Microeconomic theory

Lecture 7

Producer Theory

- Start with a single firm facing given prices
- Production set describes technology, not resources
- Comparative statics involve only substitution effects
- *Exogenous*: prices
- *Endogenous*: output and input demands

- First look at aggregate behavior and the fundamental theorems of welfare economics

Primitives:

1. Commodity space $\mathbb{R}^K$

   In contrast to consumer theory, also negative numbers are possible.

   For any $y = (y_1, ..., y_K) \in \mathbb{R}^K$,

   - Input implies $y_i < 0$.
   - Output implies $y_i > 0$.

2. Production set $Y \subset \mathbb{R}^K$:

   Summary of the technologically feasibe outcomes.

Any $y \in Y$ is feasible, any $y \notin Y$ is not.

3. With prices $p = (p_1, ..., p_K)$, profit is $p \cdot y$ for any $y \in Y$.

4. Behavioral assumption:

   Maximize profit in Y, given p.

5. Y completely general language to describe production possibilities. Possible assumptions include

   (a) Y is non-empty and closed.

(b) Y is convex.

(c) $y \in Y \cap \mathbb{R}^K_+$ implies $y = 0$, i.e. positive output requires input, and inactivity is feasible.

(d) $y - y' \in Y$ for all $y \in Y$ and $y' \in \mathbb{R}^K_+$, i.e. free disposal.

(e) $y \in Y$ and $-y \in Y$ imply $y = 0$, i.e. irreversibility: a committed production cannot be undone.

(f) $y \in Y$ implies $\alpha y \in Y$ for all $\alpha \in [0, 1]$, i.e. decreasing returns to scale. Conversely,

- Increasing returns to scale: $y \in Y$ implies $\alpha y \in Y$ for all $\alpha \in [1, \infty)$.

- Constant returns to scale: $y \in Y$ implies $\alpha y \in Y$ for all α.

(g) $y + y' \in Y$ for all $y, y' \in Y$, i.e. free entry.

- Alternative ways of describing the technology set:

1. General case: *Transformation function* $F : \mathbb{R}_+^K \rightarrow \mathbb{R}$ such that

   $$Y = \{y \in \mathbb{R}^K : F(y) \leq 0\}.$$

   F is 0 on the frontier of Y, i.e. $\partial Y = \{y \in \mathbb{R}^K : F(y) = 0\}$ is the *transformation frontier*. The slope of the level curves of F are called the marginal rate of transformation.

2. Single output -case: *Production function* $f : \mathbb{R}_+^{K-1} \rightarrow \mathbb{R}_+$ where

   - the Kth good reflects the output $q \in \mathbb{R}_+$.

   - $y = (y_1, ..., y_{K-1}) \in \mathbb{R}_+^{K-1}$ the vector of inputs.

- Then

$$Y = \left\{ (-y, q) \in \mathbb{R}_{+}^{K} : q \leq f(y) \right\}.$$

- Note that with single output Y is *convex* only if f is *concave*.

Profit Maximization Problem (PMP)

$$\max_{y \in Y} p \cdot y.$$

- Observe: No budget constraint.
- Question: When is the problem well posed (i.e. when does it have a solution)?
- Denote the value function to PMP by $\pi(p)$.
- $\pi(p)$ is called the profit function.

- Let $y(p)$ denote the set of optimal choices at price p.

- There is a duality between $\pi(p)$ and Y : If Y is convex, then

$$Y = \left\{ y \in \mathbb{R}^K : p \cdot y \leq \pi(p) \text{ for all } p \in \mathbb{R}^K_{++} \right\}.$$

Revealed Profit Approach

- For any $y, y' \in Y$, we know that if $y \in y(p)$ and $y' \in y(p')$, then

$$\begin{aligned} p \cdot y &\geq p \cdot y', \text{ and} \\ p' \cdot y' &\geq p' \cdot y. \end{aligned}$$

Let

$$\Delta p = \left(p' - p \right) \text{ and } \Delta y = \left(y' - y \right).$$

Then the inequalities can be written as:

$$-p \cdot \Delta y \geq 0 \text{ and } p' \cdot \Delta y \geq 0.$$

Summing these two inequalities gives the Law of Supply:

$$\Delta p \cdot \Delta y \geq 0$$

Optimal production

- Assume the single output model $q = f(y)$.

- Denote the (strictly positive) input prices by $w = (w_1, ..., w_{K-1})$.

- The problem reduces to

$$\max_{y \in \mathbb{R}_+^{K-1}} pf(y) - w \cdot y. \tag{1}$$

- FOCs: for all $k = 1, ..., K - 1$,

$$\begin{aligned} \frac{\partial f(y)}{\partial y_k} &\leq \frac{w_k}{p}, \text{ and} \\ \frac{\partial f(y)}{\partial y_k} &= \frac{w_k}{p}, \text{ if } y_k > 0. \end{aligned}$$

- Marginal rate of substitution:

$$MRTS_{kj} = \frac{\partial f(y) / \partial y_k}{\partial f(y) / \partial y_j}.$$

Slope of the *isoquant* $\{y' \in \mathbb{R}_+^{K-1} : f(y') = q\}$ at y.

- At the optimum,

$$MRTS_{kj} = \frac{w_k}{w_j}$$

- The following characterizes the solution (also more generally when Y is closed and satisfies the free disposal property.).

Proposition 1 (Properties of $\pi(p,w)$) *Let $y(p,w)$ be the solution to (1) and $\pi(p,w) = pf(y(p,w))$.*

1. $\pi(\cdot)$ is homogenous of degree one.

2. $\pi(p,w)$ is convex.

3. $y(p, w)$ is homogenous of degree zero.

4. If Y is convex, then $y(p, w)$ is convex valued. If Y is strictly convex then $y(p, w)$ is either empty or single valued.

5. If $y(p, w)$ is single valued at (p, w), then $\pi(p, w)$ is differentiable at (p, w) and $D\pi(p, w) = (f(y(p, w)), y(p, w))$. (Hotelling's lemma; use the envelope thrm).

6. If $y(p, w)$ is a function and differentiable at (p, w), then $D(f(y(p, w)), y(p, w)) = D^2\pi(p, w)$ is a symmetric and positive semidefinite.

- From properties 2 and 5 we get immediately:

$$\frac{\partial f(y(p,w))}{\partial p} \geq 0 \text{ and } \frac{\partial y_k(p,w)}{\partial w_k} \leq 0, \text{ for all } k = 1, ..., K-1.$$

- Interpretation: If the price of an output increases, then the supply increases: "Law of Supply".

- Also: If the price of an input increases, the demand for the input decreases:."Law of Input Demand".

Cost minimization

- For each quantity of output, q, find the least cost input combination that yields q.

- The problem:

$$\min_{z \in \mathbb{R}_+^{K-1}} w \cdot z$$
$$\text{s.t. } q = f(z).$$

- Denote the solutions by $x(w, q)$, i.e. the *conditional factor demands*.

- The value function is the *cost function*, $c(w, q)$

$$c(w, q) = w \cdot z(w, q).$$

- $z(w, q)$ is completely analogous to $h(p, u)$ in consumer theory and $c(w, q)$ is analogous to $e(p, u)$.

Proposition 2 (Properties of $c\left(w,q\right)$) *Assume a single output and that Y is closed and satisfies the free disposal property. Then,*

1. c is homogenous of degree 0 in w and nondecreasing in q.

2. c is concave in w.

3. if $\{z \geq 0 : f(z) \geq q\}$ is convex for all q, then $Y = \{(-z, q) : w \cdot z \geq c(w, q)$, for all $w \in \mathbb{R}_{++}^{K-1}\}$

4. $z\left(w, q\right)$ is homogenous of degree 0 in w

5. if $\{z \geq 0 : f(z) \geq q\}$ is convex , then $z(w, q)$ is a convex set; if $\{z \geq 0 : f(z) \geq q\}$ is strictly convex, then $z(w, q)$ is a function

6. if $z(w, q)$ is a function, then $z(w, q)$ is differentiable at w and satisfies $D_w c(w, q) = z(w, q)$ (Shepard's Lemma; envelope thrm)

7. if $z(w, q)$ is diffentiable at w, then $D_w z(w, q) = D_w^2 c(w, q)$ is symmetric and negative semidefinite with $D_w z(w, q)\, w = 0$

8. if f is homogenous of degree 1, then c and z are homogenous of degree 1 in q

9. if f is concave, then c is convex in q.

...back to optimal production

- Choose the optimal level of production.

$$\max_{q\in\mathbb{R}} pq - c\left(w,q\right).$$

- FOC:

$$p = \frac{\partial c\left(w,q\right)}{\partial q}.$$

  For competitive firms, marginal cost equals price.

- Once the cost minimizing input is determined, the problem of optimal production one dimensional!

Big Difference between Consumer and Producer Theory:

- Preference representation u is unique only up to increasing transformations.

- Production function f is a unique description of technology.

- Conclusion: Not only ordinal but also cardinal differences have meaning under f. E.g. concavity of f matters!

Aggregation - the general case

- Since there are only substitution effects along the production frontier, the aggregation theory for the supply side is straighforward.

- Let $Y_1,....,Y_J$ be the collection of production sets with profits and supply correspondences $\pi_j(p)$ and $y_j(p)$ of firms $j = 1, ..., J$.

- The aggregate supply

$$y(p) = \sum_{j=1}^{J} y_j(p) = \left\{ y \in \mathbb{R}^K : y = \sum_{j=1}^{J} y_j, \text{ for } y_j \in y_j(p) \text{ for all } j \right\}$$

- The properties of $y_j(p)$ are preserved under addition. In particular, $Dy(p) = D^2\pi(p)$ is a symmetric and positive semidefinite.

- The Law of (aggregate) Supply follows:

$$\Delta p \cdot \Delta y \geq 0.$$

- Let Y be the aggregate production set:

$$Y = Y_1 + ... + Y_J = \{y \in \mathbb{R}^K : y = \sum_j y_j, \text{ for some } y_j \in Y_j, \ j = 1, .., J\}$$

- Let $\pi^*(p)$, $y^*(p)$ be the corresponding profits and supply correspondences.

Proposition 3 *For all* $p \in \mathbb{R}^K_{++}$,

1. $\pi^* (p) = \sum_j \pi_j (p)$,

2. $y^* (p) = \sum_j y_j (p)$.