Microeconomic Theory

Lecture 4

Choice under uncertainty

- Uncertainty an unseparararable part of individual decision making.
- But what is uncertainty?
- How can it be quantified?
- How we should think about choice under uncertainty?

Approaches towards probability

- How should one assess the following probabilities?
  - 'This coin toss results in Heads'
  - 'Social Democrats will be the largest party in the next election'
  - 'Rome is more northern than Madrid'
- Classical view: Probability of an event is the long run frequency of the occurrence of the event in a sequence of independent experiments
- Subjectivist view: There is no other meaning to the probability of an event except as a feature of a decision maker's preferences in a choice situation.

- In the subjectivist view, probability can be deduced from choice behavior.

- Hence in classical view, only the first result can have a probabilistic meaning whereas in the subjectivist view all of these statements can have a probabilistic interpretation.

- Clearly the subjective view is the more relevant one for economic theory.

Probabilities

- Consider first a finite set of possible outcomes or consequences C.
- To talk about random experiments on C, we define *events*.
- Events are a family $\mathcal{A}$ of subsets of C. $\mathcal{A}$ satisfies:

  1. $C \in \mathcal{A}$,
  2. $A \in \mathcal{A}$ implies that $A^C \in \mathcal{A}$,
  3. $A_i \in \mathcal{A}$ implies that $\cup_{i=1}^{\infty} A_i \in \mathcal{A}$.

- Probability is a non-negative real valued function on $\mathcal{A}$. In general, we require:

  1. $P(\emptyset) = 0$,

  2. $P(C) = 1$,

  3. $P(A \cup B) = P(A) + P(B)$ if $A \cap B = \emptyset$, for $A, B \in \mathcal{A}$,

  4. $P\left(\cup_{i=1}^{\infty} A_i\right) = \Sigma_{i=1}^{\infty} P(A_i)$ if $A_i \cap A_j = \emptyset$ for all $i \neq j$, and $A_i \in \mathcal{A}$ for all i.

- If C is a finite set, then let $\mathcal{A} = 2^C$.

- For a finite C, the set

$$\mathcal{L} = \left\{ (p_c)_{c \in C} : \sum_{c \in C} p_c = 1 \text{ and } p_c \geq 0 \text{ for all } c \in C \right\}$$

contains all the relevant information. We call $\mathcal{L}$ the set of *simple lotteries* with a typical element $L = (p_c)_{c \in C}$.

Objective of this class: **Find useful representation of preferences on $\mathcal{L}$**

- Distinctions made in literature:

  1. Probabilities are exogenously given $\leftrightarrow$ risk; von Neumann - Morgenstern.

  2. Probabilities subjectively evaluated $\leftrightarrow$ uncertainty; Savage

- – Combination of the two (horse race/coin flipping/roulette wheel): Anscombe - Aumann.

von Neumann - Morgenstern

- We assume here for simplicity that C is finite.

Axiom 1 $\succsim$ *is a rational preference relation on* $\mathcal{L}$.

- Observe that $\mathcal{L}$ is a convex set (why?) and hence it makes sense to talk about *compound lotteries*. Take $L, L' \in \mathcal{L}$. A compound lottery $L^{\alpha} \in \mathcal{L}$ is obtained by setting:

$$L^{\alpha} = \alpha L + (1 - \alpha) L'.$$

- The meaning of this is that if $L = (p_c)_{c \in C}$, and $L' = (p'_c)_{c \in C}$, then $L^{\alpha} = (\alpha p_c + (1 - \alpha) p'_c)_{c \in C}$.

- We formulate the continuity axiom slightly differently from abstract choice theory:

Axiom 2 *(Archimedean Axiom) Take* $L, L', L'' \in \mathcal{L}$ *such that* $L \succ L' \succ L''$. *Then the sets* $\{\alpha \in [0,1] : \alpha L + (1-\alpha) L'' \succsim L'\}$ *and* $\{\alpha \in [0,1] : L' \succsim \alpha L + (1-\alpha) L''\}$ *are closed.*

- That is, together with completeness this implies that small changes in probabilities do not affect the (strict) orderings of lotteries. This form of *continuity* is implied by the standard notion of continuity in Euclidean spaces.

- The condition implies a bound on how good or bad some outcomes can be: no outcome can be incomparably painful.

- Continuity and rationality imply the existence of a continuous representation $U : \mathcal{L} \to \mathbb{R}$.

- How to describe behavior? $\mathcal{L}$ is in general too complicated.

- We want to deduce the existence of an assignment of utility numbers $u : C \to \mathbb{R}$ that allow a straightforward represention of $\succsim$.

- We call U the *von Neumann-Morgenstern utility function* of the decision maker if, for some $u(\cdot)$, it holds that $U(L) = \sum_{c \in C} p_c u(c)$, for all $L = (p_c)_{c \in C}$.

- The next axiom is the key to the representation result.

Axiom 3 *(Independence Axiom) For any $L, L' \in \mathcal{L}$, we have*

$$L \succsim L' \text{ if and only if } \alpha L + (1-\alpha) L'' \succsim \alpha L' + (1-\alpha) L''$$

for all $\alpha \in [0,1]$ and for all $L'' \in \mathcal{L}$.

- It should be clear that an axiom of this type makes only little sense for choice under certainty (recall the definition of separable preferences).

- Does it make sense for theory of choice under uncertainty? Independence of irrelevant (counterfactual) alternatives.

- Normative vs. positive implications.

Theorem 4 *(Expected Utility Theorem) A rational preference* $\succsim$ *on* $\mathcal{L}$ *satisfies the Archimedean and Independence axiom if and only if there exists a utility function* $u : C \to \mathbb{R}$ *such that*

$$L \succsim L' \text{ if and only if } \sum_{c \in C} p_C u(c) \geq \sum_{c \in C} p'_C u(c).$$

Furthermore, u *and* u' *are such representations if and only if* $u' = \beta u + \gamma$ *where* $\beta > 0$.

Proof. i) It is easy to verify that the axioms must be satisfied if a representation exists.

ii) We show that the axioms imply the existence of such a representation. Denote by 1_c the degenerate lottery that assigns probability 1 on consequence $c \in C$.

Consider first 1_{c_n}. Since C is finite, there exist worst and best outcomes $c_\circ$ and c° such that $1_{c^\circ} \succsim 1_c \succsim 1_{c_\circ}$ for all c. By independence axiom, $L \succsim 1_{c_\circ}$ for all $L \in \mathcal{L}$ and $1_{c^\circ} \succsim L$ for all $L \in \mathcal{L}$. If $1_{c_\circ} \sim 1_{c^\circ}$, we can take u to be any constant function and the theorem is proved. Assume thus that $1_{c^\circ} \succ 1_{c_\circ}$. Choose $u(c_\circ) = 0$ and $u(c^\circ) = 1$.

We argue next that for each $L \in \mathcal{L}$, there is a unique $\alpha(L)$ such that $L \sim \alpha(L) 1_{c^\circ} + (1 - \alpha(L)) 1_c$.

By Archimedean axiom,
$\{\alpha : \alpha \mathbf{1}_{c^\circ} + (1-\alpha)\, \mathbf{1}_{c_\circ} \succsim L\}$ and $\{\alpha : L \succsim \alpha \mathbf{1}_{c^\circ} + (1-\alpha)\, \mathbf{1}_{c_\circ}\}$ are closed.

By the completeness of $\succsim$, the union of these sets is $[0,1]$, which is connected, and hence the intersection of the sets must be nonempty.

By the independence axiom and the assumption that $\mathbf{1}_{c^\circ} \succ \mathbf{1}_{c_\circ}$,

$$\alpha \mathbf{1}_{c^\circ} + (1-\alpha)\, \mathbf{1}_{c_\circ} \succ \alpha' \mathbf{1}_{c^\circ} + \left(1-\alpha'\right) \mathbf{1}_{c_\circ} \text{ if } \alpha > \alpha'.$$

(why?) Thus

$$\{\alpha : \alpha \mathbf{1}_{c^\circ} + (1-\alpha)\, \mathbf{1}_{c_\circ} \succsim L\} \cap \{\alpha : \alpha \mathbf{1}_{c^\circ} + (1-\alpha)\, \mathbf{1}_{c_\circ} \preceq L\}$$

must be a singleton. Choose $\alpha\,(L)$ for the unique element in the intersection.

We claim that there is $u : C \to \mathbb{R}$ such that $\alpha(L) = \sum_{c \in C} p_c u(c)$. All we need is that for all L, L' and $\gamma \in [0,1]$ we have:

$$\alpha\left(\gamma L + (1-\gamma) L'\right) = \gamma\alpha(L) + (1-\gamma)\alpha\left(L'\right).$$

(Why is this sufficient?). This is the case since

$$\begin{aligned}
& \gamma L + (1-\gamma) L' \\
\sim\ & \gamma[\alpha(L) 1_{c^\circ} + (1-\alpha(L)) 1_{c_\circ}] + (1-\gamma)[\alpha\left(L'\right) 1_{c^\circ} + \left(1-\alpha\left(L'\right)\right) 1_{c_\circ}] \\
\sim\ & [\gamma\alpha(L) + (1-\gamma)\alpha\left(L'\right)]1_{c^\circ} + [\gamma(1-\alpha(L)) + (1-\gamma)\left(1-\alpha\left(L'\right)\right)]1_{c_\circ}.
\end{aligned}$$

And thus

$$\alpha\left(\gamma L + (1-\gamma) L'\right) = \gamma\alpha(L) + (1-\gamma)\alpha\left(L'\right).$$

Thus we can take $u(c) = \alpha(1_{c_n})$ and $U(L) = \alpha(L)$.

iii) If u' and u'' are the Bernoulli utility functions, let U' and U'' be the corresponding von Neumann-Morgenstern utility functions. The claim is proved if we prove it for U' and U''.

As before, let $\alpha'(L)$ solve

$$U'(L) = \alpha'(L)\, U'(1_{c^\circ}) + \Big(1 - \alpha'(L)\Big)\, U'(1_{c_\circ}).$$

Thus

$$\alpha'(L) = \frac{U'(L) - U'(1_{c_\circ})}{U'(1_{c^\circ}) - U'(1_{c_\circ})}.$$

But now since U'' is also a representation, we have

$$U''(L) = \alpha'(L)\, U''(1_{c^\circ}) + \Big(1 - \alpha'(L)\Big)\, U''(1_{c_\circ}).$$

Plugging in the value of $\alpha'(L)$ and rearranging, we get:

$$U''(L) = \beta U'(L) + \gamma,$$

where

$$\beta = \frac{U''\left(1_{c^\circ}\right) - U''\left(1_{c_\circ}\right)}{U'\left(1_{c^\circ}\right) - U'\left(1_{c_\circ}\right)}$$

and

$$\gamma = U''\left(1_{c_\circ}\right) - U'\left(1_{c_\circ}\right)\frac{U''\left(1_{c^\circ}\right) - U''\left(1_{c_\circ}\right)}{U'\left(1_{c^\circ}\right) - U'\left(1_{c_\circ}\right)}.$$

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- Observations: A huge simplification for use in descriptive applications.

- Can be put to use in normative exercises: i.e. construct more complicated preferences from simple preferences.

- Can be generalized to a large extent $\Rightarrow$ game theory.

- Evolutionary defence: Dutch book -argument.

Problems with the expected utility -theory

- Fails in experiments
  - Independence axiom: Allais' Paradox

C (spanning the three prize columns)

	2 500 000	500 000	0
L_1	0	1	0
L_1'	0,1	0,89	0,01

	2 500 000	500 000	0
L_2	0	0,11	0,89
L_2'	0,1	0	0,90

  - Response: Relax the independence axiom to accommodate this behavior $\Rightarrow$ non-expected utility.

- Betweenness: Dekel (1986), *JET*.

- Rank-Dependent Utility: Quiggin (1982), *JEBO*.

  – Ellsberg's Paradox show failure of probability model.

- Recall the distinction between risk and uncertainty.

- Ambiguity aversion: Choice in the face of non-additive probabilities.

- Minmax preferences vs. Choquet integrals.

Other Extensions:

- Preference for flexibility: Kreps (1979) and Dekel, Lipman and Rustichini (2002).
- Preference for commitment: Gul and Pesendorfer (2002).
- Loss Aversion: Kahnemann and Tverksy (1979).

Risk

- From now on, consequences are monetary amounts: $x \in \mathbb{R}_+$ is the final wealth of the decision maker.
- Analyze different $u : \mathbb{R}_+ \to \mathbb{R}$.
- $F(x)$ denotes the distribution function of a monetary lottery.
- Discrete or continuous.

- Expected Utility theorem:

$$U(F) = \int_{x \in \mathbb{R}_+} u(x) dF(x).$$

Or for discrete distributions:

$$U(F) = \sum_{x \geq 0} u(x) p(x),$$

where $p(x)$ is the mass function of the lottery.

- When do the integrals and sums above converge? (cf. St.Petersburg's paradox).

- Note the symmetry in the formula between $u(x)$ and $F(x)$.

- When considering expected utility, we can consider variations in each of these two components.

  1. Risk attitudes: Fix $F(x)$ and compare different $u(x)$.

  2. Riskiness of lotteries: Fix $u(x)$ and compare different $F(x)$.

Definition 5 *The* certainty equivalent $c(F,u)$ *of a lottery* F *for a decision maker with utility function* u *is defined by*

$$u(c(F,u)) = \int u(x)dF(x)$$

- We can discuss attitudes towards risk by comparing the certainty equivalents of a fixed lottery under different utility functions.

Definition 6 *A decision maker with a utility function u is* risk averse *if, for all F,*

$$c(F, u) \leq \int x dF(x).$$

- It is easy to prove that

Proposition 7 *Utility function u is* risk averse *if and only if it is concave.*

- Risk loving attitudes are defined with the opposite inequalities.

- Can second derivatives be used to measure risk aversion? But not robust against linear transformations of u!

Definition 8 *The Arrow-Pratt measure of absolute risk aversion,* $r_A(x, u)$ *of utility function* u *at wealth level* x *is given by:*

$$r_A(x, u) = -\frac{u''(x)}{u'(x)}.$$

- $r_A(x, u)$ reflects the curvature of u at x without accounting linear transformations.

- The following results shows that $r_A(x, u)$ is a good measure of risk aversion.

- When is u_1 unambigiously more risk averse than u_2?

Proposition 9 *The following are equivalent:*

i) $r_A(x, u_2) \geq r_A(x, u_1)$ *for all* x.

ii) $c(F, u_2) \leq c(F, u_1)$ *for all* $F(x)$.

iii) There is a concave function $\phi(\cdot)$ *such that* $u_2(x) = \phi(u_1(x))$.

- The more-risk-averse-than relation is a partial ordering of utiity functions: it is not possible compare all u and u' in this sense.

- Do wealthier individuals take bigger risks than others?

Definition 10 *u exhibits* decreasing *(constant)* absolute risk aversion, *DARA, (CARA, resp.) if $r_A(x, u)$ is a decreasing (constant, resp.) function of x.*

- If u exhibits DARA, then the decion maker is less sensitive towards risk when his wealth increases, i.e. he is willing to pay less to get rid of risk as his wealth increases.

Proposition 11 *The following are equivalent:*

i) u exhibits DARA.

ii) if $x_2 < x_1$, *then there is a concave function* $\phi(\cdot)$ *such that* $\phi(u(x_1 + z)) = u(x_2 + z)$ *for all* z.

iii) if $u_x(z) = u(x + z)$ *for all* z, *then* $x - c(F, u_x)$ *is decreasing in* x, *for all* F.

- Thus a person with DARA utility function is less risk-averse when richer.

- A stronger condition: u exhibits *decreasing* (constant) *absolute risk aversion*, DRRA, (CRRA, resp.) if $r_R(x, u) := xr_A(x, u)$ decreases in x.

- Requires decreasing risk aversion subject to proportional gambles: e.g. willingness to risk all ones wealth to double it.

- Strong condition, DRRA implies DARA: $r'_A(x, u) < -r_A(x, u)/x$ for all x.

- How do changes in the distribution affect expected payoff?

- Distribution F *first order stochastically dominates* G if, for all nondecreasing functions u we have

$$\int u(x)dF(x) \geq \int u(x)dG(x).$$

- This is equivalent to saying that $F \leq G$. Thus shifting probability mass forward increases expected payoff.

- But first order stochastic dominance may concern the mean values of the lotteries and hence be independent of their risk properties.

- Distribution F *second order stochastically dominates* G if they have the same mean

$$\int x dF(x) = \int x dG(x),$$

and for all nondecreasing, concave functions u we have

$$\int u(x) dF(x) \geq \int u(x) dG(x).$$

- Shift from F to G is a *mean preserving spread*, disliked by any risk-averter.

Subjective probabilities - Savage

Building Blocks:

- Consequences, States of the World, Acts.

- Consequences as before, C.

- State of the World: Complete list of all relevant data for the problem at hand, Ω.

- Acts: Functions $f : \Omega \to C$.

- Preferences are defined on $\mathcal{F}$, the set of all possible acts.

- From the preferences, we can deduce:

  1. A subjective probability assessment $p(\omega)$ on Ω.

  2. A utility function on $u : C \to \mathbb{R}$ such that for all $f, f' \in \mathcal{F}$,

  $$f \succsim f' \text{ if and only if } \int u(f(\omega))\, dp(\omega) \geq \int u\left(f'(\omega)\right) dp(\omega).$$

  or

  $$f \succsim f' \text{ if and only if } \sum_{\omega \in \Omega} u(f(\omega))\, p(\omega) \geq \sum_{\omega \in \Omega} u\left(f'(\omega)\right) p(\omega).$$

  Notice that the end result looks pretty much the same as in von Neumann-Morgenstern theory. Hence we'll be agnostic about subjective vs. objective uncertainty.

- A conclusion: if one believes the Savage axioms, then one can recover the beliefs and and utility functions of the individuals with sufficient data.

- Since also the probabilities need to be accounted, more axioms needed (than under vNM). As a consequence, the characterization complex.

Anscombe-Aumann (1963)

- Assume that acts are

$$f : \Omega \rightarrow \Delta\left(C\right),$$

  or in words, acts assign objective lotteries to states of the world.

- Technically simpler than the fully subjective view. Same end result.

State-dependent preferences

- We'll use sometimes in applications state-dependent utility functions:

$$u : \Omega \times C \to \mathbb{R}.$$

  This is a convenient device to handle situations where C is directly chosen by the decision maker, and ω is chosen exogenously "by nature".

- Expected utility formula in this framework is:

$$U(c) = \sum_{\omega \in \Omega} u(\omega, c)\, p(\omega).$$