Available online at www.sciencedirect.com
ScienceDirect
Journal of Economic Theory 146 (2011) 672–698
JOURNAL OF Economic Theory
www.elsevier.com/locate/jet

Dynamic coalitional equilibrium ☆

Hannu Vartiainen *

Department of Economics, Turku School of Economics, 20014 Turku, Finland

Received 23 August 2009; final version received 15 August 2010; accepted 20 August 2010

Available online 7 October 2010

Abstract

We study coalition formation processes of Konishi and Ray (2003) [27]. It is shown that an absorbing and deterministic process of coalition formation that also forms an equilibrium – satisfies a coalitional one-deviation property – does exist if one allows the process to be history dependent. All such dynamic equilibrium processes of coalition formation are characterized. Absorbing outcomes of dynamic equilibrium processes are also identified. It is shown that they always constitute a subset of the largest consistent set of Chwe (1994) [11]. A procedure that identifies a dynamic equilibrium process of coalition formation in finite time is constructed.

JEL classification: C71; C72; C78

Keywords: One-deviation principle; Coalition formation; History dependence

1. Introduction

As discussed by Ray [35], models of farsighted coalitional behavior suffer from the so-called *prediction problem*: the eventual profitability of a coalitional blocking cannot be predicted unless one knows the coalitional behavior that the blocking triggers. Since further blockings should be evaluated according to the same criterion as the original one, there may not be a concrete

☆ I am very grateful to an associate editor and three referees for comments and suggestions that improved the paper significantly. I also thank Jean-Jacques Herings, Klaus Kultti, Ariel Rubinstein, Hannu Salonen, Juuso Välimäki and, in particular, Debraj Ray, for useful advices and discussions.
* Fax: +358 2 481 4299.
E-mail address: hannu.vartiainen@tse.fi.

0022-0531/$ – see front matter
doi:10.1016/j.jet.2010.10.006

final step that defines the payoff, and from which to start the recursion.[1] This is why static solution concepts e.g. the core, are often in trouble.[2] They restrict coalitional behavior in a way that is against its fundamental dynamic nature. The aim of much of the modern literature of coalition formation, originated by Rosenthal [38] and further developed e.g. by Greenberg [19], Chwe [11], Xue [41], and Konishi and Ray [27], is to cope with the problem by associating dynamic elements into the solution.[3]

Konishi and Ray [27] (henceforth KR) is an important paper in the recent literature on coalition formation. They develop an extension of the one-shot coalition formation framework that solves the prediction problem in an elegant way. In KR, payoffs of the agents do not materialize only once but they accumulate over time depending on the evolution of the coalitional blockings. A key benefit of this approach is that it allows consistent modeling of farsighted coalitional behavior: current coalitional move is justified by the prediction of the coalitional behavior that the move induces. KR's coalitional solution – *equilibrium process of coalition formation* – requires that no coalition benefits from a single deviation in the process. They show that a (possibly) randomized or cycling equilibrium process of coalition formation always exists.

This paper shows that a version of a *deterministic* and *absorbing* equilibrium process of coalition formation exist if one drops the assumption that the processes are Markovian, *i.e.*, if one allows coalition formation processes to be *history dependent*.[4] Such process of coalition formation specifies which – if any – coalition blocks the current outcome on the table after the history of blockings, and which outcome does the blocking coalition take on the table for the next round. We characterize all history dependent, absorbing and deterministic equilibrium processes of coalition formation, and compare them to the other solutions in the literature.

The key benefit of the assumption that coalition formation processes are deterministic and absorbing is that such processes are easy to interpret in the classic one-shot framework. If blocking behavior is interpreted as coalitional negotiation prior to an agreement, it is not clear which outcome the coalitions could "agree upon" if there is no outcome that remains unblocked. This is the case when the process is cycling. When the process is absorbing, however, there will be a finite sequence of transitory blockings after which the outcome on the table is no longer blocked. Such outcome can be interpreted to become implemented in the one-shot sense.

Formally, we apply a version of the solution concept of KR to the classic one-shot framework in which coalitions negotiate by blocking the outcome on the table. When the outcome on the table is not blocked, it becomes implemented. We look at deterministic and *terminating* processes of coalition formation that implement an outcome in finite time after any history blockings. In the context of KR, they can be interpreted as deterministic and absorbing processes of coalition formation. To highlight that a coalition formation process may now depend on the history, and

[1] Shenoy [39] is one of the first contributions on blocking cycles. Harsanyi [23] and Aumann and Myerson [1] make a clear case for the importance of farsightedness in coalitional analysis.

[2] Barberà and Gerber [5] is an impossibility result concerning static concepts.

[3] Other recent models of farsighted coalition formation include Ambrus [2], Herings et al. [22], Ray and Vohra [36], Mariotti [29], Barberà and Gerber [4], Bloch and Gomez [8], Diamantoudi and Xue [15], and Conley and Konishi [12]. Relatedly, Chatterjee et al. [10], Bloch [7], Ray and Vohra [37], and Gomes and Jehiel [20] model coalition formation noncooperatively via a protocol.

[4] That history dependency in increases the number of noncooperative equilibria is well known (see e.g. Chatterjee et al. [10]). To the best of our knowledge, however, Hyndman and Ray [24] and Mariotti [29] are the only papers that analyze history dependency in the coalitional set up. KR discusses on history dependent coalition formation processes but do not analyze them in detail.

since there are some technical differences to the solution of KR (see below), we term our solution as the *dynamic* equilibrium process of coalition formation.

We characterize dynamic equilibrium processes of coalition formation by identifying a collection of finitely long play paths – finite, since our focus is on terminating processes – that describe how the play proceeds after different histories. Such collection of paths, called a *consistent path structure*, has the following properties: the final outcome of a path is weakly preferred to any outcome along the path for the coalition that is active at that outcome, and a deviation by the active coalition from a path leads to another one in the collection of paths whose final outcome is not preferred by the deviating coalition relative to that of the original path. Our characterization result says that any dynamic equilibrium process of coalition formation can be associated to a consistent path structure, and vice versa. Moreover, we demonstrate that it is without loss of generality to focus on path structures containing only *finitely* many paths.

A consistent path structure and, hence, a dynamic equilibrium process of coalition formation is shown to exists. We identify the largest consistent path structure (in the sense of set inclusion). The set of final elements of the paths in this structure contains all the outcomes that can be implemented via any dynamic process of coalition formation. We show that this set is a subset of the largest consistent set of Chwe [11].

The fundamental difference between our solution and the consistent set of Chwe [11] is that where the latter focuses on the stability of an outcome, we test the stability of the *path* that leads to the implemented outcome. Since, as a special case, we also test the stability of the path at its final step, it turns out that our solution also implies the stability criterion of Chwe [11]. This explains why our solution can be thought as a refinement of the largest consistent set.

Our characterization allows us to establish results concerning the computability of our solution. This is an important issue in our context since the basis of our solution – the set of all finite play paths – is typically infinite. We construct an explicit procedure that identifies the outcomes that are implementable via any dynamic equilibrium process of coalition formation. This procedure terminates in finite (bounded) time. Moreover, we show that there is essentially no loss of generality in assuming that the processes of coalition formation are finitistic in the sense that they have only finite memory (they are finite state Markov chains), and that they implement an outcome in bounded time. Thus starting from any history, there is an upper bound on the number of rounds in which the outcome becomes implemented.

The Coase theorem famously asserts that, in the absence of deadlines and other barriers on negotiation, interaction between agents always leads to efficiency. However, we demonstrate via an example that dynamic equilibrium processes of coalition formation are not always efficient (in our example every dynamic equilibrium process of coalition formation induces only inefficient outcomes).

To see how history dependency works, let us consider the "roommate problem" in KR (Example 10). There are three players $\{1, 2, 3\}$, three choices $\{x, y, z\}$, and payoffs are

	x	y	z
Player 1	1	0	a
Player 2	a	1	0
Player 3	0	a	1

where $a \in (0, 1)$. The game is depicted in the commuting diagram in Fig. 1(a), where $x \to_{\{2,3\}} y$ means that coalition $\{2, 3\}$ may replace x on the table by taking y on the table, etc. An outcome on the table is implemented if it is not blocked by the eligible coalition.

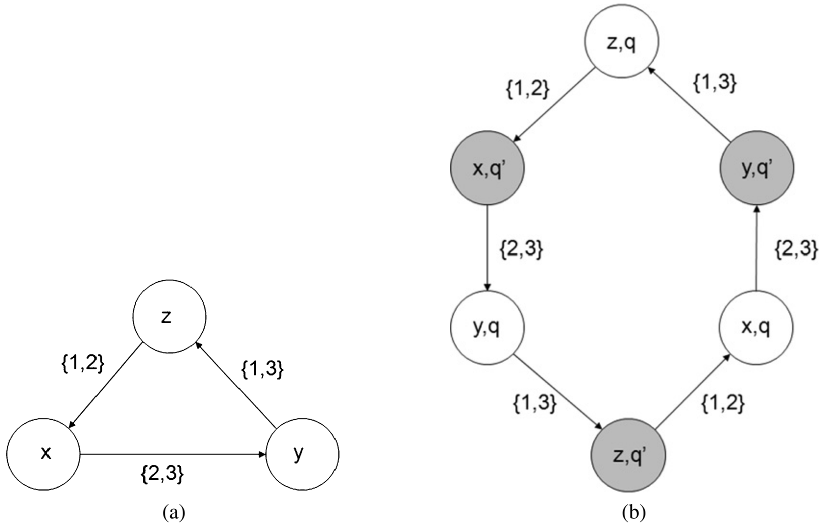

Fig. 1.

KR demonstrate that this game lacks an absorbing and deterministic equilibrium process of coalition formation, *i.e.*, a plan that specifies optimally whether an eligible coalition should block the current outcome on the table or not, given how the process behaves in the future (a randomized or, depending the value of parameter a, a cyclic equilibrium process of coalition formation does exist). We now argue that the game has an absorbing deterministic equilibrium process of coalition formation if one lets the process have memory.

Let histories affect the coalitional process via two *phases* q and q', as follows. Divide the set of outcome-phase configurations $\{x, y, z\} \times \{q, q'\}$ into two groups, *terminal* configurations $\{(x, q), (y, q), (z, q)\}$ and *transitory* ones $\{(x, q'), (y, q'), (z, q')\}$. Let the phase transitions from one configuration to another be determined as depicted in Fig. 1(b): if the process moves away from a terminal configuration, then the phase changes from q to q', and if the process moves away from a transitory configuration, then the state changes from q' to q. Finally, let the phase dependent coalitional process implement the status quo outcome if the current configuration is terminal, and let the process move to the next configuration by changing the outcome on the table if the current configuration is transitory.

The constructed history dependent process of coalition formation reflects optimality in the sense that no coalition benefits from a one-time deviation in it. In a terminal configuration, say (x, q), a deviation leads to implementation of z via a transitory configuration (y, q'). This change is not profitable from the viewpoint of the deviating coalition $\{2, 3\}$. In a transitory configuration, say (y, q'), a deviation would lead to the implementation of y (as this is the only alternative to the continuation to (z, q)), instead of z. Again, this change is not profitable from the viewpoint of the deviating coalition $\{1, 2\}$.

What is crucial in this construction is that there are different continuations from a given status quo outcome depending on the way the status quo is reached. Hence the process must have memory.

To see how the solution restricts outcomes that can be implemented, think of the case of Prisoners Dilemma. There are four outcomes CC, CD, DC, DD, and payoffs

		2: C	2: D
1	C	3, 3	0, 4
	D	4, 0	1, 1

The play can be moved from any outcome to another by the grand coalition, between outcomes on a row by the row player 1, and between outcomes on a column by the column player 2. As discussed by Chwe [11], the largest consistent set contains only the cooperative outcome CC. Hence also any dynamic equilibrium process of coalition formation implements CC. To see why, note that any dynamic equilibrium process of coalition formation can only implement CC or DD. Otherwise, under CD (DC), the row (column) player would benefit from a one-time deviation. This is since the row player must be active if the equilibrium play moves away from DD, and hence the deviation cannot make him worse off relative to DD. Of the two potentially implementable outcomes CC and DD, the only feasible one is CC since otherwise the grand coalition would benefit from a one-time deviation to the process, given that the only possible continuation path from CC can only lead to the implementation of CC – the other alternative DD makes both the players worse off. Hence the one-time deviation principle implies that a terminating process of coalition formation implements CC even if the process may be history dependent.

There are two technical differences to the way the solution is formulated in KR (apart from history dependency). First, our focus is on the properties of the solution that reflect coalitional optimality. These considerations are captured by a coalitional version of the one-deviation principle. As a consequence, we only require a weak coalitional preference for transition from the current status quo to the next one (as opposed to KR who require *strict* preference when the status quo outcome is not an efficient move for all coalitions). Second, when interpreted the solution in the real-time framework of KR, we evaluate the intertemporal payoffs of the players by the limit-of-the-means criterion rather than discounting. In the modeling of KR, this case can be seen as the full farsightedness-benchmark. These differences notwithstanding, the key aspects of the solutions (*i.e.*, the coalitional optimality and farsightedness) are the same.

It is important to recognize that our focus on deterministic and absorbing coalition formation processes is not a restriction on the players' rationality. Rather, it is an additional condition on top of the other conditions that reflect coalitional rationality (due to KR). The purpose of this extra condition is to make the solution applicable in one-shot coalitional games. Note that our solution concept requires coalitional rationality also at off-equilibrium histories.

There are interesting connections to other coalitional solutions concepts, in particular to the largest consistent set of Chwe [11], the conservative stable standard of behavior of Greenberg [19] and Xue [41], and the full coalitional equilibrium of Mariotti [29]. Formal discussion of the connections to this literature is found in Section 6.

Independently of this study, Flesch et al. [18] prove that a subgame perfect equilibria always exists in a class of perfect information recursive games. While their existence result does not imply (nor is it implied by) our existence result, the core arguments are similar. An additional contribution of our paper is, however, the complete characterization of equilibria in terms of play paths and, in particular, the maximally large equilibria. We also give the equilibrium a "solution" interpretation by characterizing the outcomes that are implementable within it, and compare it to the other solutions in the literature.

Section 2 defines the model and the solution concept. Section 3, which constitutes the main part of the paper, characterizes dynamic equilibrium processes of coalition formation and demonstrates its existence. Section 4 establishes a finite procedure to compute a dynamic equilibrium process of coalition formation and Section 5 discusses the finitistic aspects of the solution. Section 6 compares our solution to the aforementioned solutions in the literature. Section 7 comments briefly the efficiency of the solution, and Section 8 concludes with some discussion.

2. Coalitional game

A *coalitional game* (see Rosenthal [38], Greenberg [19], Chwe [11]) is defined by a list

$$\Gamma = \langle N, X, (F_S)_{S\subseteq N}, (\succsim_i)_{i\in N}\rangle,$$

where N is a finite set of players, X is a nonempty finite set of states or nodes or outcomes, a choice set $F_S(x) \subseteq X \cup \{\emptyset\}$ such that $\emptyset \in F_S(x)$ specifies the set of actions or effectiveness relations $F_S(x)$ available to a coalition $S \subseteq N$ at node $x \in X$. Each player $i \in N$ has a preference relation $\succsim_i$ over the set of outcomes X.

The game is played in the following manner: There is an initial status quo x^0. At period $t = 0, 1, \dots,$ coalition S can challenge the current status quo outcome x^t by demanding an outcome $y \in F_S(x^t)$. In such case, y becomes the new status quo at period $t + 1$. If no coalition challenges x^t, then x becomes implemented. Only *one* coalition may be active at a time.

This model embodies several classic models, such as characteristic function games, games in strategic form or majority voting games. However, it can also encompass games in partition function form (e.g. Banerjee et al. [3], Bogomolnaia and Jackson [9], Papai [34], Diamantoudi and Xue [14] and [15], Ray and Vohra [36], Barberà and Gerber [4]), or networks (e.g. Jackson and Wolinsky [26], Dutta and Mutuswami [13], Jackson and van den Nouweland [25], Page et al. [33]). For concrete examples of games falling into these categories, see Chwe [11] or Ray [35]. In its general abstract form, the game has been analyzed by Chwe [11], Xue [40], and Konishi and Ray [27]. The origins of this modeling tradition were laid down by Rosenthal [38] and Greenberg [19].

In the remainder of this paper, we make the following assumption which simplifies the exposition: For all $x, y \in X$, there is at most one coalition S such that $y \in F_S(x)$. This coalition is denoted by $S(x, y)$.

The only role of this assumption is to reduce the notational burden, so that when moving from x to y we do not need to specify which coalition induces the move.[5] To see why the assumption is without loss of generality note that it actually only requires that each outcome be indexed by the coalition that brought the outcome as the status quo (and the initial outcome is indexed by the empty set).[6] Hence the assumption imposes no restrictions on the underlying physical structure. In particular, it does *not* affect any of the results.[7]

Paths. A *path* is a *finite* sequence $(x_0, \dots, x_K)$ of outcomes such that $x_{k+1} \in \bigcup_S F_S(x_k)$, for all $k = 0, \dots, K$. The *length* of the path $(x_0, \dots, x_K)$ is K. Denote the set of all paths by

[5] The assumption allows presenting the histories of play in terms of the nodes alone. Otherwise, a history should also specify which coalitions have been active along the play path.

[6] If Y is the underlying physical outcome space, then we may let $X = Y \times 2^N$ such that $(y, S) \in X$ if and only if y is the physical outcome on the table and S is the most recent active coalition, and $S = \emptyset$ if $y = x^*$.

[7] The only consequence of the assumption is that some of the outcomes may have multiple representations in the set of outcomes. Since no restrictions are imposed on preferences over outcomes, this does not affect the characterization nor the existence results.

$$\mathcal{X} = \left\{ (x_0, \dots, x_K)\colon\ x_{k+1} \in \bigcup_S F_S(x_k),\ \text{for all } k = 0, \dots, K,\ \text{for all } K = 0, 1, \dots \right\}.$$

Further, denote the set of paths that start from node y by

$$\mathcal{X}_y = \{ (x_0, \dots, x_K) \in \mathcal{X}\colon\ x_0 = y \}.$$

For any collection $\mathcal{B} \subseteq \mathcal{X}$ of paths, denote the subcollection of paths that start from node y by

$$\mathcal{B}_y = \mathcal{B} \cap \mathcal{X}_y.$$

A path is abbreviated by $\bar{x} = (x_0, \dots, x_K)$. By our expositional assumption, a path $\bar{x}$ also implicitly defines the coalitions that are active along the play. Denote the final element x_K of the path $(x_0, \dots, x_K)$ by

$$\mu\big[(x_0, \dots, x_K)\big] = x_K.$$

Process of coalition formation. Denote the set of finite *histories* by $H := \mathcal{X}_{x^0}$. A deterministic *process of coalition formation* PCF σ is a function $\sigma : H \to X \cup \{\emptyset\}$. The interpretation of a PCF is that if $\sigma(h, x) = y \in X$, then the coalition $S(x, y)$ changes the status quo from x to y, and if $\sigma(h, x) = \emptyset$, then no coalition is active and x becomes implemented. Thus, a PCF specifies which – if any – coalition is active at a given history, and which outcome is the new status quo.

Denote, in the usual way, by $\sigma^t(h)$ the tth iteration of σ starting from h, *i.e.*, $\sigma^0(h) = \sigma(h)$ and $\sigma^t(h) = \sigma(h, \sigma^0(h), \dots, \sigma^{t-1}(h))$, for all $t = 1, \dots$. A PCF σ is *terminating* if, for any $h \in H$ there is $T_h < \infty$ such that $\sigma^{T_h+1}(h) = \emptyset$. That is, after any history h, the process will implement the outcome $\sigma^{T_h}(h)$.

Let $\bar{\sigma}(h)$ denote the sequence of status quos in X that is induced by the strategy σ from the history h onwards

$$\bar{\sigma}(h) = \big(\sigma^0(h), \sigma^1(h), \dots\big).$$

If σ is terminating, then $\bar{\sigma}(h)$ is finite and $\mu[\bar{\sigma}(h)]$ is well defined, for all h. Specifically, for a terminating PCF σ, if coalitional action $a \in X \cup \{\emptyset\}$ is chosen at history $(h, x) \in H$, then

$$\mu[\bar{\sigma}(h, x, a)] = \begin{cases} \mu[\bar{\sigma}(h, x, y)], & \text{if } a = y \in X, \\ x, & \text{if } a = \emptyset. \end{cases} \tag{1}$$

In particular, $\mu[\bar{\sigma}(h, \sigma(h))] = \mu[\bar{\sigma}(h)]$.

The solution. Our primary question is whether equilibrium reasoning is compatible with the idea that a deterministic PCF is terminating. Our equilibrium condition, which is an amended version of the solution in KR, is defined next.

Use the following notation for group preferences. For any $S \subseteq N$, and $x, y \in X$,

$$\begin{aligned} & y \succ_S x \quad \text{if } y \succ_i x,\ \text{for all } i \in S, \\ & y \succsim_S x \quad \text{if } y \succsim_i x,\ \text{for all } i \in S. \end{aligned}$$

Fix a PCF σ and a history (h, x). We say that coalition S has a *weakly preferred* move $a \in F_S(x)$ from x if $\mu[\bar{\sigma}(h, x, a)] \succsim_S x$. Further, a move $a \in F_S(x)$ is *efficient* for coalition S if there is no $b \in F_S(x)$ such that $\mu[\bar{\sigma}(h, x, b)] \succ_S \mu[\bar{\sigma}(h, x, a)]$.

Now we specify our equilibrium condition. To avoid repetition, we use the abbreviation of the term in the main body of the paper.

Definition 1 *(DEPCF).* A deterministic terminating process of coalition formation σ is a *dynamic equilibrium process of coalition formation* (DEPCF) if, for all $(h, x) \in H$,

1. $\sigma(h, x) \in X$ implies that $\sigma(h, x)$ is an efficient and weakly preferred move from x for coalition $S(x, \sigma(h, x))$.
2. $\sigma(h, x) = \emptyset$ implies that $\emptyset$ is an efficient move from x, for any coalition S.

Thus a coalition is allowed to change the current status quo only if all its members agree on the move to the new status quo and cannot find any strictly better alternative, given that the PCF is followed ever after. Moreover, if there is a strictly profitable move for some coalition, then the status quo must change.

Our solution can be interpreted as a coalitional version of the one-deviation principle: after any history there is no profitable one-shot deviation for any coalition. Note that this also implies that there cannot be a *finitely* long deviation that is profitable for all the coalitions along the deviation sequence.

To understand the novel aspects of the solution concept, it may be useful to divide possible solutions into two cases. In case 1, the coalitional action is only dependent to the current status quo outcome. Then, starting from any given status quo outcome, the implemented outcome is uniquely achieved in finite sequence of coalitional moves. This case corresponds the equilibrium notion of KR. However, in case 2, our equilibrium process may depend on the past history of play. That is, the continuation play path and the eventually implemented outcome does no longer depend on the status quo alone. However, since the history is always uniquely defined, the process does determine uniquely and in finite steps (since the process is terminating) the outcome that is to be implemented. This also applies to the initial history. The interpretation of the process is now analogous to that of a strategy in infinitely long noncooperative games.

Note also that the equilibrium condition of KR is slightly more stringent in that it requires that if there is a strictly profitable move for some coalition, then the equilibrium move must also be *strictly* preferred for some (not necessarily the same) coalition.[8] Our condition only requires the move to be weakly preferred. Closer discussion of the relation to KR and other solutions is found in Section 4.

Inducible paths and implementable outcomes. In the remainder of this study, we characterize DEPCFs and verify their existence. Since the play path as well as the eventually implemented outcome may be sensitive to the choice of the initial status quo, which is usually somewhat arbitrary, it is natural to focus on *paths that are inducible* in DEPCF independently of how their first element has been reached. Our main interest is in outcomes that are *implementable* in DEPCF after some initial history.

More formally, paths that are induced in a PCF σ after some history of play are

$$\bar{\sigma}(H) = \{\bar{x} \in \mathcal{X}\colon\ \bar{\sigma}(h) = \bar{x} \text{ and } h \in H\}.$$

Denoting the set of final outcomes of a collection paths $\mathcal{B}$ by $\mu[\mathcal{B}] = \{\mu[\bar{x}]\colon\ \bar{x} \in \mathcal{B}\}$, the set of outcomes that are implementable in a terminating PCF σ is written as

$$\mu[\bar{\sigma}(H)] = \{x \in X\colon\ \mu[\bar{x}] = x \text{ and } \bar{x} \in \bar{\sigma}(H)\}.$$

---

[8] That is, $\mu[\bar{\sigma}(h, x)] \succ_S x$ for some S.

Any outcome in $\mu[\bar{\sigma}(H)] \subseteq X$, and nothing outside, can be implemented in a PCF σ by changing the initial history.

3. Characterization

A path $(x_0, \dots, x_K) \in \mathcal{X}$ is said to be *feasible* if

$$x_K \succsim_{S(x_{k+1}, x_k)} x_k, , \quad \text{for all } k = 0, \dots, K-1. \tag{2}$$

That is, a path is feasible if following the path is not worse for any member of the active coalitions than stopping the game would be, provided that the final outcome of the path will be reached. Denote the set of feasible paths by $\mathcal{F} \subseteq \mathcal{X}$. Property 1 of Definition 1 means that any path that is played in DEPCF is necessarily feasible. It is therefore convenient to work directly in terms of feasible paths.

Now we characterize DEPCFs in terms of the primitive data alone, *i.e.*, in terms of feasible paths. To this end, we define a dominance relation over the paths. Recall that $\mathcal{F}_y$ is the set of feasible paths that originate from node y.

Definition 2 *(Path Dominance).* A path $\bar{y} \in \mathcal{F}$ *dominates* a path $\bar{x} \in \mathcal{X}$ at the kth step, denoted by $\bar{y} \rhd_k \bar{x}$, if $y \in F_S(x_k)$, $\bar{y} \in \mathcal{F}_y$, and $\mu[\bar{y}] \succ_S \mu[\bar{x}]$, for $S = S(x_k, x_{k+1})$.

That is, if outcome x_k is reached along the path $\bar{x}$, then there is an active coalition at x_k that benefits from moving the play to path $\bar{y}$ under the hypothesis that the final element of $\bar{y}$ is implemented and reached rather than that of $\bar{x}$.

Definition 3 *(Consistent Path Structure).* A collection of feasible paths $\mathcal{C} \subseteq \mathcal{F}$ is a *consistent path structure* (CPS) if,

(i) $\mathcal{C}_x$ is nonempty, for all $x \in X$,
(ii) for any $\bar{x} \in \mathcal{C}$, there are *no* k and y such that $\bar{y} \rhd_k \bar{x}$ for all $\bar{y} \in \mathcal{C}_y$.

A consistent path structure can be given a "blocking" interpretation as follows. (i) For any initial status quo node there is a feasible path in the consistent path structure that guides how the play evolves. (ii) If an active coalition along the play path deviates from the path to a new node, then there is a path *in* the consistent path structure that starts from the new status quo and ends in an outcome that does *not* improve the payoffs of the members of the deviating coalition relative to what would be obtained if the original path had been followed.

We now claim that a consistent path structure features stability in the sense required by DEPCF. The core of the argument is that any path of a consistent path structure is sustainable *if* any deviant coalition can be punished. However, according to Definition 3(ii), punishment *is* feasible if it is true that any path of a consistent path structure can be played. Hence an element of a consistent path structure is robust against one-shot (finitely many) deviations in a consistent way.

Intuitively, a consistent path structure does not aim to rule out outcomes only with confidence, as does the largest consistent set of Chwe [11]. Rather, the idea is to identify precisely the paths that are playable in equilibrium. The challenge is to answer what will happen after a (counterfactual) deviation since the knowledge of this makes the deviation unprofitable for the deviating coalition. Since this test should be passed by all paths, including the one that is followed after the (counterfactual) deviation and so on, there are no paths that are "unreasonable".

Now we verify the above argument formally, *i.e.*, that for any consistent path structure there exists a DEPCF. Let $\mathcal{C}$ be a consistent path structure. Identify a function ξ on $\mathcal{C} \times \mathbb{N} \times X$ such that $\xi(\bar{x}, k, y) \in \mathcal{C}_y$ and $\xi(\bar{x}, k, y) \not\triangleright_k \bar{x}$, for all $(\bar{x}, k, y)$ such that $y \in F_{S(x_k, x_{k+1})}(x_k) \setminus \{x_k\}$. Since $\mathcal{C}$ satisfies Definition 3, such a function does exist.

We now construct a deterministic and terminating PCF $\sigma^* : H \to X$ that is based on the function ξ. It is convenient to describe σ^* as a deterministic Markov chain $(\sigma^* : Q, g, \bar{x}^0)$, where Q is a set of *states* on which the process σ^* operates, g is a *transition function* from $Q \times X$ to Q, and $\bar{x}^0 \in \mathcal{C}_{x^0}$ is an *initial path* (which exists by Definition 3(i)). In Section 5 we shall argue that it is without loss of generality to assume that the set of states Q is *finite*.

Let the set of states Q consist of pairs of paths and integers as follows:

$$Q = \{(\bar{x}, k)\colon\ \bar{x} = (x_0, \ldots, x_K) \in \mathcal{C} \text{ and } 0 \leqslant k \leqslant K\}. \tag{3}$$

Start with the path $\bar{x}^0$. Let the transition function g satisfy, for any $\bar{x} = (x_0, \ldots, x_K) \in \mathcal{C}$, for any $k = 0, \ldots, K - 1$, and for any $y \in X$,

$$g((\bar{x}, k), y) = \begin{cases} (\bar{x}, k + 1), & \text{if } y = x_{k+1}, \\ (\xi(x, k, y), 0), & \text{if } y \neq x_{k+1}. \end{cases} \tag{4}$$

Proceeding recursively from x^0, the set of histories H is partitioned by the set Q.

Let the PCF σ^* be conditional on the current state $(\bar{x}, k) \in Q$, where $\bar{x} = (x_0, \ldots, x_K)$, such that:

$$\sigma^*(\bar{x}, k) = \begin{cases} x_{k+1}, & \text{if } k < K, \\ \emptyset, & \text{if } k = K. \end{cases} \tag{5}$$

That is, the PCF continues along the path $\bar{x} = (x_0, \ldots, x_K)$ and implements x_K when the end of the path is reached.

Lemma 1. *σ^* is a dynamic equilibrium process of coalition formation.*

Proof. Termination: Take $(q, y) \in Q \times X$ and let $g(q, y) = (\bar{x}, k) \in Q$, where $\bar{x} = (x_0, \ldots, x_K)$. Then, applying (4) and (5) recursively, σ^* implements an outcome in at most $K - k$ steps. Since (q, y) was an arbitrary element of $Q \times X$, σ^* is terminating.

To obtain equilibrium condition (1), take any path $\bar{x} = (x_0, \ldots, x_K)$ and state $(\bar{x}, k) \in Q$. Suppose that $\sigma^*(\bar{x}, k)$ is not an efficient move from x_k for coalition $S = S(x_{k+1}, x_k)$. Then there is $y \in F_S(x_k) \setminus \{x_{k+1}\}$ that induces a path $\xi(\bar{x}, k, y)$ such that $\mu[\xi(\bar{x}, k, y)] \succ_S \mu[\bar{x}]$. But by the definition of dominance, $\xi(\bar{x}, k, y) \triangleright_k \bar{x}$ which contradicts part (ii) of a definition of CPS $\mathcal{C}$. Moreover, by construction, $\bar{x} \in \mathcal{F}$, which implies that $\mu[\bar{x}] \succsim_S x_k$.

Equilibrium condition (2) is now obtained by replacing $k < K$ with $k = K$, coalition $S(x_{k+1}, x_k)$ with any coalition S, and $y \in F_S(x_k) \setminus \{x_{k+1}\}$ with $y \in F_S(x_K)$, and following the first sentence of the proof of condition (1). □

To fully characterize equilibrium strategies, consistent path structures need to be completed in the following sense: A consistent path structure $\mathcal{C}$ is *complete* if $(x_0, \ldots, x_K) \in \mathcal{C}$ implies $(x_k, \ldots, x_K) \in \mathcal{C}$, for all $k = 0, \ldots, K$. That is, following a path in $\mathcal{C}$ is consistent with staying on a path in $\mathcal{C}$. Note that the completion is a purely expositional operation; existence of a complete $\mathcal{C}$ or its uniqueness is never an issue once $\mathcal{C}$ is specified.

Given a complete consistent path structure $\mathcal{C}$ and strategy σ^* defined on it, let $\bar{\sigma}^*(q)$ denote the path followed once $q \in Q$ has materialized. By construction,

$$\bar{\sigma}^*(\bar{x}, k) = (x_k, \dots, x_K), \quad \text{for all } (\bar{x}, k) \in Q.$$

In particular, by the construction of σ^*,

$$\bar{\sigma}^*(\bar{x}, 0) \in \mathcal{C}.$$

Thus, by the definition of completeness of $\mathcal{C}$,

$$\bar{\sigma}^*(\bar{x}, k) \in \mathcal{C}, \quad \text{for all } (\bar{x}, k) \in Q.$$

Thus $\bar{\sigma}^*(Q) \subseteq \mathcal{C}$. Moreover, since $\bar{\sigma}^*(\bar{x}, 0) = \bar{x}$, for all $\bar{x} \in \mathcal{C}$, it follows that $\mathcal{C} \subseteq \bar{\sigma}^*(Q)$. We therefore have

$$\bar{\sigma}^*(Q) = \mathcal{C}. \tag{6}$$

That is, for any consistent path structure $\mathcal{C}$, we can find a DEPCF that induces the corresponding complete $\mathcal{C}$.

Now we prove the converse of Lemma 1, which is that a consistent path structure characterizes behavior in any DEPCF, *i.e.*, that any collection of equilibrium paths is equivalent to a consistent path structure.

Lemma 2. *Let σ be a dynamic equilibrium process of coalition formation. Then there is a complete consistent path structure $\mathcal{C} \subseteq \mathcal{F}$ such that $\bar{\sigma}(H) = \mathcal{C}$.*

Proof. Take any $(h, x) \in H$. Then a finite path $\bar{\sigma}(h, x)$ exists since σ is terminating. We check both the defining conditions of CPS.

(i) By construction, $\bar{\sigma}(h, x) \in \mathcal{X}_x$. Since σ is a DEPCF, $\bar{\sigma}(h, x) \in \mathcal{F}$. Thus $\bar{\sigma}(h, x) \in \mathcal{F}_x$.

(ii) Let $\bar{\sigma}(h, x_0) = (x_0, \dots, x_K) = \bar{x}$. First, take any $k < K$ and any $y \in F_S(x_k) \setminus \{x_{k+1}\}$, for $S = S(x_k, x_{k+1})$. Since x_{k+1} is efficient for S, it must be the case that $\mu[\bar{y}] \not\succ_S \mu[\bar{x}]$, where $\bar{\sigma}(h, x_0, \dots, x_k, y) = \bar{y}$. That is, for any choice of k and y there is $\bar{y} \in \bar{\sigma}(H) \cap \mathcal{F}_y$ such that $\bar{y} \not\triangleright_k \bar{x}$.

Second, take $k = K$ and any $y \in F_S(x_K)$, for $S = S(x_K, y)$. Since $\emptyset$ is efficient for S, it must be the case that $\mu[\bar{y}] \not\succ_S \mu[\bar{x}] = x_K$, where $\bar{\sigma}(h, x_0, \dots, x_k, y) = \bar{y}$. That is, for any choice of y there is $\bar{y} \in \bar{\sigma}(H) \cap \mathcal{F}_y$ such that $\bar{y} \not\triangleright_K \bar{x}$.

Finally, we argue that the CPS is also complete. Let $(x_0, \dots, x_K) = \bar{\sigma}(h)$. It suffices to show that $(x_1, \dots, x_K) = \bar{\sigma}(h, x_0)$. But this follows from the recursive structure of $\bar{\sigma}(h, x_0) = (\sigma(h), x_1, \dots, x_K)$. □

One the one hand, by Lemma 1 and condition (6), any consistent path structure can be supported by a DEPCF. On the other hand, by Lemma 2, a DEPCF induces behavior consistent with a consistent path structure. We compound these observations in the following characterization.

Theorem 1. *A deterministic and terminating process of coalition formation σ is a dynamic equilibrium process of coalition formation if and only if there is a complete consistent path structure $\mathcal{C}$ such that $\mathcal{C} = \bar{\sigma}(H)$.*

Theorem 1 permits us also to characterize the outcomes that are implementable within a DEPCF. The set of outcomes that are implementable within a consistent path structure $\mathcal{C}$ coincide with the set of outcomes that are inducible via a DEPCF and, conversely, the set of outcomes that are implementable within an equilibrium coincide with the set of outcomes that are inducible within $\mathcal{C}$. Summing up these results provides a useful corollary.

Corollary 1. *There is a dynamic equilibrium process of coalition formation* σ *such that* $\mu[\bar{\sigma}(H)] = B$ *if and only if there is a consistent path structure* $\mathcal{C}$ *such that* $\mu[\mathcal{C}] = B$.

This result does not, however, say anything about the existence of a consistent path structure nor how it can be identified. The next section identifies the maximal consistent path structure, and shows that it always exists. This guarantees the existence of a solution.

4. Existence

The aim of this subsection is to prove that a consistent path structure and, hence, a DEPCF does exist. To this end, we need to develop some new concepts. The concepts, which are crucial for the existence result, are inspired by their cousins in the social choice literature.[9] For expositional reasons, we shall use abbreviations of the concepts in the main body of the paper.

Let $\mathcal{B}$ be a set of paths.

Definition 4. Path $\bar{x} \in \mathcal{X}$ is *covered* in $\mathcal{B}$ via node y if there is k such that $\bar{y} \rhd_k \bar{x}$, for all $\bar{y} \in \mathcal{B}_y$.

That is, a path $\bar{x}$ is covered via node y in set $\mathcal{B}$ of paths if, at some particular node (the "kth") of $\bar{x}$, the members of the active coalition profit by directing the play to node y rather than continuing along $\bar{x}$, no matter which paths in the set $\mathcal{B}$ set is followed after the deviation (under the hypothesis that the final element of played path is implemented).

Denote by $uc(\mathcal{B})$ the subset of elements in $\mathcal{B}$ that are *uncovered* in $\mathcal{B} \subseteq \mathcal{X}$, *i.e.*,

$$uc(\mathcal{B}) = \{\bar{x} \in \mathcal{B}\colon \ \bar{x} \text{ is not covered in } \mathcal{B}\}.$$

By construction, $uc(\mathcal{B}) \subseteq \mathcal{X} \cup \{\emptyset\}$.

We now strengthen the concept by iterating the uncovered-operator until no paths are left to be covered. Set $UC^0 = \mathcal{F}$, and let $UC^{t+1} = uc(UC^t) \subseteq \mathcal{F}$, for all $t = 0, \ldots.$ The *ultimate uncovered set* $UUC \subseteq \mathcal{F}$ satisfies $UUC := UC^\infty$.

Our aim is to show that UUC is a consistent path structure. That is, in addition to the conditions stated in Definition 3, we need to verify that UUC is a well-defined concept, and that under all status quos it contains a path that gives guidance how to proceed. It suffices to show that UUC is reached in finitely many iterations, and that UC_y^t is nonempty for all y and t.

Lemma 3. *The ultimate uncovered set* UUC *does exist. Moreover,* UUC_y *is nonempty, for all* $y \in X$.

The proof of this important result is relegated to Appendix A. The difficult part is to show that UC_y^t is nonempty for all y and t.

We now state our existence result: the ultimate uncovered set UUC is a consistent path structure.[10] By Lemma 3, it suffices to show that no element in UUC is covered in UUC.

Theorem 2. *The ultimate uncovered set* UUC *is a consistent path structure.*

---

9 See Fishburn [17], Miller [31], and Dutta [16]. Laslier [28] is an in-depth survey. The existence result in Chwe [11] is based on a similar iterative argument.

10 See also Flesch et al. [18].

Proof. By construction, $UUC \subseteq \mathcal{F}$. Thus it suffices to check parts (i) and (ii) of Definition 3.

(i) By Lemma 3, UUC_y is nonempty, for all $y \in X$.

(ii) By the construction of UUC, $uc(UUC) = UUC$. Thus $\bar{x} \in UUC$ is not covered in UUC, *i.e.*, there is no y and k such that $\bar{y} \rhd_k \bar{x}$, for all $\bar{y} \in UUC_y$. □

The next result shows that UUC is the (unique) maximal consistent path structure in the sense of set inclusion.

Theorem 3. *The ultimate uncovered set UUC contains as a subset any consistent path structure.*

Proof. Let $\mathcal{C}$ be a CPS. Take any $\bar{x} \in \mathcal{C}$.

1st iteration: Since $\bar{x}$ satisfies Definition 3(ii), and $\mathcal{C} \subseteq \mathcal{F}$, it follows that $\bar{x}$ is not covered in $UC^0 = \mathcal{F}$. Hence $\bar{x} \in uc(\mathcal{F}) = UC^1$. Since $\bar{x}$ was arbitrary, $\mathcal{C} \subseteq UC^1$.

2nd iteration: Since $\bar{x}$ satisfies Definition 3(ii), and $\mathcal{C} \subseteq UC^1$, it follows that $\bar{x}$ is not covered in UC^1. Hence $\bar{x} \in uc(UC^1) = UC^2$. Since $\bar{x}$ was arbitrary, $\mathcal{C} \subseteq UC^2$.

$\vdots$

Tth iteration: Since $\bar{x}$ satisfies Definition 3(ii), and $\mathcal{C} \subseteq UC^{T-1}$, it follows that $\bar{x}$ is not covered in UC^{T-1}. Hence $\bar{x} \in uc(UC^{T-1}) = UC^T$. Since $\bar{x}$ was arbitrary, $\mathcal{C} \subseteq UC^T =: UUC$. □

Since the UUC is obtained via a well-defined recursive process, there is no question about its existence. By Theorems 2 and 1, one can construct a DEPCF on UUC. Hence we can conclude that a dynamic equilibrium process of coalition formation is guaranteed to exist.

Corollary 2. *There is a dynamic equilibrium process of coalition formation σ such that $\bar{\sigma}(H) = UUC$.*

By Theorem 3, we have also characterized outcomes that are implementable with any equilibrium.

Corollary 3. *The set of outcomes that are implementable via any dynamic equilibrium process of coalition formation is contained in $\mu[UUC]$.*

5. Computational considerations

The problem with game theoretic solution concepts is often their computability. Such questions are particularly acute here since the set of paths $\mathcal{X}$ typically contains an infinite number of elements. It is not clear *a priori* whether coalitional equilibria, or the outcomes they implement, can be identified via a procedure that terminates in finite time. The applicability of such a solution would be questionable.[11]

We now show that these concerns are unwarranted. A convenient algorithm for computing the relevant elements of the ultimate uncovered set, *i.e.*, the largest consistent path structure, is provided. This algorithm, which terminates in finite time, thus generates a complete description of the outcomes that can be implemented via a dynamic process of coalition formation.

[11] See Dutta [16] for a related and inspiring discussion.

In order to establish the results, we describe the information related to a path in the simplest possible form. We say that $\bar{y} = (y_0, \dots, y_L) \in \mathcal{X}$ is a *reduction* of $\bar{x} = (x_0, \dots, x_K) \in \mathcal{X}$ if $x_0 = y_0$, $x_K = y_L$, and $\{(y_l, y_{l+1})_{l=0}^{L-1}\} \subseteq \{(x_k, x_{k+1})_{k=0}^{K-1}\}$. Then $\bar{y}$ is a *full reduction* of $\bar{x}$ if it is a reduction of $\bar{x}$ and if the only reduction of $\bar{y}$ is $\bar{y}$ itself. That is, $\bar{y}$ contains a minimum amount of edges of $\bar{x}$ that are needed to travel from x_0 to x_K. Note that the reduction relation is transitive. Since the paths contain at most finitely many distinct elements, each $\bar{x}$ has a full reduction. However, a full reduction of $\bar{x}$ need not be unique.

For any set $\mathcal{B}$ of paths, denote by $fr(\mathcal{B})$ the collection of *all* full reductions of the elements in $\mathcal{B}$, *i.e.*, the elements of $\mathcal{X}$ that are full reductions of $\mathcal{B}$. Note that any path $(x_0, \dots, x_K)$ that contains a cycle such that $x_k = x_l$ for some $k < l$, cannot be a full reduction since $(x_0, \dots, x_k, x_{l+1}, \dots, x_K)$ is a reduction of $(x_0, \dots, x_K)$ but not vice versa. Thus, since any fully reduced path is acyclic and since X contains finitely many elements, the set of fully reduced paths $fr(\mathcal{X})$ can be identified in finite time.

The full reduction-operation preserves two important aspects of consistent paths structures: the initial status quo and the feasibility. More formally,

$$\mathcal{B} \subseteq \mathcal{F}_x \quad \text{implies} \quad fr(\mathcal{B}) \subseteq \mathcal{F}_x. \tag{7}$$

The following observation is now easily deduced from (7).

Lemma 4. *If $\mathcal{C}$ is a consistent path structure meeting Definition* 3*, then so is $fr(\mathcal{C})$.*

Thus, when identifying consistent path structures – or outcomes that can be implemented via them – it is in general sufficient to focus on consistent path structures that are composed of fully reduced feasible paths $fr(\mathcal{F})$. In particular, the fully reduced form of the feasible ultimate uncovered set $fr(UUC)$ is a consistent path structure.

Now we describe a finite procedure that identifies the fully reduced ultimate uncovered set $fr(UUC)$. Identify $fr(\mathcal{F})$. Define recursively the uncovered set and its iterations on the set of fully reduced paths: $UC_{FR}^0 = fr(\mathcal{F})$ and $UC_{FR}^{j+1} = uc(UC_{FR}^j)$ for all $j = 0, \dots$. The fully reduced ultimate uncovered set is then defined by $UUC_{FR} = UC_{FR}^\infty$. Since, by the argument made in Lemma 6, only finitely many iterations are needed for UC_{FR}^t to converge. UUC_{FR} can be identified in finite time.

Thus UUC_{FR} can be identified in finite time. Our aim is to show that UUC_{FR} is a CPS and contains as subsets all fully reduced CPSs. This implies that the endpoints of the paths in UUC_{FR} are the outcomes that are implementable in any dynamic equilibrium process of coalition formation.

Our first task is to show that UUC_{FR} coincides with $fr(UUC)$ and is contained in UUC.

Lemma 5. $fr(UUC) = UUC_{FR} \subseteq UUC$.

Proof. Claim 1. $UC_{FR}^t \subseteq UC^t$, for all t.

Proof. Let t be the first stage in which $\bar{x} \in UC_{FR}^t \setminus UC^t$ for some $\bar{x}$. Hence $UC_{FR}^{t-1} \subseteq UC^{t-1}$. But then, by the definition of covering, since $\bar{x}$ is covered in UC^{t-1}, it must be covered in UC_{FR}^{t-1}. But this contradicts $\bar{x} \in UC_{FR}^t$. □

Claim 2. $fr(UC^t) \subseteq UC^t$, for all t.

Proof. Let t be the first stage in which there is $\bar{x} \in fr(UC^t) \setminus UC^t$. Since $\bar{x} \in fr(UC^t)$, $\bar{x}$ must be a full reduction of some $\bar{y} \in UC^t$. But by the definition of full reduction, the assumption that $\bar{x}$ is covered in UC^{t-1} contradicts the assumption that $\bar{y}$ is not covered in UC^{t-1}. □

Claim 3. $UC^t_{FR} = fr(UC^t)$ for all t.

Proof. Since the full reduction of a set of fully reduced paths is the set itself, it follows by Claim 1 that $UC^t_{FR} = fr(UC^t_{FR}) \subseteq fr(UC^t)$, for all t. For the other direction, let t be the first stage in which $\bar{x} \in fr(UC^t) \setminus UC^t_{FR}$ for some $\bar{x}$. Hence, $fr(UC^{t-1}) \subseteq UC^{t-1}_{FR}$. But then, by the definition of covering, since $\bar{x}$ is covered in UC^{t-1}_{FR} it must be covered in $fr(UC^{t-1})$. By the definition of full reduction, $\bar{x}$ is also covered in UC^{t-1}. Hence $\bar{x} \notin UC^t$. But then $\bar{x} \in fr(UC^t) \setminus UC^t$, which contradicts Claim 2. Thus $fr(UC^t) \subseteq UC^t_{FR}$, for all t. □

Claim 4. $fr(UUC) = UUC_{FR} \subseteq UUC$.

Proof. Combining Claims 1 and 3, we have $fr(UC^t) = UC^t_{FR} \subseteq UC^t$ for all t. The fact that only finitely many iterations are needed gives the result. □

From Lemmata 4 and 5 it now follows that also the largest consistent path structure can be described directly in terms of fully reduced paths via the iterative procedure that identifies UUC_{FR}.

Theorem 4. *UUC_{FR} is a consistent path structure. Moreover, UUC_{FR} contains as a subset any fully reduced consistent path structure.*

Proof. By Lemma 4, $fr(UUC)$ is a CPS. By Lemma 5, $fr(UUC) = UUC_{FR}$. Thus UUC_{FR} is a CPS.

By Theorem 2, UUC contains any CPS $\mathcal{C}$ as a subset. Thus, by Lemma 5, UUC_{FR} contains $fr(\mathcal{C})$ as a subset. □

Furthermore, since the final element of a path is invariant with respect to the full reduction-operation, *i.e.*, $\mu[\mathcal{B}] = \mu[fr(\mathcal{B})]$ for any $\mathcal{B} \subseteq \mathcal{X}$, it follows that the outcomes that can be implemented with UUC coincide with the outcomes that can be implemented with UUC_{FR}.

Theorem 5. *The set of outcomes that are implementable via any dynamic equilibrium process of coalition formation is contained in $\mu[UUC_{FR}]$.*

Proof. By Corollary 3, Lemma 5, and since $\mu[UUC_{FR}] = \mu[UUC]$. □

We have thus constructed a procedure that specifies, in finitely many steps, the outcomes that can be implemented in *any* DEPCF. The collection of paths identified by the procedure also contains the reduced forms of the paths of all DEPCFs and, moreover, it constitutes itself a DEPCF.

6. Finitistic equilibria

We have assumed that the processes of coalition formation are history dependent and implement an outcome in finite time. These assumptions are problematic if (a) the process has to have infinite memory, (b) there is no upper bound on how long it will take to implement an outcome. In this section we collect the results of the previous sections to argue that neither of these concerns is warranted.

Recall that the dynamic process of coalition formation σ^* used in Lemma 1 uses a state space Q that has the same cardinality as the consistent path structure $\mathcal{C}$ on which it is built. By Lemma 4, a full reduction $fr(\mathcal{C})$ of $\mathcal{C}$ is also a consistent path structure. In terms of real consequences, however, $fr(\mathcal{C})$ and $\mathcal{C}$ are equivalent, *i.e.*, $\mu[fr(\mathcal{C})] = \mu[\mathcal{C}]$. Since fully reduced paths are necessarily acyclic, it follows that $fr(\mathcal{C})$ contains finitely many elements. Since, by Theorem 1, any consistent path structure can be identified with a DEPCF, we can conclude that, in terms real of consequences (implemented outcomes) there is no loss of generality to assume that the process is a finite state Markov chain.

Corollary 4. *There is a dynamic equilibrium process of coalition formation* (*DEPCF*) *that is a finite state Markov chain. Moreover, for any DEPCF there is another DEPCF, equivalent in terms of implementable outcomes, that is a finite state Markov chain.*

Moreover, since DEPCFs can be identified with consistent path structures, any property of paths in a consistent path structure translates directly to an observation concerning equilibria. Since $fr(\mathcal{C})$ contains finitely many paths that are, at most, finitely long, the final element of any path in $fr(\mathcal{C})$ can be reached in finitely many steps. Hence Theorem 1 implies the following corollary.

Corollary 5. *There is a dynamic equilibrium process of coalition formation* (*DEPCF*) *that implements an outcome in uniformly bounded time after any history. Moreover, for any DEPCF there is another DEPCF, equivalent in terms of implementable outcomes, that implements an outcome in uniformly bounded time after any history.*

7. Relation to other models

In this section, we relate our solution to some existing equilibrium notions of coalition formation.

7.1. Blocking in real time

In this section we relate our one-shot coalitional game to the model of real time blocking of KR (Konishi and Ray [27]) (see also Ray [35]).[12] Now X is interpreted as a set of *states* and u_i is a utility function of player $i \in N$ over X. Analogously to the model above, $F_S(x) \subseteq X$ is the set of states achievable by a one-step coalitional move in state x by a coalition S. It assumed that $x \in \bigcap_S F_S(x)$ to guarantee that staying in x – once reached – is possible.

Let H be the set of all histories of states $(x_0, \dots, x_t)$ such that $x_0 = x^*$. A deterministic process of coalition formation PCF is now a function $p : H \to X$, capturing transitions from one state to another. Letting $p^0(h) = p(h)$ and $p^t(h) = p(h, p^0(h), \dots, p^{t-1}(h))$, for all $t = 1, \dots,$ denote by $\bar{p}(h) = (p^0(h), p^1(h), \dots)$ the chain of states that will materialize along the PCF p,

[12] Hyndman and Ray [24] and Gomez and Jehiel [20] are other papers with real time coalitional negotiation. However, there the negotiation is noncooperative, based on a protocol.

starting from history h. All playable chains are denoted by $\bar{p}(H) = \{\bar{p}(h)\colon h \in H\}$. A PCF p is *absorbing* if for any history h there is an integer $T_h < \infty$ such that there is $x \in X$ such that

$$p^t(h) = x, \quad \text{for all } t > T. \tag{8}$$

The absorbing state of the PCF p starting from history h, denoted by $\alpha[\bar{p}(h)]$, is then well defined for all h.

We evaluate the players' intertemporal payoffs by the *limit-of-the-means* criterion:

$$V_i(h) = \lim_{T\to\infty} \frac{\sum_{t=0}^{T} u_i(p^t(h))}{T}.$$

Such payoff exists whenever PCF p is absorbing:

$$V_i(h) = u_i\big(\alpha[\bar{p}(h)]\big), \quad \text{for all } h \in H.$$

Profitable moves and efficiency concepts are defined analogously to the one-shot game discussed before. That is, given an absorbing PCF p and a history (h, x), coalition S has a weakly preferred move $y \in F_S(x)$ from x if $V_i(h, x, y) \geqslant u_i(x)$ for all $i \in S$. Further, a move $y \in F_S(x)$ is efficient for coalition S if there is no $z \in F_S(x)$ such that $V_i(h, x, z) > V_i(h, x, y)$ for all $i \in S$.

Given these notions, we may now define a modified version of the equilibrium criterion of KR. A dynamic, deterministic, absorbing PCF p forms an equilibrium if the following two conditions hold, for any $(h, x) \in H$:

1. If $p(h, x) \in X\backslash\{x\}$, then $p(h, x)$ is an efficient and weakly preferred move from x for $S(x, p(h, x))$.
2. If $p(h, x) = x$, then x is an efficient move from x for all S.

It is clear that this version of dynamic equilibrium process of coalition formation is equivalent with the one defined for terminating processes. That is, the set of states that are absorbing after some history in some equilibrium of the KR model of real time blocking coincides with the set of outcomes that are implementable after some history in a corresponding equilibrium of the one-shot game.

However, as hinted above, our solution differs from KR in two ways (apart from allowing the process to be history dependent). First, part 2 of the condition in KR is slightly stronger as it requires that if the status quo state x is not efficient for some coalition, then the equilibrium move must be an efficient and *strictly* preferred move from x for some coalition (we only demand weak preference, by part 1 of the condition).[13] This additional desideratum would affect the sufficiency but not the necessity of the characterization in Theorem 1. Second, in the original formulation of KR, intertemporal preferences are captured by the discounting criterion. Formally, given a discount factor $\delta \in (0, 1)$, player i's value function V_i^δ is derived as the solution to the recursive expression (see Ray [35])

$$V_i^\delta(h, x) = (1-\delta)u_i(x) + \delta V_i^\delta\big(h, x, p(h, x)\big). \tag{9}$$

KR define coalitional preferences and efficiency criteria and, *a fortiori*, the solution vis-à-vis these value functions.[14]

[13] The motivation for this weakening is our focus on properties of the solution that stem from the one-deviation principle.
[14] In KR, a history dependent version of weak preference reads $V_i^\delta(h, x, y) \geqslant V_i^\delta(h, x)$ for all i and for all S. However, given the definition of V_i^δ, this is equivalent to $V_i^\delta(h, x, y) \geqslant u_i(x)$ for all i and for all S.

We should add that there are games in which the way intertemporal payoffs are evaluated does make a qualitative difference (even in the limit).[15]

7.2. Largest consistent set

Rosenthal [38] and Chwe [11] define the general framework of coalition formation used in this work. Chwe's [11] aim is to rule out outcomes whose stability could never be an issue. His solution, the *largest consistent set* (LCS), always exists and is one of the most frequently used solutions in applications. KR show that the set of absorbing states of absorbing deterministic equilibrium processes that have the Markovian property is a subset of LCS. We now demonstrate that relaxing the Markovian restriction does not affect the result: the set of outcomes that are implementable via DEPCF is a subset (sometimes strict) of LCS.

To Chwe's [11] solution, an outcome $y \in X$ *indirectly* dominates x if there is a feasible path $(x_0, \ldots, x_K) \in \mathcal{F}$ such that $x_0 = x$, $y = x_K$ (recall our the definition (2) of a feasible path). [16] Set $C \subseteq X$ is a *consistent set* if C consists of all x for which the following holds: if $z \in F_S(x)$, then there is $y \in C$ such that either $y = z$ or y indirectly dominates z and $y \not\succ_S x$. Chwe showed that a consistent set exists and the *largest* (in the sense of set inclusion) *consistent set*, LCS, is unique.

We show that, in terms of implementable outcomes, DEPCF is a refinement of LCS. We do this by first showing that the ultimate uncovered set implements a consistent set of outcomes.

Proposition 1. *$\mu[UUC]$ is a subset of the largest consistent set.*

Proof. It suffices to show that $\mu[UUC]$ is a consistent set. Suppose, on the contrary, that this is not the case. Then there is an $x \in \mu[UUC]$, an S, and a $z \in F_S(x)$ such that $\mu[\bar{y}] \succ_S x$, for all $\bar{y} \in \mathcal{F}_z \cap UUC$. But since there is $\bar{x} \in UUC$ such that $\mu[\bar{x}] = x$, this contradicts the assumption that $UUC = uc(UUC)$. That is, that $\bar{x}$ is not covered in UUC. □

Importantly, $\mu[UUC]$ need not coincide with LCS. We now show via an example that $\mu[UUC]$ can be a *strict* subset of LCS. Consider the game depicted in Fig. 2.

Here $N = \{1, 2, 3\}$, $X = \{a, b, c, d\}$. Numerical payoffs (in the order of player indices) from choices a, b, c, and d are, respectively, $(0, 0, 1)$, $(0, 1, 0)$, $(1, 0, 0)$, and $(2, 2, 2)$. LCS of this problem is $\{a, b, d\}$. However, the largest consistent path structure UUC consists the of paths $\{(a, b, c, d), (b, c, d), (c, d), (d)\}$, and hence $\mu[UUC] = \{d\}$.

7.3. Stable standard of behavior and a full coalitional equilibrium

Mariotti [29] and Xue [41] are motivated by considerations that are closely related to this paper (see also Mariotti and Xue [30]). In this section, we discuss their solutions, *full coalitional equilibrium* and *stable standard of behavior*, respectively, from the viewpoint of dynamic equilibrium processes of coalition formation.

Xue [41] arguing that the largest consistent set is too permissive, imposes a consistency criterion on paths rather than on outcomes. His starting point is the observation that – since in

[15] An example is available upon request.

[16] As Konishi and Ray [27] and Ray [35], we appeal to the weaker version of Chwe's [11] notion of indirect dominance. The original one requires strict preference.

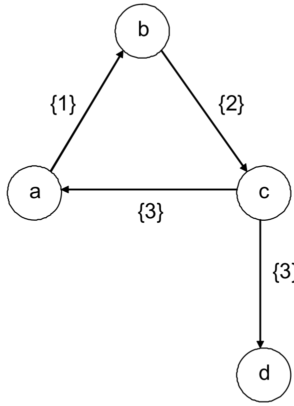

Fig. 2.

a dynamic environment play can move *in equilibrium* from one status quo to another before an outcome is implemented – what is crucial is the stability of paths rather than that of the outcomes. Xue [41] uses Greenberg's [19] foundational *Theory of Social Situations* to identify paths that are robust against deviations. This theory is based on the von Neumann–Morgenstern stable set approach. This solution concept bears similarity to the consistent path structure, the reduced form characterization of our solution. The difference between Xue's [41] solution and ours is that the former permits deviations also by inactive coalitions. This property has consequences on the existence of the solution.

To interpret our results in the framework of Xue [41], let us lay down the key ingredients of his model.[17] Assume that the alternative $x \in X$ is the current status quo. Consider a path $\bar{x}$ and some of its node x_k. Assume that a coalition S can replace x_k by some alternative $y \neq x_{k+1}$. In doing so, S is aware of that the set of feasible paths from y is $\mathcal{F}_y$. In contemplating such a deviation from y, however, members of S base their decision on the comparisons of paths that might be followed by rational and farsighted individuals at y. Let $SB(y) \subset \mathcal{F}_y$ denote this *standard of behavior*.

The following definition, which is adopted from Xue [41], describes, in our notation, the conservative approach to the stable standards of behavior.

Definition 5. A standard of behavior SB is *conservatively stable* if it is

Internally stable: for all $x \in X$, if $\bar{x} \in SB(x)$, then there is *no* y, coalition S, and k such that $y \in F_S(x_k) \setminus \{x_{k+1}\}$, and $\mu[\bar{y}] \succ_S \mu[\bar{x}]$, for all $\bar{y} \in SB(y)$,

Externally stable: for all $x \in X$, if $\bar{x} \in \mathcal{F}_x \setminus SB(x)$, then there is y, coalition S, and k such that $y \in F_S(x_k) \setminus \{x_{k+1}\}$, and $\mu[\bar{y}] \succ_S \mu[\bar{x}]$, for all $\bar{y} \in SB(y)$.

To see the relationship between our solution concept and the conservatively stable standard of behavior of Xue [41], let us spell out the key features of the feasible ultimate uncovered set in the language of internal and external stability.

[17] Referring to Greenberg's [19] Theory of Social Situations.

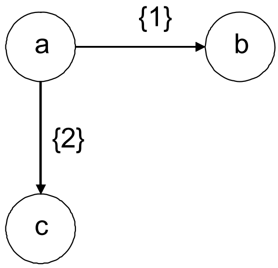

Fig. 3.

Remark 1. The ultimate uncovered set *UUC* satisfies

Internal stability: if $\bar{x} \in UUC$, then there is *no* y and k such that $y \in F_S(x_k) \setminus \{x_{k+1}\}$ and $\mu[\bar{y}] \succ_S \mu[\bar{x}]$, for $S = S(x_{k+1}, x_k)$, and for all $\bar{y} \in \mathcal{F}_y \cap UUC$.

External stability: if $\bar{x} \in \mathcal{F} \setminus UUC$, then there is y and k such that $y \in F_S(x_k) \setminus \{x_{k+1}\}$ and $\mu[\bar{y}] \succ_S \mu[\bar{x}]$, for $S = S(x_{k+1}, x_k)$, and for all $\bar{y} \in \mathcal{F}_y \cap UUC$.

Internal stability follows from the definition of a consistent path structure. External stability, in turn, is a consequence of the construction of *UUC*; every path outside *UUC* is covered in *UUC*. It is easy to see that the key difference between the solution concepts is that the notion of stable standard of behavior requires stability against arbitrary coalitional deviations, whereas the ultimate uncovered set only requires stability with respect to the active coalition. Since the restriction affects both internal and external stability, and in different directions, there is no straightforward relationship between the concepts.

From these definitions it is clear that the notions of conservative stable standard of behavior and the ultimate uncovered set coincide in games where only one coalition may change the status quo at a time. Roughly, this corresponds to the scenarios in which the coalitional game can be presented in the form of a simple recursive game (vis-à-vis interpreting coalitional preferences as a single individual's preferences).

The more stringent requirement for acceptable deviations prevents pathological blocking relationships and guarantees the existence of an ultimate uncovered set. Conversely, the leeway provided by arbitrary coalitional deviations in the context of stable standard of behavior is the reason for the occasional emptiness of the solution, as is the case in the example depicted in Fig. 3 above (due to Xue [41]), where payoffs from a, b and c are, respectively, $(0,0)$, $(2,1)$, and $(1,2)$. The unique conservative standard of behavior is empty. However, the ultimate uncovered set contains both the arcs: $\mathcal{F}_a \cap UUC = \{(a,b), (a,c)\}$ and hence $\mu[UUC] = \{b, c\}$.

To the best of our knowledge, Mariotti [29] is the only paper in the literature that explicitly models history dependent coalitional strategies. Although his set up is different from this paper, there are notable similarities in how strategic behavior is modeled. In fact, we argue that all conditions imposed on his full coalitional equilibrium-solution are also satisfied by our solution. Hence, our existence result can also be applied to Mariotti's set up.

Let us sketch Mariotti's [29] model, where players seek to form an agreement of how to play a strategic form game. The game is defined by sets A_i of actions for each player i. Preferences are defined over the set of all profiles of possible actions $A_N = \times_{i \in N} A_i$. The play proceeds by coalitions S proposing a profile of actions in $A_S = \times_{i \in S} A_i$. If coalition S serves a the proposer, then the choices of the complementary coalition $-S$ are inherited from the current status quo profile. Hence, the set of the feasible outcomes of coalition S, given the status quo profile $x \in X := A_N$, is obtained by

$$F_S(x) = \{(y_S, x_{-S}):\ y_S \in A_S\} \subseteq X.$$

Now we can apply Definition 1 to specify conditions under which an agreement is reached.

The key features of Mariotti's [29] novel approach bear similarity to ours: The strategies are assumed history dependent and the focus is, at the outset, on strategies that implement an outcome in finite time. Also a version of the one-deviation property is used as the equilibrium criterion. While Mariotti's solution, full coalitional equilibrium, is richer in the sense that it allows the coalitions to disagree on how the play will evolve in the future, it is apparent the solution *also* permits the agents to agree on how the play proceeds. This is our basic assumption: the strategy specifies a unique future play path that is known by all players. Therefore we conclude that, in the case of strategic form games, a DEPCF is a particular case of a full coalitional equilibrium of Mariotti [29]. By Corollary 2, it then follows that also *a full coalitional equilibrium exists*.[18] However, there may well be many full coalitional equilibria about which our characterization cannot say much.

8. Efficiency

A recurrent theme in coalitional analysis is efficiency. A classic argument that goes under the name of the Coase theorem says that an outcome that results from unrestricted bargaining will always be efficient: otherwise it would be blocked by another outcome that all players prefer. This intuition is not sufficient in the current framework.

Rosenthal's centipede game is an example of a game in which the dynamic equilibrium process of coalition formation does not implement an efficient outcome (in finite games of perfect information, the equilibrium coincides with the backwards induction-solution). But the centipede game is not a good example since the Coase theorem appeals to the coalitions' inability to *commit* to a bad outcome. This implicitly assumes that coalitional moves do not have irrecoverable consequences, and that the game cannot be finite. This desideratum is met for example by games that are *irreducible* in a sense that from any node there is a path to any other node.

It turns out that irreducibility is not sufficient for efficiency. Consider the following anarchistic economy, a variation of Prisoners Dilemma that was discussed in the introduction. There are two players 1 and 2. Assume that initial utility allocation is unequal, say 4 to player 1, and 1 to player 2. The worse off player, in this case 2, can pillage utility units from the other player, in this case from player 1. Pillaging is costly and as a result 2 receives only one half of what he pillages. After pillaging, the players can increase, by common effort, both of their payoffs by one unit. This results in payoffs 1 and 4 for players 1 and 2, respectively.[19] Then the roles of the players are reversed in a sense that now the worse off player is 1 and he may pillage player 2's utility units. The game is depicted in Fig. 4 where the directed edges show potential coalitional moves (numbers inside the nodes show the payoffs). The graph constitutes a circuit. Nodes $(3,0)$ and $(0,3)$ are Pareto dominated by the next nodes in the circuit. Nevertheless the largest consistent set, and hence $\mu[UUC]$ by Proposition 1, consists only of the low payoff nodes $(3,0)$ and $(0,3)$. Thus *every* DEPCF only results in Pareto dominated outcomes.

[18] There are minor differences in the defining inequalities of the two solutions concepts. However, such differences do not affect the qualitative nature of the results.

[19] Pillaging by 2 is not possible when payoffs are $(3,0)$ since pillaging itself requires some resources.

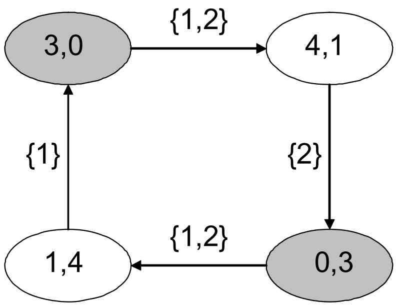

Fig. 4.

One can argue that the inefficiency here is due to artificially restricted blocking relationships. A stronger requirement than irreducibility is *completeness*: for any y and x there is S such that $y \in F_S(x)$. But again, there are irreducible and complete games where $\mu[UUC]$ contains *only* Pareto-dominated outcomes. An example is depicted in Appendix B.

A third, yet stronger condition on the graph would be that $F_S(x) = X$ *for all* S, so that at each node all coalitions can move the play to any node. In such case there would be no *a priori* restrictions on what a coalition can achieve. It is easy to show that this restriction would not preclude any Pareto-optimal outcome from being in $\mu[UUC]$. However, Pareto-optimality is still not guaranteed. Modify, for example, the game depicted in Fig. 4 by allowing each coalition $\{1\}, \{2\}$, and $\{1,2\}$ induce any outcome from any outcome. Then $\mu[UUC]$ would contain all the outcomes but there would also be a consistent path structure $\mathcal{C}$ such that $\mu[\mathcal{C}] = \{(3,0),(0,3)\}$. Hence the corresponding equilibrium process could only implement Pareto dominated outcomes. Therefore, we have to conclude that there is nothing inconsistent in the idea of an inefficient outcome being implemented, even if the bargaining opportunities are unrestricted.[20]

9. Discussion

We have shown that a dynamic version of the absorbing, deterministic equilibrium process of coalition formation of Konishi and Ray (2003) always exists. The crucial assumption is that the process may now depend on the history of the play. Another important assumption is that only one coalition may be active in blocking the underlying status quo. This assumption may be criticized.

Consider an example due to Xue [41], depicted in Fig. 5.

Now $N = \{1,2\}$, $X = \{a,b,c,d\}$, and numerical payoffs (in the order of players' indices) from a, b, c, and d are, respectively, $(6,0)$, $(7,4)$, $(5,10)$, $(10,5)$. LCS contains the same set of outcomes $\{a,c,d\}$ that is implemented in the unique DEPCF. However, as Xue [41] argues, this set is too large since d should not ever be chosen: when a is the status quo, the "predicted" outcomes are $\{a,d\}$, the latter when the coalition $\{1,2\}$ induces d. But once the coalition $\{1,2\}$ is about to form, player 2 would renege on the contract and choose option c instead. Hence, d should not be considered a good prediction.

[20] Konishi and Ray [27] establish conditions on EPCF under which Pareto-optimality is guaranteed. Diamantoudi and Xue [14] demonstrate that the top coalition-property of Banerjee et al. [3] guarantees Pareto-optimality of the conservatively stable standard of behavior in the class of hedonic games. It can be showed, by the argument made in the previous section, that this holds also for dynamic equilibrium processes of coalition formation in this class of games.

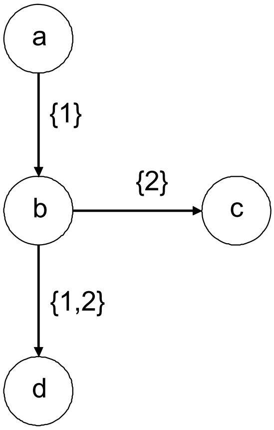

Fig. 5.

This is a valid argument in many circumstances. A way to cope with the criticism is to require robustness against *nested* coalitional deviations.[21] Part 1 of Definition 1 would then read

1′. $\sigma(h, x) \in X$ implies $\sigma(h, x)$ is an efficient and weakly preferred move from x for *all* coalitions $S \subseteq S(x, y)$.

To recover our key results with nested deviations, we would only need to define a new path dominance operator, $\rhd^*$, this time with respect to all subsets of the active coalition:

A path $\bar{y} \in \mathcal{F}$ $*$-dominates a path $\bar{x} \in \mathcal{X}$ at the kth step, denoted by $\bar{y} \rhd_k^* \bar{x}$, if $\mu[\bar{y}] \succ_S \mu[\bar{x}]$, for all $\bar{y} \in \mathcal{F}_y$, for some $y \in F_S(x_k)$, for some $S \subseteq S(x_k, x_{k+1})$.

By replacing "$\rhd$" with "$\rhd^*$" in the definition of a consistent path structure and the property "$S = S(x_k, x_{k+1})$" with "for any $S \subseteq S(x_k, x_{k+1})$" in the proofs, the existence of a solution would be obtained along the same avenues as in the current version. The new solution would, again, implement a set of outcomes that is a subset of LCS.

But even if one wants to incorporate the idea of nested deviations into the framework, it is not clear why one should do it through the solution concept. An alternative, and perhaps cleaner, way to model this is to assume that the game form Γ itself reflects nested deviations. This would be more in line with the standard position in game theory that all relevant behavioral *opportunities* should be specified through the game form (and not by the solution concept).[22] In the coalitional set up, this principle implies that the solution concept should only describe the optimal behavior of a single behavioral unit, a coalition.

More formally, to amend nested deviations into the game form, think of forming a derived game Γ' from the original one Γ that has two kinds of nodes, physical X (as in the game Γ) and transitional Z (in the present framework, transitional nodes could be modeled as "bad" out-

[21] Some nestedness assumption is often used to guarantee the existence. The classic papers in the field, Hart and Kurz [21], Bernheim et al. [6], and Ray and Vohra [36], are cases in point.

[22] See e.g. Osborne and Rubinstein [32].

comes). The derived game Γ' would now have the property that if $y \in F_S(x)$, then there are a transitional nodes $z_0, \ldots, z_{|S|+1}$ such that $z_0 = x$ and $z_{|S|+1} = y$, for some indexing j of the subsets of S, and such that $S^j \subset S$ would be the only potentially active coalition under z_j with $F'_{S^j}(z_j) = F_{S'}(x) \cup \{z_{j+1}\}$, for all $j = 0, \ldots, |S|$. Then the derived game would offer each subset of the original game's active coalition a chance to redirect the play in the direction it prefers, if it does not like the choice made by the originally active coalition. By the nestedness of S^j and S, such a derived game could clearly be constructed. In a dynamic equilibrium process of coalition formation, this would guarantee that no subset of the original active coalition would not want to deviate further.

Appendix A

In this appendix, we prove Lemma 3: that the ultimate uncovered set *UUC* is a well-defined concept, and that UUC_x is nonempty, for all x. The former claim is automatically satisfied if only finitely many iterations are needed to reach the ultimate uncovered set.

Lemma 6. *There is $T < \infty$ such that UC^T is the ultimate uncovered set UUC.*

Proof. Call a family of paths

$$\left\{(x_0, \ldots, x_K) \in \mathcal{X}\colon \left\{(x_k, x_{k+1})\right\}_{k=0}^{K-1} = B \text{ and } x_K = y\right\}$$

a *dominance class,* parametrized by $B \subseteq X \times X$ and $y \in X$. Since X is a finite set, the cardinality of distinct dominance classes is finite, and they partition $\mathcal{X}$.

A dominance class contains all the relevant information concerning dominance: If two paths $\bar{x}$ and $\bar{x}'$ belong to the same dominance class, then $\bar{x}$ is covered in UC^t if and only if $\bar{x}'$ is covered in UC^t, for any t. Since all paths in the same dominance class become covered at the same covering round t, and since there are finitely many dominance classes, the number of covering rounds to reach *UUC* must be finite. □

Given Lemma 6, to prove that UUC_x is nonempty it suffices that UC_x^t is nonempty for all t.

Lemma 7. *UC_x^t is nonempty, for all $x \in X$ and for all $t = 0, 1, \ldots$.*

Proof. Take any $S \subseteq N$, $x \in X$, and $t \in \{0, 1, \ldots\}$. Denote

$$C_S(x, t) = \left\{y \in F_S(x)\colon (x) \text{ is covered in } UC^t \text{ via } y\right\}.$$

Further, let $D_S(x, t)$ contain any $y \in C_S(x, t)$ having the property that for any $\bar{y} \in UC_y^t$ and for any $z \in C_S(x, t)$ there is $\bar{z} \in UC_z^t$ that is not preferred to $\bar{y}$ for all members of S:

$$D_S(x, t) = \left\{y \in C_S(x, t)\colon \bar{y} \in UC_y^t \text{ and } z \in C_S(x, t) \text{ imply there is } \bar{z} \in UC_z^t \text{ s.t. } \mu[\bar{z}] \not\succ_S \mu[\bar{y}]\right\}. \tag{10}$$

By the transitivity of $\succsim_S$, $D_S(x, t) = \emptyset$ if and only if $C_S(x, t) = \emptyset$.

Claim 0. Let $y \in D_S(x, t)$. Then $(x, \bar{y})$ is not covered at $k = 0$ in UC^t, for any $\bar{y} \in UC_y^t$.

Proof. If $(x, \bar{y}) \in UC_y^t$ is covered at $k = 0$ in UC^t, then there is $z \in F_{S(x,y)}(x)$ such that

$$\mu[\bar{z}] \succ_{S(x,y)} \mu[\bar{y}], \text{ for all } \bar{z} \in UC_z^t. \tag{11}$$

Since $y \in D_{S(x,y)}(x,t) \subseteq C_{S(x,y)}(x,t)$ we have $\mu[\bar{y}] \succ_{S(x,y)} x$. This means that (x) is covered in UC^t via z. But together with (11) this contradicts (10). □

Denote

$$G(x,t) = \bigcup_S D_S(x,t).$$

Claim 1. $G(x,t) = \emptyset$ if and only if $(x) \in UC^{t+1}$, for any $x \in X$ and for any $t = 0, 1, \ldots$.

Proof. By the definition of covering, if (x_J) is covered in UC^τ for any $\tau \leqslant t$, then it is covered in UC^t. □

Call a path $(x_0, \ldots, x_J)$ a *G-path under t* if $x_{j+1} \in G(x_j, t)$ for all $j = 0, \ldots, K-1$. If, moreover, $G(x_J, t) = \emptyset$, then $(x_0, \ldots, x_J)$ is a *terminal* G-path under t.

Claim 2. For any $t = 0, 1, \ldots$, if $(x_0, \ldots, x_J)$ is a terminal G-path under t, then $(x_0, \ldots, x_J) \in UC^{t+1}$.

Proof. Let $(x_0, \ldots, x_J)$ be a terminal G-path under t. By Claim 0, $G(x_J, t) = \emptyset$ implies that $(x_J) \in UC^{t+1}$. Since $x_{j+1} \in G(x_j, t)$, for all $j = 0, \ldots, J-1$, it follows by using Claim 1 and iterating backwards on $j = 1, \ldots, J$ that $(x_0, \ldots, x_J) \in UC^{t+1}$. □

Claim 3. For any $t = 0, 1, \ldots$ and for any $x \in X$, there is a terminal G-path $(y_0, \ldots, y_J)$ under t such that $y_0 = x$.

Proof. By Claim 2, the claim implies that $(y_0, \ldots, y_J) \in UC^{t+1}$. Let, on the contrary of the claim, there be t that is the first stage in which the claim does not hold. For any x, denote $G^0(x,t) = \{x\}$, and define recursively

$$G^n(x,t) = \{z\colon z \in G(y,t) \text{ and } y \in G^{n-1}(x,t)\}, \quad \text{for any } n = 1, \ldots.$$

Since the claim does not hold in stage t, there is a particular x such that $G(z,t) \neq \emptyset$, for all $z \in G^n(x,t)$, for all n. Viewing G as the support of a Markov process with state space X, denote by $V \subseteq X$ the ergodic set of this process, *i.e.*, V is the unique minimal subset of X in the sense of set inclusion such that $V = \bigcup_{v \in V} G(v,t)$ and $G^n(x,t) \subseteq V$ for all $n \geqslant n_V$ for some n_V. We now show a contradiction by proving that V cannot exist.

Subclaim 4. We first argue that $(v) \notin UC^t$ for all $v \in V$. Suppose on the contrary that $(v) \in UC^t$ and $v \in V$. By the definition of V, there is a G-path $(v_0, \ldots, v_L)$ under t such that $v = v_0 = v_L$. By using Claim 1 and iterating backwards on $\ell = 1, \ldots, L-1$, it follows that $(v_1, \ldots, v_L) \in UC^t$. But then, since $v_0 = v_J$, (v_0) is not covered in UC^t via v_1, a contradiction.

Subclaim 5. Now we argue that the opposite of Subclaim 3.1 cannot be true either. Suppose that $(v) \notin UC^t$ for all $v \in V$. Take any element of V, say y_0. Since $(y_0) \notin UC^t$, we have

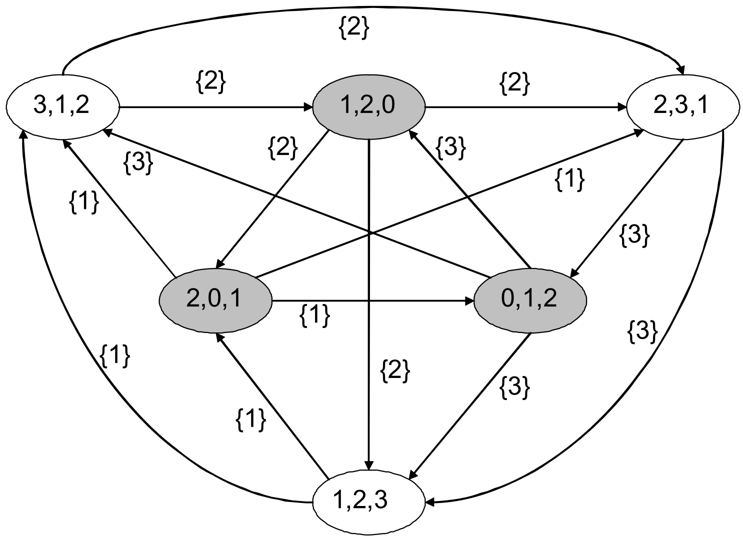

Fig. 6.

$G(y_0, t-1) \neq \emptyset$. By the maintained assumption, there is a terminal G-path $(y_0, \ldots, y_J)$ under $t-1$ such that, by Claim 2, $(y_0, \ldots, y_J) \in UC^t$. By Claim 1, $y_1 \in G(y_0, t-1)$ and $\bar{z} \in UC^t_{y_1} \subseteq UC^{t-1}_{y_1}$ imply $(y_0, \bar{z}) \in UC^t_{y_0}$. By the definition of V, there is a G-path $(v_0, \ldots, v_L)$ such that $y_0 = v_0 = v_L$. Thus by using Claim 1 and iterating backwards on $\ell = 0, \ldots, L-1$, it follows that $(v_0, \ldots, v_{L-1}, \bar{z}) \in UC^{t+1}_{y_0} \subseteq UC^t_{y_0}$ for all $\bar{z} \in UC^t_{y_1}$. This implies, since $v_1 \in G(y_0, t)$, that also $y_1 \in G(y_0, t)$. By the definition of V, then, $y_1 \in V$. Iterating this way on $j = 2, \ldots, J-1$ it follows that $y_J \in V$. But by $(y_0, \ldots, y_J) \in UC^t$ it also follows that $(y_J) \in UC^t$, and a contradiction is proved. □

Claims 2 and 3 now establish the proof. □

Appendix B

Consider the three players/six alternatives game, depicted in Fig. 6. The directed graph describes the possible coalitional blockings between the nodes. The graph is complete but, for simplicity, we only depict blockings to one direction, *i.e.*, $(3, 1, 2) \rightarrow_{\{2\}} (1, 2, 0)$ means that also $(1, 2, 0) \rightarrow_{\{1,3\}} (3, 1, 2)$, etc.. Again, $\mu[UUC]$ consists of the shaded nodes as any Pareto-optimal, non-shaded node is covered via a shaded node.

References

[1] R. Aumann, R. Myerson, Endogenous formation of links between players and of coalitions: An application of the Shapley value, in: A. Roth (Ed.), The Shapley Value: Essays in Honor of Lloyd S. Shapley, Cambridge University Press, Cambridge, New York, Melbourne, 1988, pp. 175–191.

[2] A. Ambrus, Coalitional rationalizability, Quart. J. Econ. 121 (2006) 903–929.

[3] S. Banerjee, H. Konishi, T. Sönmez, Core in a simple coalition formation game, Soc. Choice Welfare 18 (2001) 135–153.

[4] S. Barberà, A. Gerber, On coalition formation: durable coalition structures, Math. Soc. Sci. 45 (2003) 185–203.

[5] S. Barberà, A. Gerber, A note on the impossibility of a satisfactory concept of stability for coalition formation games, Econ. Letters 95 (2007) 85–90.
[6] D. Bernheim, B. Peleg, M. Whinston, Coalition-proof Nash equilibria I, Concepts, J. Econ. Theory 42 (1987) 1–12.
[7] F. Bloch, Sequential formation of coalitions in games with externalities and fixed payoff division, Games Econ. Behav. 14 (1996) 90–123.
[8] F. Bloch, A. Gomes, Contracting with externalities and outside options, J. Econ. Theory 127 (2006) 172–201.
[9] A. Bogomolnaia, M. Jackson, The stability of hedonic coalition structures, Games Econ. Behav. 38 (2002) 201–230.
[10] K. Chatterjee, B. Dutta, D. Ray, K. Sengupta, A noncooperative theory of coalitional bargaining, Rev. Econ. Stud. 60 (1993) 463–477.
[11] M. Chwe, Farsighted stability, J. Econ. Theory 63 (1994) 299–325.
[12] J. Conley, H. Konishi, Migration-proof Tiebout equilibrium: existence and asymptotic efficiency, J. Public Econ. 86 (2002) 243–262.
[13] B. Dutta, S. Mutuswami, Stable networks, J. Econ. Theory 76 (1997) 322–344.
[14] E. Diamantoudi, L. Xue, Farsighted stability in hedonic games, Soc. Choice Welfare 21 (2003) 39–61.
[15] E. Diamantoudi, L. Xue, Coalitions, agreements and efficiency, J. Econ. Theory 136 (1) (2007) 105–125.
[16] B. Dutta, Covering sets and new condorcet correspondence, J. Econ. Theory 44 (1988) 63–80.
[17] P. Fishburn, Condorcet social choice function, SIAM J. Appl. Math. 33 (1977) 295–306.
[18] J. Flesch, J. Kuipers, G. Schoenmakers, K. Vrieze, Subgame-perfection in stochastic games with perfect information and recursive payoffs, 2008, METEOR Research Memoranda 041.
[19] J. Greenberg, The Theory of Social Situations, Cambridge University Press, Cambridge, UK, 1990.
[20] A. Gomez, P. Jehiel, Dynamic process of social and economic interactions: on the persistence of inefficiencies, J. Polit. Economy 113 (2005) 626–667.
[21] S. Hart, M. Kurz, Endogenous formation of coalitions, Econometrica 51 (1983) 1047–1064.
[22] J.-J. Herings, A. Mauleon, V. Vannetelbosch, Rationalizability for social environments, Games Econ. Behav. 49 (1) (2004) 135–156.
[23] J. Harsanyi, An equilibrium point interpretation of stable sets and a proposed alternative definition, Manage. Sci. 20 (1974) 1427–1495.
[24] K. Hyndman, D. Ray, Coalition formation with binding agreements, Rev. Econ. Stud. 74 (2007) 1125–1147.
[25] M. Jackson, A. van den Nouweland, Strongly stable networks, Games Econ. Behav. 51 (2005) 420–444.
[26] M. Jackson, A. Wolinsky, A strategic model of social and economic networks, J. Econ. Theory 71 (1996) 44–74.
[27] H. Konishi, D. Ray, Coalition formation as a dynamic process, J. Econ. Theory 110 (2003) 1–41.
[28] J.-F. Laslier, Tournament Solutions and Majority Voting, Springer-Verlag, Heidelberg, New York, 1997.
[29] M. Mariotti, A model of agreements in strategic form games, J. Econ. Theory 74 (1997) 196–217.
[30] M. Mariotti, L. Xue, Farsightedness in coalition formation, in: Carlo Carraro (Ed.), The Endogenous Formation of Economic Coalitions, Edward Elgar, 2003, pp. 128–155.
[31] N. Miller, A new solution set to for tournaments and majority voting, Amer. J. Polit. Sci. 24 (1980) 68–96.
[32] M. Osborne, A. Rubinstein, A Course in Game Theory, MIT Press, Cambridge, MA, 1994.
[33] F. Page, S. Kamat, M. Wooders, Networks and farsighted stability, J. Econ. Theory 120 (2005) 257–269.
[34] S. Papai, Unique stability in simple coalition formation games, Games Econ. Behav. 48 (2004) 337–354.
[35] D. Ray, A Game Theoretic Perspective on Coalition Formation. The Lipsey Lectures, Oxford University Press, Oxford, UK, 2007.
[36] D. Ray, R. Vohra, Equilibrium binding agreements, J. Econ. Theory 73 (1997) 30–78.
[37] D. Ray, R. Vohra, A theory of endogenous coalition structures, Games Econ. Behav. 26 (1999) 286–336.
[38] R. Rosenthal, Cooperative games in effectiveness form, J. Econ. Theory 5 (1972) 88–101.
[39] P. Shenoy, On coalition formation: a game theoretic approach, Int. J. Game Theory 8 (1979) 133–164.
[40] L. Xue, Nonemptiness of the largest consistent set, J. Econ. Theory 73 (1997) 453–459.
[41] L. Xue, Coalitional stability under perfect foresight, Econ. Theory 11 (1998) 603–627.