1 Extensive Games with Perfect Information

- Nash equilibrium is about static decision making.

- This lecture covers dynamic games with complete information, when players not only anticipate what the other player will do and what the other player will think, but also how he will respond to my moves.

- Most of the applications in microeconomics belong to the domain of dynamic games.

- As dynamic games, we understand games which extend over many periods, either finitely many or infinitely many. The new concept introduced here is the notion of *subgame perfect Nash equilibrium* (SPNE), which is closely related to the *principle of backward induction*.

- The key idea is *sequential rationality*: equilibrium strategies should specify optimal behavior from any point in the game onward. The notion of subgame perfect Nash equilibrium is a strengthening of the Nash equilibrium to rule out incredible strategies. We start with finite games, in particular games with a finite horizon.

2 Definition for extensive games

- An extensive form is an explicit description of the sequential structure of the decision problem encountered by the strategic players.

- The model allows us to study situations where each player defines his plan of action at any point of time, not just in the beginning of the game.

- That is, he needs not *commit* to a certain mode of behavior in the beginning.

- A game is with *perfect information*, if each player, when making his decision, is perfectly informed of all the events and that have previously occurred.

DEFINITION (extensive form): A list $\langle I, H, P, (u_i)\rangle$ is an *extensive game* of perfect information if

1. I is the set of players i, j.

2. A set of (finite or infinite) sequences H such that (i) $\emptyset \in H$, and (ii) if $(\emptyset, a_1, ..., a_k) \in H$ then $(\emptyset, a_1, ..., a_{k'}) \in H$ for $k' < k$.

3. If $h \in H$ and there is no a such that $(h, a) \in H$, then h is a terminal history. The set of terminal histories are denoted by Z. Each terminal history is associated to a consequence, i.e. each $h \in Z$ induces a payoff $u_i(h)$.

4. A player function P that associates to each nonterminal history h a player $P(h) \in I$.

- To interpret the game form, identify a function A that associates each history h a set $A(h)$ such that $(a, h) \in H$ iff $a \in A(h)$. Then set $A(h)$ is player $i(h)$'s choice set at h.

- After any nonterminal history h player i chooses from the set $A(h)$, and history $h \in H$ is terminal if $A(h) = \emptyset$.

- The empty history $\emptyset$ is the starting point of the game.

- The game form defines the order of moves: first $P(\emptyset)$ chooses some $a_0 \in A(\emptyset)$, player $P(\emptyset, a_0)$ chooses some $a_1 \in A(\emptyset, a_0)$, and so on until a terminal history is reached.

- An extensive game with perfect information is *a tree* where at each node some player decides to which branch the play continues.

EXAMPLE (ultimatum game): Two players share a pie of size 1. Player 1 suggests a division $d \in (0,1)$. Player 2 accepts or rejects. In the former case, 1 gets d and 2 gets $1-d$. In the latter case, both get 0.

Now $I = \{1,2\}$, $P(\emptyset) = 1$, $A(\emptyset) = (0,1)$, $P(\emptyset, a_1) = 2$, $A(\emptyset, a_1) = \{\text{accept, reject}\}$, for all $a_1 \in A(\emptyset)$, and

$u_1(a_1, a_2) = a_1$ and $u_2(a_1, a_2) = 1 - a_1$, if $a_2 =$ accept,

$u_1(a_1, a_2) = u_2(a_1, a_2) = 0$, if $a_2 =$ rejects.

- A tree does not need to be finite: A game could be defined for infinite streams of actions. For an example, let players 1 and 2 announce in alternating order natural numbers $1, ..., 9$ infinitely long. Stage k number is the k decimal of a real number. Each terminal history is infinite sequence of numbers $a_1, a_2, ...$. Payoffs could be assigned to terminal histories without complications, e.g. $u_1(a_1, a_2, ...) = 1$ if $(0, a_1 a_2 ...)$ is a rational number, 0 if $(0, a_1 a_2 ...)$ is an irrational number, and $u_2 = 1 - u_1$.

- Now we define player i's strategy. A strategy specifies what a player would do whenever it is his turn to move, *even for histories that would not be reached when following the strategy.*

DEFINITION (strategy): Player i's *strategy set* is $S_i = \times_{h \in H \setminus Z : P(h) = i} A(h)$. An element s_i of S_i is player i's *strategy.*

- That is, a strategy s_i of player i is a specification of an action in each set $A(h)$ for each nonterminal history $h \in H \setminus Z$ such that $P(h) = i$.

- Note that each element of $\times_{i\in I} S_i$ specifies an action in each node. Joining the consecutive actions forms a play path. A maximal play path is a terminal history in Z.

- Denote the dependency between strategies and terminal histories by function $g : \times_{i\in I} S_i \to Z$. Thus each element of $s \in \times_{i\in I} S_i$ induces a payoff $u_i(g(s))$, and each player i ranks the joint strategies in $\times_{i\in I} S_i$ in some order, specified by u_i.

- Note that $\langle I, (S_i), (g \circ u_i)\rangle$ meets the definition of a strategic game. It is called the *reduced strategic form* of $\langle I, H, P, (u_i)\rangle$.

- How do rational players play an extensive form game with perfect information?

DEFINITION (Nash equilibrium): Strategy $s \in \times_{i \in I} S$ forms a Nash equilibrium of the perfect information extensive game $\langle I, H, P, (u_i) \rangle$ if

$$u_i(g(s_i, s_{-i})) \geq u_i(g(s'_i, s_{-i})), \text{ for all } s'_i \in S_i \text{ and for all } i \in I.$$

- Nash equilibrium of an extensive form could be thought of being played via a mediator, to who plays the game according to strategies submitted by the players (cf. the Burning money -example).

- Nash equilibrium has the undesirable property that it requires *commitment ability* from the part of the players.

Example (Predation): There is an entrant firm 1 and an incumbent firm 2. If 1 enters, then sharing the market $(1, 1)$ is better than a price war $(0, 0)$

but less profitable than monopoly $(0,2)$, which follows if 1 does not enter. The extensive form $I = \{1,2\}$, $S_1 = A(\emptyset) = \{$enter, not enter$\}$, and $S_2 = A($enter$) = \{$cooperate if 1 enters, fight if 1 enters$\}$. Payoffs are

		2 Coop.	2 Fight
1	Enter	$1,1$	$0,0$
	Not enter	$0,2$	$0,2$

The game has two Nash equilibria (Enter, Coop) and (Not enter, Fight). But what would 2 do once it sees 1 entering?

2.1 Subgame perfect equilibrium

- We aim at constructing a criterion that rules out behavior that the players should not be able to commit.

DEFINITION (subgame): The *subgame* of the extensive form game $\langle I, H, P, (u_i) \rangle$ that follows the history $h \in H$ is the extensive form $\langle I, H|_h, P|_h, (u_i|_h) \rangle$ where $H|_h$ is the set of sequences h' such that $(h, h') \in H$ and $P|_h(h') = P(h, h')$ for all $h' \in H|_h$. Finally, $u_i|_h(h') = u_i(h, h')$ for all $h' \in H|_h$.

- The notion of equilibrium we now define requires that the actions prescribed by a strategy be optimal, given the other players' strategy, after *every* history.

DEFINITION (Subgame perfect Nash equilibrium): Strategy s forms a subgame perfect Nash equilibrium (SPNE) of the perfect information extensive game $\langle I, H, P, (u_i) \rangle$ if $s|_h$ forms a Nash equilibrium of the game $\langle I, H|_h, P|_h, (u_i|_h) \rangle$, for all $h \in H \setminus Z$.

- In particular, an SPNE forms a Nash equilibrium of the subgame $\left\langle I, H|_{\emptyset}, P|_{\emptyset}, (u_i|_{\emptyset})\right\rangle$, which is just the game $\langle I, H, P, (u_i)\rangle$. Hence the set of SPNE is a *subset* of Nash equilibria of the extensive game; it is a *refinement.*

- Equivalently, an SPNE strategy maximizes each player's payoff after each history, given the other players' SPNE strategies.

- An SPNE rules out Nash equilibria that rely on incredible strategies. For example, in the Predation-game, the only SPNE is (Enter, Coop.).

EXAMPLE (Stackelberg competition): Consider again the quantity setting duopoly á la Cournot, with aggregate demand

$$\beta(a_1, a_2) = 1 - a_1 - a_2$$

and the individual profit functions

$$\begin{aligned} u_i\left(a_i, a_j\right) &= \beta(a_1, a_2)a_i \\ &= (1 - a_1 - a_2)a_i, \text{ for all } i = 1, 2. \end{aligned}$$

Suppose now that player 1 moves first. After observing the quantity choice of player 1, player 2 chooses his quantity.

Given the observed action by 1, firm 2 chooses the output that maximizes its payoff. Recall that 2's best response function is

$$BR_2\left(a_1\right) = \frac{1 - a_1}{2}.$$

In SPNE, player 1's choice problem is of the form

$$\max_{a_1} u_1\left(a_1, BR_2\left(a_1\right)\right) = \left(1 - a_1 - \frac{1 - a_1}{2}\right) a_1 = \frac{(1 - a_1)a_1}{2}.$$

The associated first-order condition:

$$u_1'\left(a_1, B\left(a_1\right)\right) = \frac{1}{2} - a_1 = 0.$$

The Stackelberg equilibrium is then

$$a_1 = \frac{1}{2},\ a_2 = BR_2(a_1) = \frac{1}{4}.$$

The payoffs are

$$u_1\left(\frac{1}{2}, \frac{1}{4}\right) = \frac{1}{8} \quad u_2\left(\frac{1}{2}, \frac{1}{4}\right) = \frac{1}{16}.$$

This example illustrates the value of commitment, which is in this game simply achieved by moving earlier than the opponent.

- However, sometimes the prediction of SPNE may be "too" strong.

EXAMPLE (Ultimatum game, continued): In the Ultimatum game, the optimal strategy for 2 is to accept *any* offer $a_1 < 1$ and he is indifferent with accepting offer $a_1 = 1$. In the unique SPNE, 1 offers $a_1 = 1$ which is accepted by 2.

EXAMPLE (Centipede game): Two players take turns to choose C or S. Choosing C means continuing the game and S to stop it. The game can continue at most K steps. Formally the game consists of sequences of $C...CS$, where the number of successive C's is less than K, and one sequence $C...C$ with length K. Player 1 starts. If he chooses S, then he gets payoff 1 and 2 gets 0. If after k C's, it is player i's turn to choose, and he chooses S, then i gets payoff $k+1$ and $j \neq i$ gets payoff $k-1$. If i chooses C, then it becomes j's turn to choose. If after $K-1$ steps i chooses C, then the game ends with i's and j's payoffs k and k, respectively.

The Centipede game has a unique SPNE where both players *always* choose S, no matter how high payoffs are waiting in the end of the game. Is this plausible?

EXAMPLE (Chain store paradox) A chain store (firm CS) has branches in cities $1, ..., K$. In city k there is a competitor, firm k. In period k firm k either enters the market or not. If firm k enters, CS either fights or cooperated. Payoffs from the city k occurrences are

		CS	
		Coop.	Fight
k	Enter	$1, 1$	$0, 0$
	Not enter	$\varepsilon, 2$	$\varepsilon, 2$

CS's eventual payoff will be the sum of all K cities payoffs. Each entrant only cares about what happens in his city.

An entrant only enters if it knows the CS does not fight. Would it pay for CS to build a reputation of toughness?

The paradox is that in SPNE, the CS can never build a reputation. In the final stage, the unique continuation SPNE is (Enter, Coop) regardless of the history up to this stage (cf. the Predation-game). Since this is known at stage $K-1$, what happens at stage K will not have any consequence on choices at that stage. Continuing this way, CS cooperates in all stages and each k enters.

- To verify that s forms an SPNE, one needs to check that at every history and every subgame no player has a profitable deviation. The number strategies that, in principle, needs to be ruled out can be very large. The next result establishes a very useful property of any SPNE of perfect information games.

LEMMA (one deviation property): A strategy s is an SPNE of a *finitely long* perfect information game $\langle I, H, P, (u_i) \rangle$ if and only if, for every player $i \in I$ and every history $h \in H$ for which $P(h) = i$ it holds true that

$$u_i(g(s_i|_h, s_{-i}|_h)) \geq u_i(g(s_i', s_{-i}|_h)),$$

where s_i' differs from s_i in at most after one history.

- That is, it suffices to check the first action of each subgame.

- To see why, note that "only if" is implied by the definition of SPNE. The "if" part follows from the finiteness of $\langle I, H, P, (u_i) \rangle$. For if there is a profitable deviation s_i' from a putative SPNE s_i so that s fails to satisfy the criteria for SPNE. Then one can track down the final stage at which

s_i' differs from s_i. At this stage, the putative SPNE collapses also by the one time deviation principle.

- This is a very useful property since under it one only needs to check that there is no profitable deviation in a *single* stage of the game, rather that go through all the strategic contingencies.

- One deviation property is usefule for checking whether a particular strategy forms a SPNE. But how does one derive the SPNE?

- If a perfect information game $\langle I, H, P, (u_i)\rangle$ is finitely long, then there is a maximal length K for all non-terminal histories. A decision by $P(h)$ at history h of length K leads to a terminal history. Since $P(h)$ is rational, he choice is well defined. By the similar argument, a decision by $P(h')$ at history h' of length $K-1$ leads to a history of length K. Since the outcome following histories of length K are well defined, and $P(h')$ is rational, $P(h')$'s choice at h' is well defined and this choice specifies an outcome. Going backwards in this way, a decision of the player at history of length 0 can be determined. This decision specifies an SPNE outcome.

- The procedure used above is called *backwards induction.* If preferences over the terminal histories are *strict*, then backwards induction leads to a *unique* SPNE.

PROPOSITION (Zermelo): Every finite complete information game has an SPNE that can be derived via backwards induction

- Is chess a finite complete information game? Does it have a unique backwards induction solution?

Simultaneous moves

- The formal model of a game can be extended to the case of simultaneous moves by assuming that function P is multivalued and coincides with I after all histories. Since P is constant, we can simply prune it away from the formal definition of the game form.

- An extensive game with perfect information and simultaneous moves is a triple $\langle I, H, (u_i) \rangle$, where *every* nonterminal history h assigns a nonempty set of actions $A_i(h)$ to each player i. Player i's strategy set is now $\times_{h \in H \setminus Z} A_i(h)$.

- If $A_i(h)$ is singleton at h, then i is a dummy player, he does not have strategic power. Function p could be replaced without loss of generality in the definition of perfect information extensive game by a system of dummy players.

- A history of the game with simultaneous moves consists now of *vectors.*

- The definition of SPNE can be applied without complications to the simultaneous moves case: a SPNE constitutes a SPNE in *each* subgame.

- One deviation principle still holds (this is the case where it is most useful) but backwards induction can no longer be applied; the game cannot be solved recursively since a single simultaneous moves stage may not be recursively solvable.

EXAMPLE (Finitely repeated prisoner's dilemma) Suppose the prisoner'd dilemma game

		2	
		C	D
	C	3,3	0,4
1	D	4,0	1,1

is repeated for K times. In the unique SPNE, players choose (D, D) is all stages. To see this, suppose there is a set of histories H^d in which a deviation which some player does not choose D. Take a maximal history h in H^d, and let i choose C. Since afterwards both players choose (D, D), no matter what i chooses at h affects the continuation play. But then C is not i's optimal choice. By the one deviation property, the suggested alternative strategy does not form a SPNE.

Dominated strategies and backwards induction

- Suppose that a finite game $\langle I, H, P, (u_i)\rangle$ is of perfect information. This game can be solved via backwards induction.

- This is related to elimination of weakly dominated strategies in the strategic game representation of the game: Any strategy that is eliminated in the process of backwards induction of an extensive game with perfect information $\langle I, H, P, (u_i)\rangle$ is also eliminated in the process of iterative elimination of weakly dominated strategies in the reduced strategic form $\langle I, (S_i), (g \circ u_i)\rangle$ of $\langle I, H, P, (u_i)\rangle$ game.

- Let $S_i = \times_{h \in H \setminus Z : P(h) = i} A(h)$ be the strategy set of player i. Assume strict preferences over the outcomes. Find the maximal length K for all non-terminal histories. For any history h of length K, a decision by $P(h) = i$ leads to a terminal history. Identify the choice $a(h)$ in $A(h)$ of player

$P(h) = i$ at h that generates him the maximal payoff. This is the strategy that survivies backwards induction at history h. Now strategy $(s_i, a(h)) \in S_i$ is weakly dominates (s_i, a') for i, for any $s_i \in S_i \backslash A(h)$. For any history h' of length $K - 1$ we can similarly identify the undominated strategies, given the previosuly undominated strategies. Iterating the argument for $K - 1$, $K - 2, ..., 0$, one obtains that the outcome that survives iterative elimination of dominated actions is precisely the same that results from the process of backwards induction.

EXAMPLE (Centipede game, cont.): Let $K = 3$. Both players chose at most two times, and hence have strategy sets $\{SS, SC, CS, CC\}$. The reduced strategic form of the game is

1 \ 2	SS	SC	CS	CC
SS	1,0	1,0	1,0	1,0
SC	1,0	1,0	1,0	1,0
CS	0,2	0,2	3,1	3,1
CC	0,2	0,2	2,4	3,3

EXAMPLE (Battle of sexes, burning money) Consider the Battle of Sexes game. Allow 1 to burn one unit of payoff before entering the game.

Not burn	L	R
U	$3,1$	$0,0$
D	$0,0$	$1,3$

Burn	L	R
U	$2,1$	$-1,0$
D	$-1,0$	$0,3$

The reduced form of the extensive game is

		2			
		L, L	L, R	R, L	R, R
1	$Burn,\ U$	$2, 1$	$2, 1$	$-1, 0$	$-1, 0$
	$Burn,\ D$	$-1, 0$	$-1, 0$	$0, 3$	$0, 3$
	$Not\ burn,\ U$	$3, 1$	$0, 0$	$3, 1$	$0, 0$
	$Not\ burn,\ D$	$0, 0$	$1, 3$	$0, 0$	$1, 3$

First eliminate $(Burn, D)$, then eliminate (R, R) and (R, L), then eliminate $(Not\ burn, D)$, then eliminate (L, R), then eliminate $(Burn, U)$. Thus the unique outcome surviving the iterative elimination of weakly dominated strategies are $(Not\ burn,\ U)$, (L, L). Thus this *forward induction argument* implies that the possibility of burning money *communicates* player 1's intentions.