FDPE Microeconomic Theory I
Fall 2007

Problem Set 6

1. A real valued function $f : \mathbb{R}^L_+ \to \mathbb{R}$ is called superadditive if for all z^1, z^2,
$$f\left(z^1 + z^2\right) \geq f\left(z^1\right) + f\left(z^2\right).$$

   a) Show that every cost function is superadditive in input prices.

   b) Using this fact, show that the cost function is nondecreasing in input prices.

2. Consider a firm which has n independent divisions. Each division uses a common factor x and an individual factor z_i.The question deals with the issue of allocating the cost of the common factor to divisions. The production function of division i is given by
$$q_i = f_i\left(z_i, x\right).$$
Assume that this function can be inverted to yield the common factor requirement
$$x_i = g_i\left(q_i, z_i\right).$$
In words, division i requires x_i units of the common factor to convert z_i units of private factor into q_i units of its output. Let w be the vector of private input prices, v the price of the common input and p the vector of output prices. The cost function of the firm is obtained from
$$\begin{aligned} c\left(q, w, v\right) &= \min_{z,x} \; w \cdot z + vx \\ \text{s.t. } x_i &\geq g_i\left(q_i, z_i\right) \text{ for all } i. \end{aligned}$$
The proposal is to allocate the price of the common resource according to the Lagrange multipliers in the above minimization problem, i.e. each division is to buy the common resource at price λ_i.

   (a) Show that these prices cover the cost, i.e.
$$\Sigma_{i=1}^n \lambda_i = v.$$

   (b) Based on these prices, consider the individual cost minimization problems of the divisions and show that the individual cost functions sum to the common cost function.

(c) Suppose that

$$f_i(z_i, x) = \min\{z_i, x\} \text{ for all } i.$$

Calculate the total cost function and determine the cost allocation.

3. (Free Entry and Returns to Scale) The aggregate production set Y satisfies "free entry" if $Y + Y \subset Y$. A technology has constant returns to scale (CRS) if $y \in Y \Rightarrow \alpha y \in Y$ for $\alpha > 0$.

   (a) Give an example that shows that CRS does not necessarily imply free entry.

   (b) Show that if Y is convex, CRS implies free entry.

   (c) Give an example that satisfies free entry but does not satisfy CRS.

   (d) Show that if Y satisfies free entry and $0 \in Y_j$ for all j, then in a competitive equilibrium, profits for all firms are zero.

4. (Robinson Crusoe) Consider an economy with 3 goods (x, y, z), 1 firm and 1 consumer. (The consumer owns the firm.) The firm has the following technology: $(z, -x, -y) \in Y$ if there exists a $\lambda \in [0, 1]$ such that

$$z \leq \lambda \min\{x, 2y\} + (1 - \lambda) \min\{2x, y\} \text{ and } x, y, z \geq 0.$$

   The household's preferences are given by

$$u(x, y, z) = xyz.$$

   The consumer's initial endowment is $(1, 1, 0)$.

   (a) Draw the isoquants of the firm's technology.

   (b) Can you find a competitive equilibrium for this economy?

5. (Harder) In models of oligopolistic competition, it is typical that the profit of a firm lagging behind the leader in the industry in terms of the quality of its product has a profit function that is first convex and then concave in any improvements to its own quality. R&D investments within a firm result normally in random improvements in the quality A possible way of modeling the R&D activity is by considering the choice of various types of projects, i.e. various distributions over quality improvements. With this as a motivation, consider the following decision model. The decision maker has a Bernoulli utility function $u(x)$ defined for $x \geq 0$, and there is an x_0 such that $u''(x) \geq 0$ for all $x \leq x_0$, and $u''(x) \leq 0$ for all $x \geq x_0$, and furthermore $u'(x) > 0$ for all x and $\lim_{x\to\infty} u'(x) = 0$. The decision maker chooses amongst all possible random distributions on $\mathbb{R}_+$.

(a) For each possible expected value μ of the random variable $\widetilde{x}$, show that the expected utility maximizing distribution has at most two points in its support, and characterize the optimal distributions.

(b) Suppose that here is a cost of increasing μ. More specifically, let $c(\mu)$ denote the cost, and assume that $c, c', c'' \geq 0$. Find the optimal distribution .