Microeconomic Theory

Lecture 5

Standard Portfolio Choice

- Risk averse decision maker. Initial wealth w_0. Decision problem: How much to invest in safe versus risky assets?

- No short sales allowed.

- $(1 + r)$ riskless return.

- $(1 + \tilde{x})$ the random return on the risky investment.

- Denote the amount of risky investment by $0 \leq \alpha \leq w_0$, and thus the safe investment is $(w_0 - \alpha)$.

- Final wealth of the decision maker:

$$(w_0 - \alpha)(1 + r) + \alpha(1 + \tilde{x}) = w_0(1 + r) + \alpha(\tilde{x} - r).$$

- Strictly concave, strictly increasing twice differentiable utility function $u(w)$.

- Expected utility from a risky investment α:

$$v(\alpha) = \mathbb{E}u(w_0(1 + r) + \alpha(\tilde{x} - r)).$$

- $v(\alpha)$ is a strictly concave function of α if $\Pr(\tilde{x} = r) < 1$ since

$$v''(\alpha) = \mathbb{E}\left((\tilde{x} - r)^2 u''(w_0(1 + r) + \alpha(\tilde{x} - r))\right) < 0.$$

- Thus FOC sufficient for maximum:

- Interior solution ($0 < \alpha < w_0$)

$$v'(\alpha) = \mathbb{E}(\widetilde{x} - r)\, u'(w_0(1 + r) + \alpha(\widetilde{x} - r)) = 0.$$

- Conrner solution ($\alpha = 0$)

$$v'(0) = \mathbb{E}(\widetilde{x} - r)\, u'(w_0(1 + r)) \leq 0.$$

  This is equivalent to $\mathbb{E}(\widetilde{x}) \leq r$.

- Hence a necessary *and* sufficient (why?) condition for risky investment is that the expected value of the investment be no larger than the safe return.

- Thus all decision makers, risk averse or not, invest some positive amount in risky assets if their expected return is larger than the safe rate.

- Consider two risk averse decision makers, $u_1$and u_2.

- Suppose that u_1 is more risk averse than u_2.

- Then $u_1(x) = \phi(u_2(x))$ for some concave function ϕ.

- We want to see how the optimal portfolio choices of u_1 and u_2 can be compared.

- Denote the optimal risky investments by α_1 and α_2 respectively. From the FOC for u_2 we have:

$$v_2'(\alpha_2) = \mathbb{E}(\widetilde{x} - r)\, u_2'(w_0(1 + r) + \alpha_2(\widetilde{x} - r)) = 0. \qquad (1)$$

- To see how the optimal risky investment of u_1 relates to α_2, evaluate the derivative of $v_1(\cdot)$ at $\alpha = \alpha_2$.

$$\begin{aligned} v_1'(\alpha_2) &= \frac{d}{d\alpha}\mathbb{E}\phi\left(u_2\left(w_0(1+r)+\alpha_2(\widetilde{x}-r)\right)\right) \\ &= \mathbb{E}\left(\widetilde{x}-r\right)\phi'\left(u_2\left(w_0(1+r)+\alpha_2(\widetilde{x}-r)\right)\right)u_2'\left(w_0(1+r)+\alpha_2(\widetilde{x}-r)\right) \end{aligned}$$

- Since $\phi'' \leq 0$, we know that for $\widetilde{x} < r$,

$$\phi'\left(u_2\left(w_0(1+r)+\alpha_2(\widetilde{x}-r)\right)\right) \geq \phi'\left(u_2\left(w_0(1+r)\right)\right)$$

and similarly for $\widetilde{x} > r$,

$$\phi'\left(u_2\left(w_0(1+r)+\alpha_2(\widetilde{x}-r)\right)\right) \leq \phi'\left(u_2\left(w_0(1+r)\right)\right).$$

Hence

$$(\widetilde{x}-r)\,\phi'\left(u_2\left(w_0(1+r)+\alpha_2(\widetilde{x}-r)\right)\right) \leq (\widetilde{x}-r)\,\phi'\left(u_2\left(w_0(1+r)\right)\right), \text{ for all } \widetilde{x}.$$

But then we know that

$$\begin{aligned} v_1'(\alpha_2) &\leq \mathbb{E}(\widetilde{x}-r)\,\phi'(u_2(w_0(1+r)))\,u_2'(w_0(1+r)+\alpha_2(\widetilde{x}-r)) \\ &= \phi'(u_2(w_0(1+r)))\,\mathbb{E}(\widetilde{x}-r)\,u_2'(w_0(1+r)+\alpha_2(\widetilde{x}-r)) = 0, \end{aligned}$$

where the last equality follows from 1. Thus by the concavity of $v_1(\alpha)$, we know that $\alpha_1 \leq \alpha_2$.

Proposition 1 *If u_1 is more risk averse than u_2, then 1 does not invest more than 2 to the risky asset .*

- This proposition also yields an immediate corollary for risky investment as a function of initial wealth.

- Let $\alpha(w_0)$ be the optimal investement under initial wealth w_0.

Proposition 2 *If u exhibits DARA, then $\alpha(w_0) \leq \alpha\left(w_0'\right)$ whenever $w_0 < w_0'$.*

Proof. Take $u_2(z) = u(z)$ and $u_1(z) = u(z-k)$ and apply the previous theorem. ■

Consumption and Savings

- Start with the simplest deterministic two-period model, and derive conclusions for optimal savings and consumption.

- Additively separable utility function.

- In other words, the consumer has a separate Bernoulli utility function for the consumption in each period $t = 0, 1$.

- The consumer receives wealth w_0 and w_1 respectively in the two periods.

- She can borrow and lend as she wishes at the risk free rate r.

- If we let s denote the savings by the consumer, then the optimization problem can be written as

$$\max_{s} \; u_0\left(w_0 - s\right) + u_1\left(w_1 + s\left(1 + r\right)\right).$$

- Observe that we can allow for negative saving (i.e. borrowing) in this model, but we require that consumption be positive in both periods (i.e. $s \leq w_0$).

- Assume throughout that $u_i\left(\cdot\right)$ are strictly concave and twice continuously differentiable for $i = 0, 1$.

- Hence if we let

$$v\left(s\right) = u_0\left(w_0 - s\right) + u_1\left(w_1 + s\left(1 + r\right)\right),$$

we see immediately that $v''(s) < 0$.

- This allows us again to locate optimal savings levels from the first order conditions.

- The optimal level of savings s^* is characterized by

$$v'(s^*) = -u_0'(w_0 - s^*) + (1+r)\,u_1'(w_1 + s^*(1+r)) = 0.$$

- If $u_0 = u_1 = u$ and $r = 0$, we see the most clearly how savings are used to smooth consumption across periods.

- From

$$u'(w_0 - s^*) = u'(w_1 + s^*),$$

we conclude by the strict concavity of u that

$$w_0 - s^* = w_1 + s^*.$$

- Hence the consumption levels in the two periods are identical.

- The other main motive of saving is to increase wealth.

- This effect can obviously only be seen when $r > 0$.

- Again in the case where $u_0 = u_1 = u$, we get

$$u'(w_0 - s^*) = (1 + r)\, u'(w_1 + s^*(1 + r)).$$

- By concavity of u, we see that consumption in the second period is larger (since the marginal utility is lower) than in the first period.

- Hence the consumer is willing to sacrifice some of the consumption smoothing for increases in wealth.

- Finally, we can totally differentiate the FOC with respect to s and w_i to get

$$\frac{ds^*}{dw_0} = \frac{u_0''(w_0 - s^*)}{\left[u_0''(w_0 - s^*) + (1+r)^2 u_1''(w_1 + s^*(1+r))\right]} > 0,$$

$$\frac{ds^*}{dw_1} = \frac{-u_1''(w_1 + s^*(1+r))}{\left[u_0''(w_0 - s^*) + (1+r)^2 u_1''(w_1 + s^*(1+r))\right]} < 0.$$

- Hence an increase in the first period income increases savings, and an increase in the second period income decreases savings.

- With these preliminaries in place, we can start the analysis of the optimal savings problem in a world of uncertainty.

- The first question that we ask is whether the optimal savings are larger in a model where the second period income is random than in the deterministic model.

Definition 3 *A utility function is prudent if adding an uninsurable zero mean risk to the second period income increases the savings.*

- To characterize prudent utility functions, let $\widetilde{w}_1 = w_1 + \widetilde{x}$, where $\widetilde{x}$ is assumed to be uninsurable and $\mathbb{E}\widetilde{x} = 0$.

- Denote the new expected utility from savings s by:

$$V(s) = u_0(w_0 - s) + \mathbb{E}u_1(w_1 + s(1+r) + \widetilde{x}).$$

- $V(s)$ inherits the curvature of the u_i functions.

- Analyze comparative static questions by evaluating the derivative of $V(s)$ at point s^* such that $v'(s^*) = 0$, i.e. at the optimal savings level of the deterministic model.

- Observe that $V'(s^*) \geq 0$ if

$$\mathbb{E}u_1'(w_1 + s^*(1+r) + \widetilde{x}) \geq u_1'(w_1 + s^*(1+r)). \tag{2}$$

- Notice that on the left hand side of the inequality, we have the expected utility from a random variable.

- On the right hand side, we have the utility from the expected value of the random variable.

- This is exactly the definition of a risk loving utility function since w_1 and $\widetilde{x}$ are arbitrary.

- As risk loving functions are convex, we deduce that 2 holds for all w_1 and $\widetilde{x}$ if and only if u_1' is convex.

- Hence we have proved the following proposition.

Proposition 4 *A utility function is prudent if and only if u_1' is convex.*

- From this point on, we could develop a theory for comparing prudence of different individuals or the prudence of a given individual at various wealth levels.

- Much of this theory has been done by Miles Kimball, and the central concept for the analysis is the coefficient of absolute prudence:

$$P(w) = \frac{-u'''(w)}{u''(w)}.$$

- We conclude this section on precautionary savings by recalling from the previous lecture the derivation for decreasing absolute risk aversion, DARA.

$$\frac{d}{dw} r^A(w) = r^A(w) \left[r^A(w) - P(w) \right].$$

- Hence there are two arguments for believing in the prevalence of prudent utility functions.

- First of all, there is direct econometric evidence on the savings behavior of individuals with various degrees of uninsurable risk positions.

- Second, there is overwhelming empirical support for DARA.

- As the formula above indicates, DARA is only possible for prudent utility functions.

First Look at a Behavioral Economics Question

Commitment and Temptation

- Is choice over time a decision problem or a game?

- Commodities are (in principle) differentiated by time, location and contingency. Why couldn't we treat decision makers in the same way?

- Will your preferences tomorrow regarding future choices congruent with your preferences today over those same choices?

- Example: Do you want one apple today or two apples tomorrow? Do you want an apple in 30 days or two apples in 31 days?

- Suppose there is some conflict between the preferences over choices at different points in time. Then an intertemporal choice problem becomes a game between decision makers at different times. How can we capture such preference reversals in a simple model?

Hyperbolic discounting (or $\beta - \delta$ -model)

- Standard model: Let x_t be the choice in period t. Let $\mathbf{x}_0 = (x_0, x_1, x_2)$ be the sequence of choices.

$$\begin{aligned} U(\mathbf{x}_0) &= \sum_{t=1}^{3} \delta^t u(x_t), \\ \text{s.t. } \sum_{t=1}^{3} x_t &\leq w. \end{aligned}$$

FOC

$$u'(x_0) = \delta u'(x_1) \text{ and } u'(x_1) = \delta u'(x_2) \text{ and } \sum_{t=1}^{\infty} x_t = w.$$

- Hyperbolic discounting (Strotz, REStud 1955): for all $s = 0, 1, 2$,

$$U(\mathbf{x}_s) = u(x_s) + \beta \sum_{t \geq s+1} \delta^t u(x_t)$$

for some $\beta < 1$.

- FOC: in period 0,

$$u'(x_0) = \beta\delta u'(x_1) \text{ and } u'(x_1) = \delta u'(x_2) \text{ and } \sum_{t=1}^{3} x_t = w.$$

However, in period 1, FOC implies

$$u'(x_1) = \beta\delta u'(x_2) \text{ and } \sum_{t=2}^{3} x_t = w - x_0.$$

Thus the hyperbolic discounter wants to *reallocate* the savings at period 1 which is in conflict with the inital efficiency.

- Example: Take $\beta = \frac{1}{2}$, $\delta = 1$ and let $u(x) = \ln x$. How does your optimal saving strategy depend on what you know about your future behavior? Would you like to commit to a plan of action at $t = 0$? How would you do that?

- A huge literature on this model. Topics include:

  - Saving for retirement (Laibson).

  - Addiction (O'Donoghue and Rabin)

  - Deadlines in optimal contracts (O'donoghue and Rabin).

- Are There Alternative Explanations for Preference for Commitment? *Temptation and Self-Control* by Gul and Pesendorfer, (Econometrica, 2001):

  - Standard neoclassical preference model on an extended domain. Let L denote the set of lotteries. $X, Y \in L$ are sets of lotteries or menus. 2^L is the set of all possible menus. Preferences are defined on 2^L. Preferences are rational, satisfying a form of continuity and independence axiom for singleton menus.

  - Preference for flexibility Kreps :

    $$\text{If } X \subset Y, \text{ then } Y \succeq X.$$

    Even if $x \succeq z$ for all $z \in Y$ and $z \in X$, we can have $Y \succ X$.

  - Temptation (Set Betweenness):

If $X \succeq Y$, then $X \succeq X \cup Y \succeq Y$.

Theorem 1 (Gul and Pesendorfer): The binary relation $\succeq$ satisfies Rationality, continuity, independence and Set Betweenness if and only if there are continuous linear functions U, u, v such that

$$U(X) := \max_{x \in X}(u(x) + v(x)) - \max_{y \in X} v(y)$$

for all X and U represents $\succeq$.

- This formulation also gives rise for a preference for flexibility. Note the differences:

- A single decision maker.

- Welfare comparisons much easier.

- Are there assumptions about own future behavior?