Microeconomic theory

Lecture 6

General Equilibrium in Exchange Economies

- In this lecture we consider transactions between individuals pursuing their own self interests in perfectly competitive markets.

- We consider exchange economies, i.e., there are no producers but only consumers who transact their endowments.

- Exchange economies display the key features of the Walrasian equilibrium, extension to cover production is not difficult.

Exchange Economies

- Construct an economy from a number of consumers:
  - Consumers maximize utility at given prices
  - Prices determine consumer's budget set as they determine the cost of consumption and also the value of consumer's endowment. That is, also income depends on prices.
- In general equilibrium analysis:
  - Behavioral assumptions: individual optimization and price taking.
  - Equilibrium concept: market clearing.
- Endogenous variables: vectors of consumption and prices.

- In equilibrium: prices balance supply and demand.

- Exchange economy formally:
  - Consumers by $h \in \{1, ..., H\}$.
  - Commodities by $l \in \{1, ..., L\}$.
  - Consumer h's continuous utility function: $u_h(x^h) : \mathbb{R}^L_+ \to \mathbb{R}$, representing h's preferences.
  - Consumer h's initial endowment $\omega^h \in \mathbb{R}^L_+$.
- An exchange economy is completely specified by the list $(u_h, \omega^h)_{h=1,...,H}$.

The Edgeworth box

We consider first basic concepts in the two-consumer case. The Edgeworth box is a useful device for analyzing this case.

Back to general case

- Given prices $p \in \mathbb{R}^L_+$ and initial endowement ω^h, consumer h's consumable income is $p \cdot \omega^h \in \mathbb{R}_+$.

- Consumer's *optimization problem* is

$$\max_{x^h} u_h(x^h)$$
$$\text{s.t.} p \cdot x^h \leq p \cdot \omega^h.$$

- Let $x^h(p) \in \mathbb{R}^L_+$ be the optimal consumption at p.
  - Note that consumer's income is determined by p, so the demand depends only on p.

- Denote net trade by $z^h(p) = x^h(p) - \omega^h$.
  - Determines if the consumer is a net seller or buyer of each good $l = 1, ..., L$.
  - The *budget constraint* can be written as
$$p \cdot z^h(p) \leq 0.$$
- Walras' law:
$$p \cdot z^h(p) = 0.$$
  Holds under locally nonsatiated preferences.

- Summing over individuals we get the aggregate demand:

$$z(p) = \sum_h z^h(p)$$

- Thus, the aggregate version of the Walras' law is:

$$p \cdot z(p) = 0$$

  i.e. the value of the aggregate net demand is zero.

- Markets clear under prices p if the demand is at most supply:

$$\sum_h x^h(p) \leq \sum_h \omega^h,$$

  or simply

$$z(p) \leq 0.$$

- **Definition 1** *A* **Walrasian equilibrium** *of an exchange economy* $(u^1, \omega^1, ..., u^H, \omega^H)$ *is a price vector* $p \in \mathbb{R}^L_+$ *and a consumption allocation* $x = (x^1, ..., x^H)$ *such that:*
  - $x^h = x^h(p)$*, for all* $h = 1, ..., H$,
  - $z(p) \leq 0$.
- Questions:
  - does a Walrasian equilibrium exist?
  - is it unique?
  - is it desirable?

- Before stating the existence, we need an important result:

Theorem 2 (Brouwer Fixed Point) *If f is a continuous function from the $L-1$ dimensional unit simplex $\Delta^{L-1} = \{q \in \mathbb{R}^L_+ : \sum q_l = 1\}$ to itself, then there is $q \in \Delta$ such that $f(q) = q$.*

- Note that since x^h is homogenous of degree zero for all h, also z is. Thus it is without loss of generality to *normalize* any prices p such that $\sum_l^L p_l = 1$. Thus it is also without loss of generality to assume that prices belong to the $L-1$ dimensional unit simplex Δ.

Theorem 3 (Existence: Arrow and Debreu, 1954) *If z satisfies Walras' law and the individual demands are continuous functions, then there exists a p such that $z(p) \leq 0$.*

Proof (sketch) Define function g_l on Δ^{L-1} such that

$$g_l(p) = \frac{p_l + \max\{0, z_l(p)\}}{\sum_k (p_k + \max\{0, z_k(p)\})}.$$

Then function $g = (g_1, ..., g_L)$ is from Δ^{L-1} to itself. By Brouwer's Theorem, there is p^* such that

$$g(p^*) = p^*.$$

We claim that $z(p^*) \leq 0$. By construction,

$$\begin{aligned} p_l^* \sum_k (p_k^* + \max\{0, z_k(p^*)\}) &= p_l^* + p_l^* \sum_k \max\{0, z_k(p^*)\} \\ &= p_l^* + \max\{0, z_l(p^*)\}. \end{aligned}$$

Thus

$$p_l^* \sum_k \max\{0, z_k(p^*)\} = \max\{0, z_l(p^*)\}$$

and, *a fortioti*,

$$z_l(p^*)p_l^* \sum_k \max\{0, z_k(p^*)\} = z_l(p^*) \max\{0, z_l(p^*)\}.$$

Summing over all l, and using Walras' law,

$$\sum_l z_l(p^*) \max\{0, z_l(p^*)\} = 0.$$

Unless $z_l(p^*) = 0$ for all l, this condition cannot hold.

- Continuity of individual demand ensured by convex (+continuous) preferences . Hence convexity guaranatees teh existence of the Walrasian equilibrium.

- Uniqueness cannot be guaranteed.

- Properties of competitive equilibrium: recall general definition of economic efficiency.

Definition 4 *A feasible outcome $a \in A$ is Pareto-efficient if there is no other feasible outcome a' such that all agents are at least as well of at a' as at a and some agent strictly prefers a' to a*

- Specializing to current setting

Definition 5 *A consumption vector $x = (x^1, ..., x^H)$ is Pareto efficient if there is no $y = (y^1, ..., y^H)$ such that s is feasible, i.e., $\sum_h y_l^h \leq \sum_h \omega_l^h$ for all $l = 1, ..., L$ and $u_h(y^h) \geq u_h(x^h)$ for all $h = 1, ..., H$ and $u_{h'}(y^{h'}) > u_{h'}(x^{h'})$ for some h'.*

Theorem 6 (First Fundamental Welfare Theorem) *Suppose that the preferences of all consumers satisfy local non-satiation. Then all competitive equilibrium allocations are Pareto efficient.*

Proof Let (x, p) be a competitive equilibrium. Suppose that y Pareto dominates x. By local non-satiation: $p \cdot y^h \geq p \cdot x^h$ for all h and $p \cdot y^{h'} > p \cdot x^{h'}$ for some h'. Summing over h gives

$$p \cdot \sum_{h=1}^{H} y^h > p \cdot \sum_{h=1}^{H} x^h = p \cdot \sum_{h=1}^{H} \omega^h,$$

where last equality follows from Walras' law. In other words,

$$\sum_{l=1}^{L} p_l \sum_{h=1}^{H} y_l^h > \sum_{l=1}^{L} p_l \sum_{h=1}^{H} \omega_l^h.$$

But then there must be a particular l such that $p_l > 0$ and

$$p_l \sum_{h=1}^{H} y_l^h > p_l \sum_{h=1}^{H} \omega_l^h.$$

Thus y is not feasible.

Theorem 7 (Second Fundamental Welfare Theorem) *Suppose that x is a Pareto efficient allocation and that preferences are monotonic. Identify a competitive equilibrium (y, p) emerging from the initial endowments ω such that $\omega = x$. Then, in fact, (x, p) is a competitive equilibrium.*

Proof Since x^h is in consumer h's budget set under p, it must be that $u_h(y^h) \geq u_h(x^h)$ for all h. Since x is Pareto efficient, necessarily $u_h(y^h) = u_h(x^h)$ for all h. Since y^h is optimal for each h, also x^h is optimal for each h. Then (x, p) is a Walrasian equilibrium.

- Implication: *All Pareto efficient allocations are competitive equilibrium allocations for some prices and some initial endowments.*

- Note that existence of competitive equilibrium (supposed in theorem) is ensured by convex preferences.

- Overall:

  - Marginal rates of substitution for individuals must be equalized at Pareto-optimal allocations

  - At a competitive equilibirium (x, p), interior optimality implies that

    $$MRS_{lk}^h = \frac{\partial u_h(x)/\partial x_k}{\partial u_h(x)/\partial x_l} = \frac{p_k}{p_l}$$

    for each individual h and goods l and k.

– Watch out for corner solutions!

Example: Cobb-Douglas Economy

- Utility functions take the form

$$u^h(x_1, ..., x_L) = \sum_{l=1}^{L} \alpha_l^h \ln x_i$$

where $0 < \alpha_l^h < 1$ for all l and h; and $\sum_l \alpha_l^h = 1$ for all h.

- Let $H = L$ and initial endowments be given by $\omega^1 = (1, 0, ..., 0)$, $\omega^2 = (0, 1, 0, ..., 0)$, ..., and $\omega^H = (0, ..., 0, 1)$.

- At h's optimum,

$$\frac{\alpha_k^h / x_k^h}{\alpha_l^h / x_l^h} = \frac{p_k}{p_l},$$

i.e.

$$p_l x_l^h = \left(\frac{\alpha_l^h}{\alpha_k^h}\right) p_k x_k^h$$

By Walras' law,

$$\sum_l p_l x_l^h = p_h.$$

Thus

$$\frac{p_k x_k^h}{\alpha_k^h} = p_h.$$

and finally h's demand for k :

$$x_k^h(p) = \alpha_k^h \left(\frac{p_h}{p_k}\right).$$

Market clearing in good $l = 1, ..., L$ implies

$$\sum_h x_l^h(p) = 1.$$

Now there are H equations from which the H unkonwns (prices) can be solved.

- E.g. $H = 2$:

$$\begin{aligned} \alpha_1^1 + \alpha_1^2 \left(\frac{p_2}{p_1}\right) &= 1, \\ \alpha_2^1 \left(\frac{p_1}{p_2}\right) + \alpha_2^2 &= 1, \end{aligned}$$

which yields the equilibrium allocations

$$(x_1^1, x_2^1) = (\alpha_1^1, 1 - \alpha_2^2) \text{ and } (x_1^2, x_2^2) = (1 - \alpha_1^1, \alpha_2^2).$$

Core

- The theory of competitive markets is salient about *how* the equilibrium is reached.

- Underlying is the idea that as the economy grows, individual agents are less able to affect on prices and the Walrasian equilibrium prevails.

- The *Core* was the first attempt to formalize the effect of large markets on the equilibriu formation.

- It seeks to explain how the equilibrium is reached through *coalitional negotiation.*

- As above let $I = \{1, ..., H\}$ be the set of agents with initial endowments $(\omega^h)_{h\in I}$.

- An allocation $x = (x^h)_{h\in I}$ is feasible if

$$\sum_{h\in I} x^h \leq \sum_{h\in I} \omega^h.$$

Definition 8 *A coalition $S \subseteq I$ blocks a feasible allocation x if there is a $(y^h)_{h\in S}$ such that*

$$\sum_{h\in S} y^h \leq \sum_{h\in S} \omega^h$$

and such that $u_h(y^h) > u_h(x^h)$ for all $h \in S$.

- A feasible allocation that blocked by a coalition is not agreeable since all the coalition members are better if they reject the allocation and form a subsystem where they reallocate their own goods among themselves.

Definition 9 *The* **Core** *is the set of feasible allocations* x *that no coalition blocks.*

- The Core, if it exists, is necessarily unique (why?).

- Since I is a coalition, any allocation in the Core must be *Pareto efficient*. Since $\{h\}$ is a coalition, any allocation in the Core must be *individually rational*. Thus the Core allocations are contained by the *contract curve* (allocations that are PO and IR).

- Any *Walrasian allocation x must in the Core*: For suppose a coalition S blocks it via allocation $y_S = (y^h)_{h \in S}$. Let p be the Walrasian price. Then, since $u_h(y^h) > u_h(x^h)$ for all $h \in S$, must have

$$p \cdot y^h > p \cdot \omega^h.$$

Thus also

$$p \cdot \sum_{h \in S} y^h > p \cdot \sum_{h \in S} \omega^h,$$

implying

$$\sum_{h \in S} y_l^h > \sum_{h \in S} \omega_l^h$$

for at least one l, contradicting the hypothesis that S blocks via y.

- Increase the size of the economy by *replicating* the agents and their endowments for n times: $nI = \{11, ..., 1H, 21, ..., 2H, ..., n1, ..., nH\}$ each jh

possessing endowment ω^h. The n times replicated economy is called the n-*replica* of the original economy.

- Allocation $(x^{jh})_{jh \in nI}$ has the *equal treatment property* if $x^{jh} = x^{ih}$ for all $j, i = 1, ..., n$.

- An allocation $(x^{jh})_{jh \in nI}$ that has the equal treatment property treats all similar agents similarly. Moreover, if it is feasible in the nI agents problem, then it can be expressed in terms of a feasible allocation of a *single* generation problem $(x^h)_{h \in I}$. In such case, $(x^h)_{h \in I}$ is the *type allocation* of $(x^{jh})_{jh \in nI}$.

- Any allocation in the Core meets equal treatment if preferences are convex and strongly monotonic.

Theorem 10 (Core Convergence) *Let preferences be convex and strongly monotonic. Then the feasible type allocation* $(x^h)_{h \in I}$ *is in the Core of the* $n-$*replica economy for all* $n = 1, 2, ...$ *only if it is a Walrasian equilibrium allocation.*

Proof: In $H = 2$ case, let y not be a Walrasian allocation but in the Core for all replications $n = 1, 2, ...$ Since y satiasfies equal treatment, it is feasible in the $n = 1$ case:

$$y^1 + y^2 = \omega^1 + \omega^2.$$

Assume, without loss, that $u_1(y^1) > u_1(x^1)$ for a Walrasian allocation x. Then there is rational number $a/b \in (0, 1)$ where a and b are integers such that

$$u_2(\frac{a}{b}y^2 + \frac{b-a}{b}\omega^2) > u_2(\omega^2).$$

Take a coalition consisting of a type 1 agents and b type 2 agents. Reallocate so that each type 1 agent still gets y^1 and each type 2 gets $\frac{a}{b}y^2 + \frac{b-a}{b}\omega^2$. Since the contribution of goods of type 2 agents to the coalition is

$$-b(\frac{a}{b}y^2 + \frac{b-a}{b}\omega^2 - \omega^2) = -a(y^2 - \omega^2),$$

and since the type 1 receive the amount

$$-a(y^1 - \omega^1),$$

the reallocation is feasible for the coalition. But then the coalition blocks y.

Production Economies

- Firms are indexed by $j \in \{1, ..., J\}$
- Technologies given by production sets $Y^j \subset \mathbb{R}^L$
- Consumer h owns share θ^{hj} of firm j

- An allocation is $(x, y) = (x^1, ..., x^H, y^1, ..., y^J)$ such that $x^h \in \mathbb{R}^L_+$ for all h and $y^j \in Y^j$ for all j

- An allocation is feasible if

$$\sum_h x^h = \sum_h \omega^h + \sum_h y^h$$

- An allocation (x, y) is Pareto-efficient if there is no other feasible allocation (x', y') such that for all h

$$x^h \succeq_h x'^h \text{ and } x^{h'} \succ_{h'} x'^{h'} \text{ for some } h'$$

Definition 11 *A competitive equilibrium in a production economy consists of a price vector* $p^* \in \mathbb{R}^L_+$ *and an allocation* (x^*, y^*) *such that*

1. $\{y^{*j}$ *solves* $\max_{y \in Y^j} p \cdot y$ *for all* j

2. $\{x^{*h}$ *solves* $\max_x u(x)$ *s.t.* $p \cdot x \leq p \cdot \omega^h + \sum_j \theta^{hj}(p \cdot y^j)$

3. $\{\sum_h x^{*h} = \sum_h \omega^h + \sum_j y^{*j}$

- Existence, local uniqueness and optimality properties can be established along similar lines to exchange economies

- Extending the model
  - sequence of dates, e.g. macro and dynamic finance models

- number of contingencies important in applied work.
- externalities and public goods

- Problems arise with
  - missing markets
  - incomplete information
  - more realistic firm behaviour