Microeconomic Theory

Lecture 2-3

Consumer with preferences

- From lecture 1: If $\succsim$ is rational, i.e. complete and rational, then $c^*(\cdot, \succsim)$ satisfies WA.

- Let $x(p, w) = c^*(B(p, w), \succsim)$.

- If $x(p, w)$ is homogenous of degree 0 in (p, w) and satisfies Walras' Law, then $x(p, w)$ also satisfies compensated law of demand and hence the Slutsky matrix is negative semidefinite.

- Since $B(p, w) = B(\lambda p, \lambda w)$, for all $\lambda > 0$,.$x(p, w)$ is homogenous of degree 0.

- For Walras' Law, we need new assumptions.

Axiom 1 *Preferences* $\succsim$ *are* monotonic *if, for all* $x, y \in X$,

$$x_i \geq y_i, \text{ for all } i \text{ imply } x \succsim y.$$

They are strongly monotonic *if, for all* $x, y \in X$,

$$x \geq y \text{ and } x \neq y \text{ imply } x \succ y.$$

Axiom 2 *Preferences* $\succsim$ *are locally* nonsatiated *if for all* $x \in X$ *and for all* $\delta > 0$*, there exists* $y \in X$ *such that*

$$\|y - x\| < \delta \text{ and } y \succ x.$$

- Local satiation is implied by strong monotonicity but not vice versa.

- Local non-satiation of $\succsim$ implies Walras' law, thus strict monotonicity implies it.

- Other classical assumptions guarantee further structure, and stronger results.

Axiom 3 *Preferences* $\succsim$ *are* convex *if for all* $x, y, \in X$ *and for all* $t \in [0, 1]$,

$$x \succsim y \text{ implies } (tx + (1 - t)y) \succsim y.$$

They are strictly convex *if for all* $x, y, \in X$ *and for all* $t \in (0, 1)$,

$$x \succsim y \text{ implies } (tx + (1 - t)y) \succ y.$$

- Recall the upper and lower sets $U(x, \succsim) = \{y \in X : y \succsim x$ and $L(x, \succsim) = \{y \in X : x \succsim y\}$.

Axiom 4 *Preferences* $\succsim$ *are* continuous *if, for all* $x \in X$, *the sets* $U(x, \succsim)$ *and* $L(x, \succsim)$ *are closed.*

- Recall the definition of quasi-concave functions:

Definition 5 *Let f be defined on the convex set $X \subset \mathbb{R}^n$. It is a* quasiconcave *function if and only if*

$$f(tx_1 + (1-t)x_2) \geq \min[f(x_1), f(x_2)]$$

for every $x_1, x_2 \in X$,and $0 \leq t \leq 1$.

Theorem 6 *Let f be defined on the convex set $X \subset \mathbb{R}^n$. It is quasiconcave if its upper sets $U(f, \alpha) = \{x : x \in X, f(x) \geq \alpha\}$ are convex sets for every real α.*

Proposition 7 *Let rational preferences $\succsim$ be continuous, convex, and strongly monotonous. Then there is a continuous utility function u which represents those preferences. Moreover, u is quasiconcave and convex valued.*

- Notice from here the connection between quasiconcavity of a representation and the convexity of the underlying preferences. Note also that a concave function is quasiconcave but not vice versa.

- If $\succsim$ is strictly convex, then $x(p, w)$ is a singleton for all p, w.

- Assume from now on:
  - Budget set $B(p, w) = \{x \in X : p \cdot x \leq w\}$ and $p \gg 0$.
  - Monotonous, strictly convex, and continuous preferences $\succsim$.

- Thus there is a continuous u that represents $\succsim$ and a single valued $x(p, w)$ that solves.

$$\max_{x \in B(p,w)} u(x).$$

or, equivalently,

$$\begin{aligned} &\max_{x \geq 0} \; u(x) \\ &\text{s.t. } p \cdot x \leq w . \end{aligned}$$

- Does $x(p, w)$ always exist?

Proposition 8 *(Weierstrass). Let $f : X \rightarrow \mathbb{R}$ be a continuous function and X a compact set. Then f attains its maximum on x, i.e. there is a point $x^* \in X$ such that*

$$f(x^*) = \max_{x \in X} f(x).$$

- Recall: $X \subset \mathbb{R}^n$ is compact if and only if it is closed and bounded.

- It is easy to see through counterexamples that continuity, boundedness and closedness are all required for the result.

- We have assumed $p \gg 0$, so that $B(p, w)$ is a compact convex set, and u is continuous, hence a utility maximizer does exist.

- How to characterize $x(p, w)$?

- Assuming an interior solution, construct a Lagrangian

$$\mathcal{L}(x, \lambda) = u(x) - \lambda(p \cdot x - w),$$

where $\lambda \in \mathbb{R}$ is the Lagrange multiplier.

- If $x^* = x(p, w)$ solves the Lagrangian, then it meets the first order conditions

$$\begin{aligned} \frac{\partial u(x^*)}{\partial x_l} - \lambda p_l &= 0 \text{ for all } l = 1, \ldots L, \\ p \cdot x^* &= w. \end{aligned}$$

- Thus

$$\frac{\partial u(x^*) / \partial x_l}{\partial u(x^*) / \partial x_k} = \frac{p_l}{p_k}.$$

- The ratio p_l/p_k is the *marginal rate of substitution* between goods l and k at x^*.

- On the ther hand

$$\frac{\partial u(x^*)}{\partial p_l} = \lambda p_l$$

- Lagrange multiplier λ gives the marginal (shadow) value of relaxing the constraint, i.e. the marginal value of wealth w. To see this, recall that by the first order condition,

$$D_x \mathcal{L}(x^*, \lambda) = Du(x^*) - \lambda p = 0,$$

  and by Engel aggregation

$$p \cdot D_w x(p, w) = 1.$$

  Thus, by the chain rule,

$$\begin{aligned} D_w u(x(p, w)) &= Du(x^*) \cdot D_w x(p, w) \\ &= \lambda[p \cdot D_w x(p, w)] \\ &= \lambda. \end{aligned}$$

- Define the *indirect utility function* by condition

$$v(p, w) = u(x(p, w)).$$

- What are the properties of $v(p, w)$ implied by the utility maximization problem?

- Conversely, if $v(\cdot, \cdot)$ is known, can we recover the utility function $u(\cdot)$, i.e. the preferences $\succsim$?

- Let u represent monotonic, strictly convex, and continuous preferences $\succsim$. Then $v(\cdot, \cdot)$ is:
  - Homogenous of degree 0.
  - Increasing in w, nonincreasing in p.
  - Continuous.
  - Quasiconvex (that is $-v(p, w)$ is quasiconcave, or $\{(p, w) : v(p, w) \leq v\}$ is a convex set).
- If $\succsim$ is strongly monotonic, then v is strictly increasing in w.

Proposition 9 *If $v(p,w)$ satisfies i-iv, then there exists a monotonic, continuous, and quasiconcave $u(x)$ such that $v(p,w) = u(x(p,w))$.*

- We can solve for such a $u(x)$ from the problem

$$\min_{p \in \mathbb{R}^L_{++}} \quad v(p,w)$$
$$\text{s.t.} p \cdot x = w.$$

Proposition 10 *(Roy's Identity) Given an indirect utility function* $v(p,w)$ *, the Walrasian demand* $x(p,w)$ *can be recovered from*

$$x_l(p,w) = -\frac{\partial v(p,w)/\partial p_l}{\partial v(p,w)/\partial w}.$$

- To see this, note that

$$\begin{aligned}
\frac{\partial v(p,w)}{\partial p_l} &= \frac{\partial u(x(p,w))}{\partial p_l} \\
&= \frac{\partial u(x(p,w))}{\partial x_l} \cdot D_{p_l} x(p,w) \\
&= \lambda p_l \cdot D_{p_l} x(p,w) \\
&= -\lambda x_l(p,w)
\end{aligned}$$

  where the third equality follows from optimizing the Lagrangean and the last one from Cournot aggregation. Moreover, recall the wealth effect:

$$\frac{\partial v(p,w)}{\partial w} = \frac{\partial u(x(p,w))}{\partial w} = \lambda.$$

Duality

- Given utility function $u(\cdot)$, denote by $h(p,u)$ the outcome that solves the expedinture minimizing problem given utility of at least u and prices p:

$$\min_{x \in \mathbb{R}^L_+} p \cdot x$$
$$\text{s.t. } u(x) \geq u.$$

- Notice that even though the feasible set is not bounded, the problem has a solution when $p \in \mathbb{R}^L_{++}$.

- $h(p,u)$ is called the *Hicksian* or *compensated demand function*.

- Denote the value function of the expedinture minimization problem by $e(p,u)$; the *expenditure function*.

$$e(p,u) = p \cdot h(p,u).$$

- The following observation is key to the development that follows.

Proposition 11 *Fix a price vector* $p \in \mathbb{R}^L_{++}$.

- 1. If $x^* = x(p,w)$, then $x^* = h(p,u(x^*)) = h(p,v(p,w))$.

  2. If $x^* = h(p,u)$, then $x^* = x(p,p \cdot x^*) = x(p,e(p,u))$.

- Summarizing:

$$x(p, w) = h(p, v(p, w)) \text{ and } h(p, u) = x(p, e(p, u)).$$

- Obviously then also:

$$w = e(p, v(p, w)) \text{ and } u = v(p, e(p, u)).$$

Proposition 12 *Let u represent monotonic, strictly convex, and continuous preferences $\succsim$. Then, for all (p, w)*

$$h(p, u) = D_p e(p, u).$$

- To see this,

$$\begin{aligned} D_p e(p, u) &= D_p[p \cdot h(p, u)] \\ &= h(p, u) + [p \cdot D_p h(p, u)]^T \\ &= h(p, u) + [\lambda^{-1} D_p u(h(p, u)) \cdot h(p, u)] \\ &= h(p, u). \end{aligned}$$

  where the second equality follows from the first order condition, and the third from the fact that $u(h(p, u)) = u$ for all p.

- Hicksian demand $h(\cdot, \cdot)$ thus satisfies

  – Adding up: $p \cdot h(p, u) = w$.

  – Homogeneity of degree 0 in prices: $h(\alpha p, u) = h(p, u)$ for all p, u, and scalars $\alpha > 0$.

– Convexity: if $\succsim$ is convex, then $h(p, u)$ is a convex set; if $\succsim$ is strictly convex, then $h(p, u)$ is a function.

– Matrix $D_p h(p, u)$ negative semidefinite, *symmetric*, and satisfies $D_p h(p, u) p = 0$.

- Expedinture function $e(\cdot, \cdot)$ satisfies:
  - $e(p, u)$ is homogenous of degree 1 in p.
  - Increasing in u and non-decreasing in p_l for all l.
  - *Concave* in p.
  - Continuous in p, u.

- $D_p h(p,u)$ can be computed from $x(p,w)$ which is observable and thereby potentially testable. How to express $D_p h(p,u)$ in terms of $x(p,w)$?

- Recall :

$$h(p,u) = x(p, e(p,u)).$$

Therefore (Slutsky Equation):

$$\begin{aligned} & D_p h(p,u) \\ = \ & D_p x(p, e(p,u)) + D_w x(p, e(p,u)) D_p e(p,u) \\ = \ & D_p x(p, e(p,u)) + D_w x(p, e(p,u)) h(p,u)^T \\ = \ & D_p x(p,w) + D_w x(p,w) x(p,w)^T, \end{aligned}$$

where we have set $w = e(p,u) = e(p, v(p,w))$.

- Recall the properties of $S(p, w)$, the substitution matrix defined in the choice-based approach.

- $D_p h\,(p, u)$ is symmetric but $S(p, w)$ need not be.

- The restrictions under the preference-based approach are stronger, so we obtain additional observable implications.

1. Homothetic Preferences

2. Quasilinear Preferences

3. Additively Separable Preferences

Definition 13 *A continuous rational preference relation* $\succsim$ *is said to be homothetic if for all* $\alpha > 0$, *we have* $x \succsim y \iff \alpha x \succsim \alpha y$.

Exercise 14 *Show that for homothetic preferences,* $x(p, \alpha w) = \alpha x(p, w)$.

Engel curves are rays through origin.

Let $u(x)$ be a representatition of $\succsim$.

Exercise 15 *If $u(x)$ is linearly homogenous (i.e. $u(\alpha x) = \alpha u(x)$ for all x), then $\succsim$ is homothetic.*

Clearly all representations of $\succsim$ cannot be linearly homogenous.

Exercise 16 *If $\succsim$ is homothetic, then there exists a representation $u(x)$ that is linearly homogenous.*

Hint: Let $e = (1, ..., 1) \in \mathbb{R}^L_+$. Show that for each $x \in \mathbb{R}^L_+$, there is a unique $\lambda(x) \in \mathbb{R}$, such that $x \sim \lambda(x) e$. Show that if $\succsim$ is homothetic, then $\lambda(x)$ is linearly homogenous.

Denote now the first unit vector by $e_1 = (1, 0, ..., 0)$.

Definition 17 *A continuous rational preference relation $\succsim$ on $(-\infty, \infty) \times \mathbb{R}^{L-1}_+$ is said to be quasilinear with respect to commodity 1 (the numeraire) if $x \sim y$ implies that $(x + \alpha e_1) \sim (y + \alpha e_1)$ for all $\alpha \in \mathbb{R}$ and $x + \alpha e_1 \succ x$ for all $\alpha > 0$.*

Here the Engel curves are lines parallel to x_1 axis.

Exercise 18 *A rational preference relation $\succsim$ on $(-\infty, \infty) \times \mathbb{R}_+^{L-1}$ is quasilinear with respect to commodity 1 if and only if it admits a utility representation of the form $u(x) = x_1 + \phi(x_2, ..., x_l)$. (Show only the if part)*

1. $v(p, w) = p_1 w + \theta(p)$.

2. $x_l(p, w) = \xi_l(p)$ for $l \in \{2, ..., L\}$.

3. $e(p, u) = \frac{u}{p_1} - \zeta(p)$.

Utility functions of the form

$$u(x) = \Sigma_{l=1}^{L} u_l(x_l)$$

are called additively separable. It can be shown that the following restriction on preferences is equivalent to additive separability:

Definition 19 *A continuous rational preference relation $\succsim$ is additively separable if for all $\mathcal{L} \subset \{1, .., L\}$ we have*

$$(x_{\mathcal{L}}, y_{-\mathcal{L}}) \succsim \left(x'_{\mathcal{L}}, y_{-\mathcal{L}}\right) \iff \left(x_{\mathcal{L}}, y'_{-\mathcal{L}}\right) \succsim \left(x'_{\mathcal{L}}, y'_{-\mathcal{L}}\right)$$

for all $y_{-\mathcal{L}}, y'_{-\mathcal{L}} \in \times\mathbb{R}_{+}^{L-n(\mathcal{L})}$, where $n(\mathcal{L})$ is the number of elements in $\mathcal{L}$.

Exercise 20 *Show that whenever a utility representation is additively separable, then the above property holds. (The converse is hard.)*

1. If u is strictly quasiconcave, then either all goods are normal or one is normal and all others are inferior.

2. If $u_l(x_l) = u_k(x_k)$ and $u''(x_l) < 0$, then all goods are normal.

3. If $u_l(x_l) = u_k(x_k)$ and $-\frac{x_l u''(x_l)}{u'(x_l)} < 1$, then $\frac{\partial x_l(p,w)}{\partial p_k} > 0$ for all $k \neq l$.

4. For all separable u, $\frac{\partial x_i(p,w)/\partial p_k}{\partial x_j(p,w)/\partial p_k} = \frac{\partial x_i(p,w)/\partial w}{\partial x_j(p,w)/\partial w}$.

5. Element s_{ij} of the slutsky matrix is of the form

$$-\frac{\lambda\left(p,w\right)}{D_w\lambda\left(p,w\right)}\frac{\partial x_i}{\partial w}\frac{\partial x_j}{\partial w} \text{ for } i \neq j,$$

   and for s_{ii} it is of the form

$$\frac{\lambda\left(p,w\right)}{p_i D_w\lambda\left(p,w\right)}\frac{\partial x_i}{\partial w}\left(1 - p_i\frac{\partial x_i}{\partial w}\right).$$

Proposition 21 *Suppose that $u(x)$ is quasiconcave and*

$$\Delta u\left(x^{*}\right)\neq 0,\ \text{for all } x\in B\left(p,w\right).$$

Then, if x^{*}satisfies the Kuhn-Tucker first-order conditions, x^{*} solves UMP.

Note:

Marginal rate of substitution (*MRS*) between goods l and k.

For all l, k such that $x_l^*, x_k^* > 0$, we have:

$$\frac{\frac{\partial u(x^*)}{\partial x_l}}{\frac{\partial u(x^*)}{\partial x_k}} = \frac{p_l}{p_k}.$$

Let $z\left(\bar{q}\right)$ denote value function depending on a parameter vector $\bar{q}$

Objective is $f(x;\bar{q})$ and $g_1\left(x(\bar{q});\bar{q}\right), ...g_m\left(x(\bar{q});\bar{q}\right)$ are constraints.

The Envelope Theorem for the constrained optimization problem says that

$$\nabla z\left(\bar{q}\right) = \nabla_q f(x(\bar{q});\bar{q}) - \sum_{m=1}^{M} \lambda_m \nabla_q g_m\left(x(\bar{q});\bar{q}\right)$$

For the UMP, this equation gives the interpretation for λ discussed above.