\pi'] + \lambda_2 v_2[a,\pi'] = 2, \qquad \text{for all } a\in[0,1].$$

Hence $v[[0,1],\pi']$ coincides with the line $((0,2/\lambda_2),(2/\lambda_1,0))$. Thus π' clearly is a member of Π. Moreover,

$$N(\pi)\lambda_1 = (1-N(\pi))\lambda_2 = \frac{1}{\pi(N(\pi))}.$$

Thus

$$\lambda_1 v_1[N(\pi),\pi'] = \frac{2N(\pi)\lambda_1}{N(\pi)\lambda_1+(1-N(\pi))\lambda_2} = 1,$$

$$\lambda_2 v_2[N(\pi),\pi'] = \frac{2(1-N(\pi))\lambda_2}{N(\pi)\lambda_1+(1-N(\pi))\lambda_2} = 1.$$

Now $\lambda v[\Delta,\pi']$ is a convex hull of points $(0,2)$, $(2,0)$, and $(0,0)$. By symmetry, $\lambda v[F(\pi'),\pi']=(1,1)$. Since $F(\pi')\in[0,1]$, necessarily $N(\pi)=F(\pi')$. Since

$$v[\Delta,\pi]\subseteq\lambda v[\Delta,\pi'],$$

it follows by Lemma 1 that

$$v_i[a,\pi]\le\lambda_i v_i[a,\pi'], \qquad \text{for all } a\in[0,1], \qquad \text{for all } i=1,2.$$

Thus also

$$v_i[p,\pi]\le\lambda_i v_i[p,\pi'], \qquad \text{for all } p\in\Delta, \qquad \text{for all } i=1,2.$$

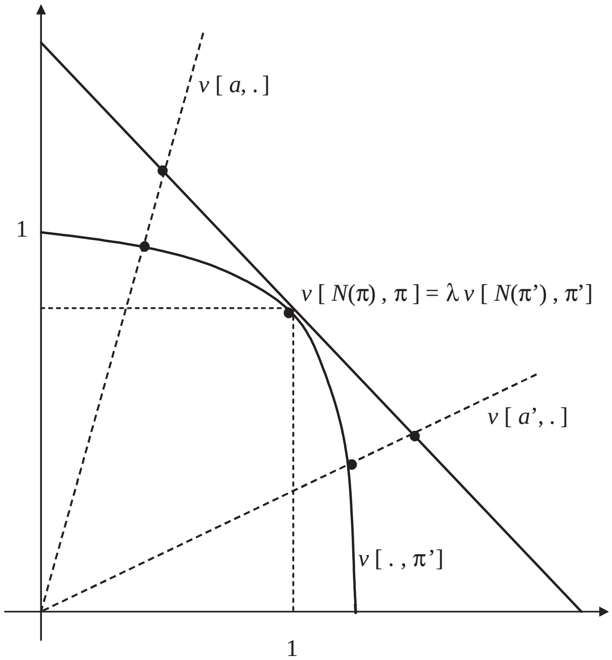

Fig. 2. Maskin monotonicity of the Nash solution.

As $v[N(\pi),\pi]=\lambda v[N(\pi),\pi']$, it follows that

$$L_i(N(\pi),\pi)\subseteq L_i(N(\pi),\pi'), \qquad \text{for } i=1,2.$$

By monotonicity, $N(\pi)=F(\pi)$. □

To understand the necessity part of proof, take any $a\in[0,1]$. Then *any* payoff vector in the line running through the origin with slope $a/(1-a)$ can be induced by some $\pi(a)$. In Fig. 2, we have constructed function π' such that the normalized symmetric Pareto frontier $\lambda v[\cdot,\pi']$ runs through the point $v[N(\pi),\pi]$ and lies above the frontier $v[\cdot,\pi]$. Under π, any symmetric rule $F(\pi')$ satisfies $F(\pi')=N(\pi')$. By monotonicity, then, $F(\pi)=N(\pi)$.

4. Remarks

It is natural to interpret the surplus sharing problem as a closed form representation of a hold-up problem. Motivation for the hold-up framework stems from contractual incompleteness: since parties have problems in describing physical contingencies, details are left out from the contract. Hence some relevant decisions are made via bargaining.

More specifically, a hold-up problem typically assumes an ex ante stage, where project specific investments are sunk, and an ex post stage, where bargaining over profits takes place. Since the ex post bargaining procedure determines how profitable it is for individual player to invest at the ex ante stage, it also affects the level of investment and, a fortiori, to the induced surplus. It is typical to assume Nash bargaining at the ex post stage.

Instead of asking how the ex post profits are shared, we study how to divide the *overall* surplus of the project, including the ex ante investment costs. Thus our model answers how to agree at the ex ante stage *on* the bargaining procedure at the ex post stage. But what if the details of the physical environment are not describable at the ex ante stage? As argued by Maskin and Tirole (1999a), implementation only requires that *payoff* contingencies are describable. Thus a surplus sharing rule should be implementable as long as payoff contingencies can be contracted upon.

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