Comparing Complete Information All-Pay Auctions*

Hannu Vartiainen†
Yrjö Jahnsson Foundation

January 18, 2007

Abstract

We study Nash equilibria of all-pay auctions when players' cost functions are potentially nonlinear. We show that with linear cost functions the revenue maximizing equilibrium of the first price all-pay auction is at least as profitable as that of the second price all-pay auction, and the winner-pay action lies in between. In an asymmetric case this order is strict. With symmetric quadratic cost functions, the order of the first price all-pay action and the second price all-pay action is reversed, and both dominate than the winner-pay auctions. Revenue differences grow as the number of bidders becomes large.

Keywords: First price and second price all-pay auctions, revenue comparisons.

JEL: D44, D72.

1 Introduction

In all-pay auctions, all bidders - not only the winner - pay their bids. An all-pay auction captures in a reduced form many relevant features of a contest, and is hence pertinent to a large class of scenarios, e.g. tournaments, rent-seeking, technological competition and R&D-races, lobbying, advertising, political campaining, education, job promotion, sports, or animal conflicts. General properties on all-pay auctions are, hence, of interest. Of importance are, e.g., who wins, what are the bids, or how much of the total value of the prize is dissipated. One is especially interested in comparing auctions.

We focus on the *first-price* and the *second-price* all-pay auctions (FPAA and SPAA, respectively) under complete information. In the former the winner pays his bid whereas in the latter he pays the second highest bid.[1] While

*I thank Semih Koray and a referee for useful comments. I am also grateful to Klaus Kultti for helpful suggestions.

†Address: The Yrjö Jahnsson Foundation, Ludviginkatu 3-5, 00130 Helsinki, Finland, Tel: +358-40-7206808, E-mail: hannu.vartiainen@helsinki.fi

[1] A dynamic version of SPAA is known as the *war of attrition.*

both all-pay mechanisms have been studied in the literature, not much is known of their comparative properties.[2] We take the position of the seller. Seller's revenue - or the expected bids - can be interpreted either literally, or as the aggregate loss of resources. In the former case, higher revenues are desirable but not necessarily in the latter case. Performance of auctions are evaluated under different hypotheses of bidders' private costs of bidding. Nonlinearity of cost functions means that there is a degree of asymmetry between the revenues of the seller and costs of the bidders. Such nonlinearity could stem e.g. from financial constraints, from opportunity costs of diverting resources away from productive activities, or, with some caveats, from risk aversion. Increasing marginal cost is the natural assumption in many scenarios.

What makes complete information scenario interesting is that the (interesting) Nash equilibria of all-pay auctions are in mixed strategies: Bidding high is profitable if all others bid low while bidding low is profitable if others bid high. Because of the randomization the final allocation of the good is not efficient.[3] Hence there is no *a priori* reason to expect the revenue equivalence á la Myerson (1981) to hold. We show that interesting comparisons between auctions can be made. Moreover, we show that the results are sensitive to the choice of cost functions.

There are $n \geq 2$ bidders, bidding for a single good. All bidders' reservation valuations for the good is 1 but their cost functions may differ.[4] We derive a general closed form expression of the seller's revenues in both all-pay auctions, and characterize the equilibrium strategies.

Sharpest results are obtained in the two-bidders case. We show that with *linear* cost functions (the completely mixed equilibrium of) SPAA is at least as profitable as FPAA, and that the (trembling hand perfect equilibria of) the standard winner-pay auctions lie in between.[5] In the asymmetric marginal costs case this ranking is strict. Thus the revenue equivalence of auctions breaks down.

However, with increasing marginal costs the revenue ordering may be reversed. We show that if cost functions are quadratic, the two-player FPAA revenue dominates SPAA, *and* both the all-pay auctions dominate the winner-pay auctions. This is related to Che and Gale (1998a) who show that a *cap*, i.e. an upper bound on feasible bids, increases revenues from FPAA. A cap can be interpreted as an upper envelope of two cost functions,

[2] See Baye *et al.* (1993, 1996), and references therein. Seminal contributions include Hillman and Riley (1989), Hendricks *et al.* (1988), and Moulin (1986).

[3] Krishna and Morgan (1997) analyze all-pay auction in the incomplete information scenario á la Milgrom and Weber (1982) which permits them to focus on pure strategies.

[4] Moldovanu and Sela (2001) study a similar set up and focus on the question of how to optimally bundle several goods in FPAA.

[5] The completely mixed equilibrium of SPAA is the unique subgame perfect equilibrium in the war of attrition -version of SPAA (see Hendrics *et al.,* 1988).

the original and a one that is (an approximation of) infinitely elastic at the point of the cap. Hence a cap imposes a degree of convexity on the cost functions.

In the n-player context, Baye *et al.* (1993, 1996) show that in the linear cost functions case, FPAA entertains may equilibria whose revenue properties differ. We use their result concerning the optimal equilibrium to show that revenue rankings of the auctions remain unchanged in the $n > 2$ situation.

However, we also show that the Baye *et al.* optimal equilibrium is sensitive to the shape of the cost function. Specifically, under quadratic cost functions, increasing the number of active bidders increases the seller's revenue (which does not hold under linear cost functions). When n approaches infinity, the size of the revenue is doubled. Also the revenue of SPAA under quadratic cost functions increases when n becomes large. While the revenue ranking of auctions remains the same as in the two-player case, the difference between the all-pay and winner-pay auctions increases.

The fact under convex cost functions the expected bids are higher in both FPAA and SPAA than in winner-pay auctions means that the expected aggregare value of bids is higher than the value of the good to the bidders. This can be interpreted as *over-dissipation* of rents (Tullock, 1980). We show that there is no upper bound on the amount of rents that may, under some circumstances, be over-dissipated.

The paper is organized as follows: Section 2 introduces the set up. Section 3 specifies seller's revenues as a function of bidders' strategies. In Section 4, the two-player case is analyzed and in Section 5 the limit case, when n approaches infinity. Section 6 concludes. The appendix gives more detailed characterizations of equilibria.

2 The Set Up

There is an indivisible object to be allocated to players $N = \{1, ..., n\}$, the "bidders". Bidder i's payoff depends on the allocation of the prize and his costly transfer, which are determined by all bidders' actions, their "bids". Bidder i's action space is $\mathbb{R}_+$ with a typical element b_i. Define allocation rule $x = (x_1, ..., x_n) : \mathbb{R}_+^n \to \{0, 1\}^n$ such that $\sum_{i=1}^n x_i(b) = 1$, for all $b = (b_1, ..., b_n) \in \mathbb{R}_+^n$. Given b, the prize is devoted to bidder i if $x_i(b) = 1$. Function $t = (t_1, ..., t_n) : \mathbb{R}_+^n \to \mathbb{R}_+^n$ specifies a transfer from each bidder contingent on a joint action b. Pair (x, t) is an *auction.*

We focus on auctions that allocate the prize to the highest bidder. Let

$$M(b) := \left\{ \arg\max_i b_i \right\}.$$

Then

$$x_i(b) = \frac{1}{\#M(b)}, \quad \text{if } i \in M(b),$$
$$x_i(b) = 0, \quad \text{if } i \notin M(b).$$

All-pay auctions differ in how transfers are determined. Denote by $b^{(2)}$ the second order statistics of sample $b_1, ...b_n$.

FPAA *First price all-pay auction*:

$$t_i^{FPAA}(b) = b_i, \quad \text{for all } i \in N.$$

SPAA *Second price all-pay auction*:[6]

$$t_i^{SPAA}(b) = \begin{cases} b^{(2)}, & \text{if } i \in M(b), \\ b_i, & \text{if } i \notin M(b). \end{cases}$$

The corresponding first and second-price *winner-pay auctions* are analogously defined with the difference that

$$t_i^{WA}(b) = 0, \quad \text{if } i \notin M(b).$$

Function $c_i : [0,1] \to \mathbb{R}_+$ describes the cost of transfer $t_i \geq 0$ to bidder i. We assume that $c_i(\cdot)$ is strictly increasing, differentiable, and unbounded, and satisfies $c_i(0) = 0$. We also assume that

$$c_1(b) \leq ... \leq c_n(b), \quad \text{for all } b \geq 0.$$

Given a payment vector $t = (t_1, ..., t_n)$, bidder i's payoff, given his bid b, is

$$u_i(b) = x_i(b) - c_i(t_i).$$

Hence, bidder's payoff from wealth is separable from the consumption of the prize. Possible nonlinearity of $c_i(\cdot)$ can be interpreted risk sensitivity. If c_i is convex (linear, concave) then i can be interpreted to be risk averse (neutral, loving, resp.) with respect to his wealth.

Denote by $\bar{b}_i$ the break-even bid of bidder i, i.e.

$$1 = c_i(\bar{b}_i), \quad \text{for all } i.$$

Denote by $F_1, ..., F_n$ a collection of independent cumulative distribution functions on $\mathbb{R}_+^n$, interpreted as bidders' *strategies*. Let suppF_i be the support of F_i.[7] If supp$F_i = \mathbb{R}_+$, then F_i is *completely mixed.* With bid b_i and

[6] The second price all-pay auction is known also as *the war of attrition.*

[7] The smallest closed set S_i such that $\Sigma_i(b) - \Sigma_i(b + \varepsilon) > 0$, for all $\varepsilon > 0$, for all $b \in S_i$.

the other bidders' strategies $F_{-i} = (F_j)_{j \neq i}$, bidder i's expected payoff is

$$\begin{aligned} \mathbb{E}u_i(b_i, F_{-i}) &= \int_{\mathbb{R}_+^{n-1}} [x_i(b) - c_i(t_i(b))] \, dF_{-i}(b_{-i}) \\ &= \textstyle\prod_{j \neq i} F_j(b_i) - \displaystyle\int_{\mathbb{R}_+^{n-1}} c_i(t_i(b)) dF_{-i}(b_{-i}). \end{aligned}$$

Strategy $F = (F_1, ..., F_n)$ constitutes a *Nash equilibrium* (NE) if

$$\mathbb{E}u_i(F) \geq \mathbb{E}u_i(b_i, F_{-i}), \quad \text{for all } b_i \geq 0, \text{ for all } i = 1, ..., n.$$

With bids $b = (b_1, ..., b_n)$, seller's revenue is

$$v(b) = \textstyle\sum_{i=1}^n t_i(b).$$

Since strategies $F = (F_i)_{i=1}^n$ are independent, the seller's expected revenues from a mechanism characterized by the transfer rule t is

$$\mathbb{E}v(F) = \textstyle\sum_{i=1}^n \displaystyle\int_{\mathbb{R}_+} t_i(b) dF_i(b).$$

Denote the expected payoff from FPAA, SPAA, or winner-pay auctions by $\mathbb{E}v^{FPAA}(F)$, $\mathbb{E}v^{SPAA}(F)$, and $\mathbb{E}v^{WA}(F)$, respectively.[8] If the strategy F is known from the context, it may be dropped.

Note that under full information, the first and the second-price winner-pay auctions are easy to solve. In the natural Nash equilibrium (trembling hand in the first-price auction and undominated in the second-price) the good is sold to the bidder with the highest willingness to pay, i.e. to the bidder with the lowest marginal cost.[9] The equilibrium price - and hence the revenue of the seller - is equal to the break-even price of the bidder with the second lowest marginal cost, $\mathbb{E}v^{WA} = \bar{b}_2$.

3 Revenues

In this section we derive a reduced form expression of the seller's revenues under FPAA and SPAA. Denote by $\mathbb{E}v^{FPAA}(F)$ and $\mathbb{E}v^{SPAA}(F)$ the expected revenues of the seller under the two auctions when the two bidders obey strategy F.

[8] Since the analysis of the two winner-pay auctions is trivial in the present complete information framework, and the relevant equilibria of them generate the same revenue to the seller, there is no need to reserve distiguished notation for both of them.

[9] Assuming that in the first price auction ties are broken in favour of the player with the lowest marginal cost.

Lemma 1 *Given the FPAA strategies* $F = (F_i)_{i=1}^n$*, the expected revenue from FPAA is*

$$\mathbb{E}v^{FPAA}(F) = \textstyle\sum_{i=1}^n \displaystyle\int_0^\infty (1 - F_i(b))\, db. \tag{1}$$

Proof. The expected transfer from i is now obtained by integrating by parts (note that bid $b = 0$ results in 0 payment):

$$\begin{aligned}\mathbb{E}t_i^{FPAA} &= \int_0^\infty bdF_i(b) + 0 \cdot F_i(0) \\ &= \int_0^\infty (F_i(\infty) - F_i(b))db \\ &= \int_0^\infty (1 - F_i(b))db.\end{aligned}$$

Since the bids are independent,

$$\begin{aligned}\mathbb{E}v^{FPAA}(F) &= \textstyle\sum_{i=1}^n \mathbb{E}t_i^{FPAA} \\ &= \textstyle\sum_{i=1}^n \displaystyle\int_0^\infty (1 - F_i(b))\, db.\end{aligned}$$

■

An implication of Lemma 1 is that if strategy F first order stochastically dominates F', i.e. $F_i \leq F_i'$ for all i, then $\mathbb{E}v^{FPAA}(F) \geq \mathbb{E}v^{FPAA}(F')$.

Lemma 2 *Given the SPAA strategies* $F = (F_i)_{i=1}^n$*, the expected revenue from SPAA is*

$$\mathbb{E}v^{SPAA}(F) = \textstyle\sum_{i=1}^n \displaystyle\int_0^\infty (1 - \textstyle\prod_{j \neq i} F_j(b))\,(1 - F_i(b))\, db. \tag{2}$$

Proof. Let, for any i,

$$G_i(b) = \textstyle\prod_{j \neq i} F_j(b), \text{ for all } b.$$

The expected transfer of bidder i who bids a is

$$\mathbb{E}t_i^{SPAA}(a) = \int_0^a bdG_i(b) + a(1 - G_i(a)).$$

Integrating the first term by parts,

$$\begin{aligned}\mathbb{E}t_i^{SPAA}(a) &= aG_i(a) - \int_0^a G_i(b)db + a(1 - G_i(a)) \\ &= \int_0^a (1 - G_i(b))db.\end{aligned}$$

The expected transfer of bidder i is then

$$\begin{aligned}\mathbb{E}t_i^{SPAA} &= \int_0^\infty \mathbb{E}t_i^{SPAA}(a)dF_i(a)\\ &= \int_0^\infty \int_0^a (1-G_i(b))dbdF_i(a).\end{aligned}$$

This yields, by integrating by parts,

$$\begin{aligned}\mathbb{E}t_i^{SPAA} &= \int_0^\infty (1-G_i(b))dbF_i(\infty) - \int_0^\infty (1-G_i(b))F_i(b)db\\ &= \int_0^\infty (1-G_i(b))(1-F_i(b))db.\end{aligned}$$

Since the bids are independent,

$$\begin{aligned}\mathbb{E}v^{SPAA} &= \textstyle\sum_{i=1}^n \mathbb{E}t_i^{SPAA}\\ &= \textstyle\sum_{i=1}^n \displaystyle\int_0^\infty (1-G_i(b))\,(1-F_i(b))\,db.\end{aligned}$$

■

4 Two Bidders

This section shows that the revenue ranking of FPAA, SPAA, and winner-pay auctions depends on the shape of the cost functions. We assume two bidders. First we characterize the Nash equilibrium strategies under the two all-pay auctions. If the opponent uses strategy F_j in FPAA (in SPAA), then the probability of i, $i \neq j$, winning with bid b in FPAA (in SPAA) is $F_j(b)$.

Proposition 1 *Let $n = 2$.*

1. *In the Nash equilibrium of FPAA,*

$$\begin{aligned}F_1(b) &= c_2(b) + 1 - c_1(\bar{b}_2), \quad \textit{for all } b \in [0, \bar{b}_2],\\ F_2(b) &= c_1(b), \quad \textit{for all } b \in [0, \bar{b}_2].\end{aligned}$$

2. *In the completely mixed Nash equilibrium of SPAA,*

$$\begin{aligned}F_1(b) &= 1 - e^{-c_2(b)}, \ \textit{for all } b \geq 0,\\ F_2(b) &= 1 - e^{-c_1(b)}, \ \textit{for all } b \geq 0.\end{aligned}$$

Analogous characterizations of the equilibria of FPAA can be found in Baye *at al.* (1993, 1996), in Hillman and Riley (1989), Che and Gale (1998a), or Kaplan and Wettstein (2006), and of the equilibria of SPAA in Hendricks *et al.* (1988), or in Moulin (1988). Below we sketch the proofs. For more comprehensive characterizations of the equilibria the reader is referred to Propositions 8 and 9 in the appendix.

FPAA: Equilibrium is in mixed strategies. Bidder 1 has payoff at least $1 - c_1(\bar{b}_2)$ since any bid above $\bar{b}_2$ makes him win with certainty. Bidders cannot have mass points at the same point since infinitesimal deviation upwards would be profitable. Thus the the lowest bid generates a zero profit. Thus bidder 2 has zero profit. If the highest bid of 1 is below $\bar{b}_2$, then 2 could guarantee postive profit. Thus 1's profit cannot exceed $1 - c_1(\bar{b}_2)$. If i has a gap B the support of his strategy, then j needs to have the same gap since otherwise j would benefit from deviation downwards. Both cannot have the same gap since then both would have an incentive to deviate at the upper boundary of the gap. Since 1's highest bid is $\bar{b}_2$ and 2's lowest bid is 0, both strategies have support $[0, \bar{b}_2]$. Thus bidders choose strategies (F_1, F_2) such that, for all $b \in [0, \bar{b}_2]$ it holds that

$$\begin{aligned} F_2(b) - c_1(b) &= 1 - c_1(\bar{b}_2), \\ F_1(b) - c_2(b) &= 0. \end{aligned}$$

SPAA: Since the strategy (F_1, F_2) is completely mixed, it must be atomless. Otherwise there is i who chooses b with positive probability. But then there is small enough $\varepsilon > 0$ such that $j \neq i$ strictly prefers $b + \varepsilon$ rather than b, contradicting the assumption that j's strategy is completely mixed. Since bidding 0 must be best response for both i, it must be that both players expected payoff is 0. Thus,

$$\int_0^b (1 - c_i(b'))dF_j(b') - c_i(b)(1 - F_j(b)) = 0, \quad \text{for all } b \geq 0, \text{ for } i = 1, 2.$$

Since this holds as an identity,

$$F_j'(b) - c_i'(b)(1 - F_j(b)) = 0.$$

Thus,

$$c_i(b) = \int_0^b \frac{F_j'(b')}{1 - F_j(b')} db' = -\ln(1 - F_j(b)),$$

or, for $i \neq j$, for all $b \geq 0$,

$$F_j(b) = 1 - e^{-c_i(b)}.$$

Hendricks *at al.* (1988) point out that SPAA always hosts an asymmetric *pure strategy* equilibrium where bidder 1 bids $b \geq \bar{b}_2$ and all other bidders bid 0. However, they show that such Nash equilibrium is never *subgame perfect* in the dynamic version of the game, known as the *war of attrition*, where bidders continue to raise their bids until only one bidder is left. The war of attrition -interpretation is natural in many economic settings, and hence we concentrate on the completely mixed Nash equilibria of SPAA.

It is now straightforward to combine Lemmata 1 and 2 with the equilibrium strategies in Proposition 1.[10]

Proposition 2 *If $n = 2$, then the expected revenues of the seller in the Nash equilibrium of FPAA and in the completely mixed Nash equilibrium of SPAA are, respectively,*

$$\begin{aligned} \mathbb{E}v^{FPAA} &= \left(1 + c_1(\bar{b}_2)\right)\bar{b}_2 - \int_0^{\bar{b}_2} \left(c_1(b) + c_2(b)\right) db, \\ \mathbb{E}v^{SPAA} &= 2\int_0^{\infty} e^{-c_1(b)-c_2(b)} db. \end{aligned}$$

Thus the seller's payoff under SPAA depends only of the *average* cost function whereas FPAA needs information of both the cost functions (in particular $\bar{b}_2$ and $c_1(\bar{b}_2)$).

Terminological and notational convention: From this on we refer the Nash equilibrium of FPAA, the completely mixed Nash equilibrium of SPAA, and the trembling hand perfect or the undominated Nash equilibrium of either of the winner-pay auctions simply as the Nash equilibrium of the game at hand. The Nash equilibrium revenue of he seller in either case is denoted, respectively, by $\mathbb{E}v^{FPAA}$, $\mathbb{E}v^{SPAA}$, or $\mathbb{E}v^{WA}$.

A Note on Caps It is illustrative to study geometrically the consequences of a *cap* on bids. Che and Gale (1998a, 2006) found out that a cap has surprising effect on the all-pay auction. To interpret a cap in our set up, approximate it with a *non-rigid* cap that makes it very costly (in a continuous way) to bid above certain level, say m.[11] The cost function implied by such non-rigid cap is the upper envelope of the the cost function and the cap (a near-vertical line segment at m).[12]

The revenue from FPAA is the sum of the areas between $c_i(\bar{b}_2)$ and the $c_i(\cdot)-$functions from 0 to $\bar{b}_2$ (recall that $c_2(\bar{b}_2) = 1$). The striped area in Fig. 1 is the revenue from 1, and the shaded area the revenue from 2. Player 1's payoff is $c_2(\bar{b}_2)$ $(= 1)$ minus $c_1(\bar{b}_2)$.

[FIGURE 1]

[10] Note that since $\Sigma_1(\beta_2) = \Sigma_1(\beta_2) = 1$ in FPAA, the upper bound the integrals in (1) is in fact β_2.

[11] Thus with non-rigid cap the cost functions are still increasing. For discussion of the rigidity of caps, see Kaplan and Wettstein (2006).

[12] For example, a cost function implied by a non-rigin cap m could be $c(b)$ for all $b \leq m$, and $c(b) + (b - m)/\varepsilon$, for all $b > m$, for small $\varepsilon > 0$.

There are two effects at play when a cap is imposed, displayed geometrically in Fig. 2. (i) A cap removes the bids from the interval $(m, \bar{b}_2]$ which decreases revenues from both bidders (narrows the striped and shaded areas). (ii) A cap extracts all surplus from 1 by shifting his maximal bid from $c_1(\bar{b}_2)$ to 1, and thus increases revenue from 1 (heightens the striped area). Since *any* cap below $\bar{b}_2$ has these effects, there is an area below $\bar{b}_2$ where the latter effect dominates, and where the seller's revenue is increased. By inspection, almost all surplus of bidder 1 can be extracted with a non-rigid cap close to $\bar{b}_2$.[13]

[FIGURE 2]

The revenue from SPAA is the area below the function $2e^{-c_1(b)-c_2(b)}$. This is reflected by the shaded area in Fig 3.

[FIGURE 3]

Imposing a cap m means that the costs at m become very large and hence the shaded area is cut close to zero at m (see Fig. 4, where $\varepsilon > 0$ is a small number). Thus also the revenues are cut.

[FIGURE 4]

[13]This is, however, not the equilibrium constructed by Che and Gale (1998). Since in their model, a cap is rigid, both bidders choose m with postive probability, and 1 earns positive profit.

Thus we conclude that caps may increase the revenues from FPAA but not from SPAA. It is interesting to compare this ranking to Che and Gale (1998b). They show that in the context of standard winner-pay auctions, caps are more favorable to the first-price auction than to the second-price auction. The intuition is that in the second-price auction the bidders bid more aggressively and are hence more constrained by a cap.

The analogue of this argument to our setting is that a cap above $\bar{b}_2$ does not affect FPAA but *any* cap affects negatively to the revenue from SPAA since in the absence of a cap the second-price structure induces the bidders to bid aggressively above their own valuation. In addition, a cap in FPAA may actually level the play field (in the asymmetric case), inducing more aggressive bidding and ultimately higher revenues.

4.1 Linear Cost Functions

In this subsection, we assume linear cost functions, i.e. $c_i(b) = \xi_i b$ for all $b \geq 0$, for some $\xi_i > 0$, for $i = 1, 2$. Then $\bar{b}_2 = 1/\xi_2$ and $c_1(\bar{b}_2) = \xi_1/\xi_2$. We argue that in an asymmetric case $\xi_1 < \xi_2$ the revenue equivalence of different auction forms no longer holds.

Proposition 3 *Let $n = 2$. Under linear cost functions $\mathbb{E}v^{SPAA} \geq \mathbb{E}v^{FPAA}$, with strict inequality when the bidders are asymmetric.*

Proof. Recall that $\bar{b}_2 = 1/\xi_2$. By Proposition 1, the expected payoff from the unique NE of FPAA is

$$\begin{aligned}
\mathbb{E}v^{FPAA} &= \bar{b}_2 \left(1 + c_1(\bar{b}_2)\right) - \int_0^{\bar{b}_2} (c_1(b) + c_2(b))db \\
&= \frac{1}{\xi_2}\left(1 + \frac{\xi_1}{\xi_2}\right) - \int_0^{\frac{1}{\xi_2}} (\xi_1 + \xi_2) b db \\
&= \frac{\xi_2 + \xi_1}{(\xi_2)^2} - \frac{\xi_1 + \xi_2}{2(\xi_2)^2} \\
&= \frac{\xi_1 + \xi_2}{2(\xi_2)^2}. \qquad (3)
\end{aligned}$$

By Propositions 9 and 2, there is a completely mixed NE of SPAA, whose expected revenue is

$$\begin{aligned}
\mathbb{E}v^{SPAA} &= 2\int_0^{\infty} e^{-(\xi_1+\xi_2)b} db \\
&= \frac{2}{\xi_1 + \xi_2}. \qquad (4)
\end{aligned}$$

Denote the average marginal cost by $\xi^A = (\xi_1 + \xi_2)/2$. Then $\xi_2 \geq \xi^A \geq \xi_1$. Now

$$\mathbb{E}v^{SPAA} = \frac{1}{\xi^A} \geq \frac{1}{\xi_2} = \bar{b}_2 \geq \bar{b}_2 \frac{\xi^A}{\xi_2} = \mathbb{E}v^{FPAA}, \qquad (5)$$

with strict inequalities when $\xi_2 > \xi_1$. ■

Thus, with linear cost function SPAA is at least as profitable to the seller as FPAA, and strictly more profitable if the marginal costs are not equal. The reason for this is that FPAA necessarily permits bidder 1 to gain surplus of value $(\xi_2 - \xi_1)/\xi_2$ whereas SPAA extracts all the surplus from all bidders.

From (3) and (4) it is easy to deduce that an increase in ξ_1 contributes *positively* to the revenue of FPAA but *negatively* to that of SPAA. Thus a decrease in ξ_1 increases the revenue gap of the two auctions. When $\xi_1 = 0$, the expected revenue from SPAA is $2/\xi_2$, and from FPAA $1/(2\xi_2)$, implying a maximum revenue gap $3/(2\xi_2)$.

Conversely, in the symmetric case, $\xi_1 = \xi_2 = \xi^A = 1/\bar{b}_2$. Since the seller extracts all the surplus it follows that (see (5)) the revenues from both all-pay auctions are equal to $\bar{b}_2$. Since the revenue to the seller from the winner-pay auctions is also $\bar{b}_2$, the general ranking of FPAA, SPAA, and the winner-pay auctions can be stated as follows:

Corollary 1 *Let $n = 2$. Under linear cost functions,*

$$\mathbb{E}v^{SPAA} \geq \mathbb{E}v^{WA} \geq \mathbb{E}v^{FPAA},$$

with strict inequalities when the bidders are asymmetric.

FPAA generates a lower revenue than the winner-pay auctions because of randomization. When the marginal costs differ, randomization entails inefficiencies. Hence the extractable payoffs are lower.

4.2 Quadratic Cost Function

We now demonstrate that the ranking of auctions in the previous section is sensitive to the choice of the cost functions. We argue that under convex cost functions the ranking is reversed. To allow closed form comparisons, we assume quadratic cost functions, i.e. $c_1(b) = c_2(b) = b^2$. Then $c_1(\bar{b}_2) = \bar{b}_2 = 1$.

By Corollary 2, the expected revenues of the seller from FPAA and SPAA are, respectively

$$\mathbb{E}v^{FPAA} = 2\left(1 - \int_0^1 b^2 db\right) = \frac{4}{3} \approx 1.33, \tag{6}$$

$$\mathbb{E}v^{SPAA} = 2\int_0^\infty e^{-2b^2} db = \sqrt{\frac{\pi}{2}} \approx 1.25. \tag{7}$$

Proposition 4 *Let $n = 2$. Under quadratic, symmetric cost functions,*

$$\mathbb{E}v^{FPAA} > \mathbb{E}v^{SPAA}.$$

Thus the revenue ordering of SPAA and FPAA is changed when compared to the linear case. However, it can be shown that with *asymmetric* quadratic cost functions the ordering of Proposition 4 may be reversed. This suggests that convexity increases the appeal of FPAA relative to SPAA whereas asymmetry of marginal costs does the converse.

To understand the source of Proposition 4, it is useful to compare this result to the linear symmetric case $c(b) = b$. As discussed in the last subsection, the expected revenue from both auctions in such case is 1. Under FPAA a bidder randomizes only in the interval $[0, 1]$, i.e. if and only if $b^2 \leq b$. However, under SPAA a bidder randomizes also in $(1, \infty)$. Thus FPAA allows a bidder to fully economize the lower costs whereas SPAA forces him to also bid in the high-cost area. Since the expected revenue from a winner-pay auction is $\bar{b}_2 = 1$, a ranking of FPAA, SPAA, and the winner-pay auctions is followed:

Corollary 2 *Let $n = 2$. Under quadratic, symmetric cost functions,*

$$\mathbb{E}v^{FPAA} > \mathbb{E}v^{SPAA} > \mathbb{E}v^{WA}.$$

Closed form solutions of $\mathbb{E}v^{SPAA}$ for cost functions that are exponential beyond the quadratic case are not available. However, numeric simulations suggest that the revenue ordering remains unchanged under more high powered cost functions. The ordering is also consistent with Che and Gale (1998b), who report that all-pay auctions revenue dominate standard winner-pay auctions under financial constraints.

This result demonstrates that when cost functions are convex the sum of the expected bids in both all-pay auctions may be higher than the expected revenue. This phenomenon of *over-dissipation of rents,* famously anticipated by Tullock (1980), is absent in the context of FPAA in the much studied case of linear cost functions.[14]

5 Large Populations

5.1 Linear Cost Functions

Now we generalize the results of the linear two-player case to $n \geq 2$ case. Hillman and Riley (1989) and Baye *et al.* (1996) show that in any Nash equilibrium of FPAA under linear cost functions and $n \geq 2$ bidders, bidder 1 extracts payoff $(\xi_2 - \xi_1)/\xi_2$ and all other bidders get zero.[15] Baye *et al.* (1996) show that there may be many equilibria with different revenue properties. However, the revenue maximizing equilibrium is familiar already from the $n = 2$ case.

[14] See e.g. Baye *et al.* (1994).

[15] Baye *et al.* (1996) assume identical (linear) cost functions but allow different valuations.

Proposition 5 (Baye, Kovenock, and de Vries, 1996) *Let $n \geq 2$. Suppose $c_i(b) = \xi_i b$ for all $b > 0$ for all i. In the most profitable NE of FPAA, only 1 and 2 are active, and strategies are $F_1(b) = \xi_2 b$, $F_2(b) = \xi_1 b + 1 - \xi_1/\xi_2$, and $F_i(b) = 1$ for all $i = 2, ..., n$.*

Thus in the most profitable equilibrium of FPAA, bidders 1 and 2 mix on $[0, \bar{b}_2]$, as they do in the $n = 2$ case, and all the others bid 0. However, as the next section shows, this result does not hold outside the linear case.

In the context of SPAA and linear cost functions, it is easy to see that the equilibrium in Proposition 1, where only 1 and 2 are active, is valid under any $n \geq 2$. This is proven in the appendix in Proposition 9. Here we sketch the proof. Recall that all bidders $k = 3, ..., n$ face a higher marginal cost. Thus if bidding against 1's strategy generates 2 a zero profit, as it does in the equilibrium constructed in Proposition 1, then no k's bid against the same 1's strategy can generate k a strictly positive payoff. Hence inactivity (bidding 0) is an optimal strategy for k.

We conclude that (i) since the $n = 2$ equilibrium strategies where only 1 and 2 are active forms an equilibrium in SPAA under any $n \geq 3$, (ii) since the $n = 2$ equilibrium strategies where only 1 and 2 are active is the revenue maximizing equilibrium in FPAA under any $n \geq 3$, and (iii) since in the $n = 2$ case the SPAA generates a higher revenue than FPAA (Proposition 3), the SPAA generates a higher revenue than FPAA under any $n \geq 2$.

Since adding players does not affect the performance of the winner-pay auctions, it follows that the revenue ordering of auctions remains unchanged under all $n \geq 2$.

Corollary 3 *Let $n \geq 2$. Under linear cost functions,*

$$\mathbb{E}v^{SPAA} \geq \mathbb{E}v^{WA} \geq \mathbb{E}v^{FPAA},$$

with strict inequalities when bidders 1 and 2 are asymmetric.

5.2 Symmetric Bidders

FPAA Assume $c_1(\cdot) = ... = c_n(\cdot) = c(b)$ and normalize $\bar{b}_2 = 1$. We construct the symmetric Nash equilibrium by replacing F_j with $\prod_{j \neq i} F_j(b)$ and assuming $F_1(b) = ... = F_n(b)$ in the FPAA part of the proof of Proposition 1. We have

$$F_i(b) = c(b)^{\frac{1}{n-1}}, \quad \text{for all } i, \text{ for all } b \in [0, \bar{b}]. \tag{8}$$

Denote the expected revenue of the seller of FPAA under n symmetric bidders by $\mathbb{E}v_n^{FPAA}$. Incorporating (8) into (1),

$$\mathbb{E}v_n^{FPAA} = n \left(1 - \int_0^1 c(b)^{\frac{1}{n-1}} db \right). \tag{9}$$

From (9) it not clear how an increase in n affects the revenue. We now identify a condition under which the revenue is bounded.

Proposition 6 *Under symmetric bidders,*

$$\mathbb{E}v_n^{FPAA} \to -\int_0^1 \ln[c(b)]\,db, \ \ \textit{as } n \to \infty. \tag{10}$$

Proof. By the l'Hospital's rule,

$$\lim_{n\to\infty} n\left(1 - c(b)^{\frac{1}{n-1}}\right) = -\ln[c(b)], \text{ for all } b \geq 0.$$

By taking the pointwise limit,

$$\begin{aligned}\lim_{n\to\infty} \mathbb{E}v_n^{FPAA} &= \int_0^1 \lim_{n\to\infty} n\left(1 - c(b)^{\frac{1}{n-1}}\right) db \\ &= -\int_0^1 \ln[c(b)]\,db.\end{aligned}$$

■

An immediate observation from (10) is that there are cost functions under which the revenue increases without a bound (try $c(b) = e^{(b^2-1)/b}$). That is, it is possible that the amount rents that are over-dissipated becomes arbitrary large as n increases. In the particular case of quadratic cost function, the limit revenue has a simple form:

$$-\int_0^1 \ln\left[b^2\right] db = 2. \tag{11}$$

SPAA Replacing F_j with $\prod_{j\neq i} F_j(b)$ and assuming $F_1(b) = ... = F_n(b)$ in the proof of SPAA part of Proposition 1 it follows that the symmetric completely mixed SPAA Nash equilibrium satisfies

$$F_i(b) = (1 - e^{-c(b)})^{\frac{1}{n-1}}, \quad \text{for all } i, \text{ for all } b \geq 0.$$

Plugging this into (2), the expected revenue of SPAA when n bidders use the symmetric completely mixed NE strategy is

$$\begin{aligned}\mathbb{E}v_n^{SPAA} &= n\int_0^\infty \left(1 - F(b)^{n-1}\right)(1 - F(b))\,db \\ &= n\int_0^\infty \left(1 - \left(1 - e^{-c(b)}\right)^{\frac{n-1}{n-1}}\right)\left(1 - \left(1 - e^{-c(b)}\right)^{\frac{1}{n-1}}\right) db \\ &= n\int_0^\infty e^{-c(b)}\left(1 - (1 - e^{-c(b)})^{\frac{1}{n-1}}\right) db.\end{aligned}$$

Unfortunately, with only very few parametrizations does this expression have a closed form solution. Little more can be said of the limiting case.

Proposition 7 *Under symmetric bidders,*

$$\mathbb{E}v_n^{SPAA}(F) \to -\int_0^\infty e^{-c(b)} \ln\left[1 - e^{-c(b)}\right] db, \quad \text{as } n \to \infty.$$

Proof. By the l'Hospital's rule,

$$\lim_{n\to\infty} n\left(1 - (1 - e^{-c(b)})^{\frac{1}{n-1}}\right) = -\ln[1 - e^{-c(b)}], \quad \text{for all} b > 0.$$

By taking the pointwise limit,

$$\begin{aligned}\lim_{n\to\infty} \mathbb{E}v_n^{SPAA} &= \int_0^\infty e^{-c(b)} \lim_{n\to\infty} n\left(1 - (1 - e^{-c(b)})^{\frac{1}{n-1}}\right) db \\ &= -\int_0^\infty e^{-c(b)} \ln[1 - e^{-c(b)}] db.\end{aligned}$$

■

We can now evaluate the limit revenue under quadratic cost function. We have, by expanding the logarithm,

$$\begin{aligned}-\int_0^\infty e^{-b^2} \ln[1 - e^{-b^2}] db &= -\int_0^\infty e^{-b^2}\left(-e^{-b^2} - \frac{-e^{-2b^2}}{2} - \frac{-e^{-3b^2}}{3} - ...\right) db \\ &= \int_0^\infty e^{-2b^2} db + \frac{1}{2}\int_0^\infty e^{-3b^2} db + \frac{1}{3}\int_0^\infty e^{-4b^2} db + ... \\ &= \frac{1}{2}\sqrt{\frac{\pi}{2}} + \frac{1}{2\cdot 2}\sqrt{\frac{\pi}{3}} + \frac{1}{2\cdot 3}\sqrt{\frac{\pi}{4}} + ... \\ &= \frac{\sqrt{\pi}}{2}\textstyle\sum_{k=1}^\infty \frac{1}{k\sqrt{k+1}} \\ &\approx 1.94. \end{aligned} \tag{12}$$

By (11), the revenue related to SPAA approaches that of FPAA when n becomes high. Since the winner-pay auctions generate payoff 1 under all $n \geq 2$, we conclude from (11) and (12) that the limit ordering of auctions under quadratic cost functions is the same as under two bidders. However, the revenue difference between the all-pay auctions and the winner-pay auctions has increased.

Corollary 4 *Under quadratic cost functions,*

$$\lim_{n\to\infty} \mathbb{E}v_n^{FPAA} > \lim_{n\to\infty} \mathbb{E}v_n^{SPAA} > \lim_{n\to\infty} \mathbb{E}v_n^{WA}.$$

6 Closing remarks

This paper has investigated equilibria in complete information all-pay auctions when the cost functions of the bidders may be non-linear. A closed form expression of the seller's revenues from the first-price and second-price all-pay auctions are derived, and comparisons are made from the viewpoint the expected revenue of the seller, i.e. the expected bids. Our analysis suggests that convexity of cost functions increases the revenues related to the first-price all-auction relative to those of the second-price all-pay auction whereas asymmetries between the bidders' cost functions does the converse. Moreover, convexity of cost functions increases the expected bids of both the all-pay auctions above those of the corresponding winner-pay auctions.

Increase in bidders increases the expected bids in all-pay auctions in the convex cost functions case and hence increase the revenue difference between all-pay and winner-pay auctions. However, the internal revenue ordering of the all-pay auctions does not seem to be sensitive to the number of bidders.

A Appendix

For the next result, assume

$$c_i(t) = \xi_i y(t), \text{ for all } t \in \mathbb{R}_+, \text{ and for all } i = 1, ..., n, \tag{13}$$

where $\xi_1, ..., \xi_n$ are positive scalars with $\xi_1 \le \xi_2 \le ... \le \xi_n$, and $y(\cdot)$ is nondecreasing, differentiable, and unbounded, and satisfies $y(0) = 0$. Let

$$\begin{aligned} \tilde{c}_1(b) &= c_1(b) + \frac{\xi_2 - \xi_1}{\xi_2}, \\ \tilde{c}_i(b) &= c_i(b), \quad \text{for } i = 2, ... \,. \end{aligned}$$

If $\xi_1 = \xi_2$, then $(\tilde{c}_1(\cdot), ..., \tilde{c}_n(\cdot)) = (c_1(\cdot), ..., c_n(\cdot))$. Let m be the largest integer such that $\xi_m \le \xi_2$.

Proposition 8 *Assume (13). Strategy $(F_i)_{i=1}^n$ constitutes a NE of FPAA if and only if there is a permutation of agents $\{2, ..., m\}$ and numbers $0 = \lambda_1 = \lambda_2 \le \lambda_3 \le ... \le \lambda_m \le \lambda_{m+1} = ... = \lambda_n \le \beta_2$ such that, for all $k = 2, ..., m$, for all $b \in (\lambda_k, \lambda_{k+1}]$,*

$$F_1(b) = \frac{\tilde{c}_2(b)}{\tilde{c}_1(b)} \left(\frac{\tilde{c}_1(b)}{\prod_{j=k+1}^n \alpha_j(0)} \right)^{\frac{1}{k-1}} \tag{14}$$

$$F_i(b) = \left(\frac{\tilde{c}_1(b)}{\prod_{j=k+1}^n \alpha_j(0)} \right)^{\frac{1}{k-1}}, \qquad \textit{for all } i = 2, ..., k, \tag{15}$$

$$F_i(b) = \alpha_i(0), \qquad \textit{for all } i = k+1, ..., n,$$

where the size of i's atom $\alpha_i(0)$ at 0, for $i = 2, ..., n$, is defined recursively by

$$\alpha_n(0) = \tilde{c}_1(\lambda_n)^{\frac{1}{n-1}}, \tag{16}$$

$$\alpha_i(0) = \left(\frac{\tilde{c}_1(\lambda_i)}{\prod_{j=i+1}^{n} \alpha_j(0)}\right)^{\frac{1}{i-1}}, \qquad \textit{for } i = 2, ..., n-1.$$

Proof of Proposition 8. *Necessity*: Let $(F_i)_{i=1}^n$ constitute a NE, and let $(u_i^*)_{i=1}^n$ be the corresponding payoff. Denote $c_i^*(b) = c_i(b) + u_i^*$, for all b and i. First, bidding more than β_2 is dominated action for all $i = 2, ..., n$. Since 1 can guarantee payoff $1 - \xi_1 y(\beta_2) = (\xi_2 - \xi_1)\xi_2^{-1}$ by bidding $\beta_2 + \varepsilon$ for any $\varepsilon > 0$, we have $u_1^* \geq (\xi_2 - \xi_1)\xi_2^{-1}$. Denote the support[16] of F_i by $S_i \subseteq [0, \beta_2]$.

Claim 0: There are no gaps in $\cup_{j \in N} S_j$.

Proof: If there was $b \in (0, \max S_j)$ for some j, but $b \notin \cup_{j \in N} S_j$, then there is i that would strictly benefit from choosing b instead of $b' = \inf\{b'' \in S_i : b'' > b, i \in N\}$, as the lower bid would not affect his winning probability but would decrease his payments.

Claim 1: Let $K = \{j : b \in S_j\}$. Then K contains at least two elements.

Proof: By Claim 0, K is nonempty. If $K = \{i\}$, then, since a lower bid does not affect his winning probability but does decrease his payments, i would strictly benefit from downgrading his bid by some $\varepsilon > 0$ (note that S_i is a closed set).

Claim 2: Suppose there is nonempty $K' \subset N$ such that all F_k, $k \in K'$, contain an atom $\alpha_k(b) > 0$ at b. Then there is $i \notin K'$such that $F_i(b) = 0$.

Proof: Under the supposition, there is i such that bidding $b+\varepsilon$, for any $\varepsilon > 0$, increases his winning probability at least the amount

$$\textstyle\prod_{j \in N \setminus K'} F_j(b) \sum_{M \subseteq K'} \frac{1}{\#M} \prod_{j \in M} \alpha_j(b), \tag{17}$$

whereas the increase in the cost is $c_i(b+\varepsilon) - c_i(b)$. By the continuity of c_i, the latter number goes to zero. Thus so does (17). This implies there is $i \notin K'$such that $F_i(b) = 0$.

Claim 3: $\inf \cup_{j \in N} S_j = 0$.

Proof: If $\inf \cup_{j \in N} S_j > 0$, then, by Claim 2, bidder i such that $\inf S_i = \inf \cup_{j \in N} S_j$ would strictly benefit from choosing $b = 0$ rather than $b \in S_i$, as this change would not affect his winning probability.

Claim 4: $\inf S_i = 0$ for all $i = 1, ..., n$.

Proof: Suppose there is i such that $\inf S_i > 0$. Then, since there are no gaps in $\cup_{j \in N} S_j$ and $\inf \cup_{j \in N} S_j = 0$, there is bidder j and bid b such that $b \in S_j$ and $b < \inf S_i$. But this implies that i would strictly benefit from bidding 0, as this change would not affect his winning probability.

[16] The smallest closed set S such that $\Sigma_i(b) - \Sigma_i(b+\varepsilon) > 0$, for all $\varepsilon > 0$, for all $b \in S$.

Claim 5: $u_j^* = 0$ for all $j \in \{2, ..., n\}$.

Proof: By Claims 2 and 4, there is i such that $F_i(0) = 0$. By Claim 4 we have $u_j^* = 0$, for all $j \neq i$. Since $u_1^* \geq (\xi_2 - \xi_1)\xi_2^{-1} > 0$, it must be that $i = 1$.

Claim 6: If $b \in \cap_{j \in K} S_j \cap (0, \beta_2]$, then $K \subseteq \{1, ..., m\}$.

Proof: Suppose not. Then by Claim 5, for all $b \in \cap_{j \in K} S_j$,

$$\begin{aligned} &\textstyle\prod_{j \in K \backslash \{i\}} F_j(b) - c_i(b) = 0, \quad \text{for all } i \in \{2, ..., m\}, \text{ and} \\ &\textstyle\prod_{j \in K \backslash \{k\}} F_j(b) - c_k(b) = 0, \quad \text{for some } k \in \{m+1, ..., n\}. \end{aligned}$$

Take $\bar{b} = \sup S_k$. Then, since $F_k(\bar{b}) = 1 \geq F_i(\bar{b})$ and $c_k > c_i$ for all $i = 2, ...m$, we have

$$\begin{aligned} \textstyle\prod_{j \in K \backslash \{i\}} F_j(\bar{b}) - c_i(b) &> \textstyle\prod_{j \in K \backslash \{i\}} F_j(\bar{b}) - c_k(\bar{b}) \\ &\geq \textstyle\prod_{j \in K \backslash \{k\}} F_j(\bar{b}) - c_k(\bar{b}) \\ &= 0. \end{aligned}$$

This violates Claim 5.

Claim 7: Define correspondence $K : [0, 1] \to N$ such that

$$K(b) = \left\{ i \in N : \textstyle\prod_{j \in N \backslash \{i\}} F_j(b) - c_i^*(b) = 0 \right\}, \text{ for all } b.$$

Then $K(\cdot)$ is upper hemi-continuous on $(0, \beta_2]$.

Proof: Take a converging sequence $b^\nu \to b$ and k such that $k \in K(b^\nu)$ for all ν.[17] We claim $k \in K(b)$. Now

$$\textstyle\prod_{j \in N \backslash \{k\}} F_j(b^\nu) - c_k(b^\nu) = u_i^*$$

Since F_j contains no atoms on $(0, \beta_2]$, it is continuous in this range. Moreover, since c_k is continuous, the left hand side converges to u_i^*. Thus the equality holds for b, too, and hence $k \in K(b)$.

Claim 8: If $i \in K(b) \cap \{2, ..., m\}$, $b \in (0, \beta_2]$, then $i \in K(b')$, $b' \in (b, \beta_2]$.

Proof: Suppose there is an interval (b', b'') such that $i \in K(b') \cap K(b'') \cap \{2, ..., m\}$ but $i \notin K(b)$ for $b \in (b', b'')$. Then $F_i(b) = F_i(b') = F_i(b'')$ for all $b \in (b', b'')$. Note that, for any b,

$$\textstyle\prod_{j \in N} F_j(b) - c_i^*(b) F_i(b) = 0, \quad \text{for all } i \in K(b). \tag{18}$$

Consequently

$$F_j(b) = \left(\frac{c_i^*(b)}{c_j^*(b)} \right) F_i(b), \quad \text{for all } i, j \in K(b). \tag{19}$$

[17] Or equivalently a converging $k^\nu \to k$ such that $k^\nu \in K(b^\nu)$ for all ν.

In particular,

$$F_j(b) = F_i(b), \quad \text{for all } i, j \in K(b) \cap \{2, ..., m\}. \tag{20}$$

Take sequence b^ν converging to b' from upwards such that $k \in K(b^\nu) \cap \{2, ..., m\}$ and $b^\nu < b''$ for all ν. Then, since K is uhc by Claim 8, $k \in K(b')$. By (20), $F_k(b^\nu) \geq F_i(b^\nu) = F_i(b')$ for all ν. Since $i \notin K(b^\nu)$,

$$\prod_{j \in N \setminus \{i\}} F_j(b^\nu) - c_i(b^\nu) < 0 = \prod_{j \in N \setminus \{k\}} F_j(b^\nu) - c_k(b^\nu),$$

or $F_k(b^\nu) < F_i(b^\nu)$, a contradiction.

Now, since S_i contains no gaps on $(0, \beta_2]$, it can only have a gap of form $(0, \lambda_i]$. Thus $K(b) \subseteq K(b')$ for all $b' \geq b$. Since K contains at least two elements in $(0, \beta_2]$, there is $i \in \{2, ..., m\}$ such that $i \in \lim_{b \to 0} K(b)$. By (18) and (19),

$$\begin{aligned}\prod_{j \in N \setminus \{i\}} F_j(b) &= F_i(b)^{|K(b)|-1} \prod_{j \in K(b) \setminus \{i\}} \frac{c_i^*(b)}{c_j^*(b)} \prod_{j \in \{1, \ldots, m\} \setminus K(b)} F_j(b) \\ &= c_i^*(b), \quad \text{for all } i \in K.\end{aligned}$$

Dividing and rearranging

$$F_i(a) = \left(c_i^*(b) \prod_{j \in K(b) \setminus \{i\}} \frac{c_j^*(b)}{c_i^*(b)} \prod_{j \in \{1, \ldots, m\} \setminus K(b)} \frac{1}{F_j(b)} \right)^{\frac{1}{|K(b)|-1}}, \quad \text{for all } i \in K(b). \tag{21}$$

In particular, for $i \neq 1$, we have

$$c_i^*(b) \prod_{j \in K(b) \setminus \{i\}} \frac{c_j^*(b)}{c_i^*(b)} \prod_{j \in \{1, \ldots, m\} \setminus K(b)} \frac{1}{F_j(b)} = \frac{c_1^*(b)}{\prod_{j \in \{1, \ldots, m\} \setminus K(b)} F_j(b)}, . \tag{22}$$

Claim 9: If $1 \in K(b') \cap K(b'')$, then $1 \in K(b)$ for all $b \in (b', b'')$, for all $b', b'' \in [0, \beta_2]$.

Proof: Suppose there is a $b' < b''$ such that $1 \in K(b') \cap K(b'')$ but $i \notin K(b)$ for $b \in (b', b'')$. Take sequence $b^\nu \in (b', b'')$ converging to b'. Since $1 \notin K(b^\nu)$, his payoff is, by (21),

$$\prod_{j \in N \setminus \{1\}} F_j(b^\nu) - c_1^*(b^\nu) = \frac{c_2(b^\nu)}{F_1(b^\nu)} - c_1^*(b^\nu) = \frac{c_2(b^\nu)}{F_1(b')} - c_1^*(b^\nu).$$

Recall that, by Claim 5, $c_j^*(b) = c_j(b)$ for all $j \in \{2, ..., m\}$ and that $c_1^*(b) = \xi_1 y(b) + u_1^*$. Since $1 \in K(b')$ and $c_j(\cdot)$'s are continuous, this number converges to zero. Thus

$$\frac{c_2(b')}{c_1^*(b')} = F_1(b'). \tag{23}$$

Similarly, take sequence in (b', b'') converging to b''. Then, by continuity,

$$\frac{c_2(b'')}{c_1^*(b'')} = F_1(b''). \tag{24}$$

Since $F_1(b') = F_1(b'')$, we have

$$\frac{1}{\xi_1 + \frac{u_1^*}{y(b')}} = \frac{1}{\xi_1 + \frac{u_1^*}{y(b'')}}.$$

But this can hold only if $y(b') = y(b'')$. Since y is increasing, this implies $b' = b''$, a contradiction.

Claim 10: $\sup S_1 = \beta_2$ and $u_1^* = (\xi_2 - \xi_1)\xi_2^{-1}$.

Proof. Let $\sup S_1 = \bar{b}$. Since F_1 is a cdf, we have $F_1(\bar{b}) = 1$. Since $u_1^* \geq (\xi_2 - \xi_1)\xi_2^{-1}$, necessarily $\bar{b} \leq \beta_2$. Suppose $\bar{b} < \beta_2$. By (23) $c_2(\bar{b}) = c_1^*(\bar{b})$ or

$$\xi_2 y(\bar{b}) = \xi_1 y(\bar{b}) + u_1^*.$$

Therefore

$$y(\bar{b}) = \frac{u_1^*}{\xi_2 - \xi_1} \geq \frac{1}{\xi_2}.$$

Since y is an increasing function, this implies $\bar{b} \geq \beta_2$, a contradiction. Since $\bar{b} = \beta_2$, we have $u_1^* = (\xi_2 - \xi_1)\xi_2^{-1}$.

By Claims 5 and 10 we now have $c_i^* = \tilde{c}_i$ for all $i = 1, ..., n$. Rank bidders $\{2, ..., m\}$ according their $\inf S_i$'s. Rename the lowest ranked bidder 2, the second lowest ranked by 3, and so on. Choose $\lambda_1 = \inf S_1$, and $\lambda_j = \inf S_j$ for all $j = 1, 2, ..., m$. Then, by Claim 1, $\lambda_1 = \lambda_2 = 0 \leq \lambda_3 \leq ... \leq \lambda_m$. Thus, by (21) we have constructed strategies (F_i) of the desired form.

The remaining task is to construct the atoms at $b = 0$. Let k be the number of active bidders, i.e. $\lambda_k < \beta_2$. Then $\lambda_k = \max\{\lambda_j : \lambda_j < 1, j = 1, ..., m\}$. Then $F_j(0) = \alpha_j(0) = 1$ for all $j = k+1, ..., m$. Since $c_j^*(b) = c_2(b)$ for all $j = 2, ..., k$ and $\prod_{j \in k+1,...,m} F_j(b) = 1$, we have, by (21),

$$\alpha_k(0) = \left(\tilde{c}_i(b) \prod_{j \in 1}^{k-1} \frac{\tilde{c}_j(b)}{c_i(b)}\right)^{\frac{1}{k-1}} = \tilde{c}_1(\lambda_k)^{\frac{1}{k-1}}.$$

Then

$$\alpha_{k-1}(0) = \left(\frac{\tilde{c}_1(\lambda_k)}{\alpha_k(0)}\right)^{\frac{1}{k-2}},$$

and, inductively,

$$\alpha_{k'}(0) = \left(\frac{\tilde{c}_1(\lambda_{k'})}{\prod_{j=k'+1}^{k} \alpha_j(0)}\right)^{\frac{1}{k'-1}}.$$

This proves the necessity.

Sufficiency: Suppose that $(F_i)_{i=1}^n$ satisfies (21) for some K. It suffices to show there is no profitable deviation by $k \in N \setminus K$. Suppose there is a profitable bid $b > 0$ for k. Bidding over β_2 is clearly dominated. Then

$$\prod_{j\in K\setminus\{i\}} F_j(b) - \tilde{c}_2(b) = 0.$$

Since k's deviation is profitable

$$\prod_{j\in K} F_j(b) - \tilde{c}_k(b) > 0.$$

By assumption $\tilde{c}_i(b) \leq \tilde{c}_k(b)$. But this implies

$$\prod_{j\in K} F_j(b) > \tilde{c}_k(b) \geq \prod_{j\in K\setminus\{i\}} F_j(b),$$

or $F_i(b) > 1$, a contradiction. ■

The next proposition allows general increasing and continuous cost functions.

Proposition 9 *Assume (13). There is a completely mixed NE of SPAA where set B of bidders completely mix only if*

$$F_i(b) = \left(\left(1 - e^{-c_i(b)}\right) \prod_{j\in B-\{i\}} \left(\frac{1 - e^{-c_j(b)}}{1 - e^{-c_i(b)}}\right)\right)^{\frac{1}{|B|-1}}, \tag{25}$$

for all $b > 0$, for all $i \in B$.

Moreover, for any $B = \{1, ..., k\}$, $k \leq n$, such equilibrium can be formed.

Proof of Proposition 9. Necessary condition: Let the constructed strategies of players in B form an equilibrium in completely mixed strategies. For any $b > 0$, let the probability of i winning be

$$G_i(b) = \prod_{j\neq i} F_j(b).$$

Since the strategy is completely mixed and atomless (see the proof Proposition 1), all bids generate i a zero payoff

$$\int_0^a (1 - c_i(b))dG_i(b) - c_i(a)(1 - G_i(a)) = 0, \quad \text{for all } a > 0, \text{ for all } i \in B. \tag{26}$$

Taking the derivative,

$$G_i'(a) - c_i'(a)(1 - G_i(a)) = 0, \quad \text{for all } a > 0, \text{ for all } i \in B. \tag{27}$$

Thus,

$$c_i(a) = \int_0^a \frac{G_i'(b)}{1 - G_i(b)} db = -\ln[1 - G_i(a)],$$

or

$$G_i(a) = 1 - e^{-c_i(a)}. \tag{28}$$

Thus,

$$F_j(a) = \left(\frac{1 - e^{-c_i(a)}}{1 - e^{-c_j(a)}} \right) F_i(a).$$

Inserting this into (28) gives

$$F_i(a) = \left((1 - e^{-c_i(a)}) \prod_{j \in B - \{i\}} \frac{1 - e^{-c_j(a)}}{1 - e^{-c_i(a)}} \right)^{\frac{1}{|B|-1}}, \quad \text{for all } i \in B$$

establishing (25).

Sufficient condition: Taking the above steps in reversed order, if F meets (25), then no player in $B = \{1, ..., k\}$ wants to deviate. We need to check that no player $i > k$ benefits from bidding above 0. By assumption, $c_i(b) \geq c_k(b)$ for all b. Moreover, the probability of i being the winner when bidding b is $G_k(b) \cdot F_k(b) \leq G_k(b)$. Letting g_k be the density of G_k, and f_k the density of F_k we have, for all $a > 0$,

$$\begin{aligned}
0 &= \int_0^a (1 - c_k(b)) dG_k(b) - c_k(a)(1 - G_k(a)) \\
&\geq \int_0^a (1 - c_i(b)) dG_k(b) - c_i(a)(1 - G_k(a)) \\
&= G_k(a) \left\{ 1 + c_i(a) - \int_0^a c_i(b)) \frac{g_k(b)}{G_k(b)} db - c_i(a) \right\} \\
&\geq [G_k(a) \cdot F_i(\cdot)] \left\{ 1 + c_i(a) - \int_0^a c_i(b)) \left(\frac{g_k(b)}{G_k(b)} + \frac{f_k(b)}{F_k(b)} \right) db - c_i(a) \right\} \\
&= \int_0^a (1 - c_i(b)) d[G_k(b) \cdot F_i(b)] - c_i(a)(1 - G_k(a) \cdot F_i(b)),
\end{aligned}$$

where the first inequality follows from $c_i(\cdot) \geq c_k(\cdot)$ and the second from $G_k(\cdot) \cdot F_k(\cdot) \leq G_k(\cdot)$ and $f_k(\cdot)/F_k(\cdot) \geq 0$. ■

References

[1] Baye, M., Kovenock, D., and de Vries, C. (1993), Rigging the lobbying process: An application of the all-pay auction, *American Economic Review* **83**, 289–294

[2] Baye, M., Kovenock, D., and de Vries, C. (1994), The solution to the Tullock rent-seeking game when $R > 2$: Mixed-strategy equilibria and mean dissipation rates, *Public Choice* **81,** 363-80.

[3] Baye, M., Kovenock, D., and de Vries, C. (1996), The All-pay Auction with Complete Information, *Economic Theory* **8,** 291-305.

[4] Che, Y-K and I. Gale (1998a) Caps on Political Lobbying, *American Economic Review* **88**, 643-651.

[5] Che, Y-K and I. Gale (1998b), Standard Auctions with Financially Constrained Bidders, *Review of Economic Studies* **65**, 1-21.

[6] Che, Y-K and I. Gale (2006), Caps on Political Lobbying: Reply, *American Economic Review* **96**, 1355-1360.

[7] Hendricks, K., Weiss, A., and Wilson, C. (1988), The War of Attrition in Continuous Time with Complete Information *International Economic Review* **29,** 663-680.

[8] Hillman, A., and Riley, J. (1989), Politically Contestable Rents and Transfers, *Economics and Politics* **1,** 17-39.

[9] Kaplan, T. and D. Wettstein (2006), Caps on Political Lobbying: Comment, *American Economic Review* **96**, 1351-1354.

[10] Konrad, K. and Schlesinger, H. (1997), Risk Aversion in Rent Seeking and Rent-augmenting Games, *Economic Journal* **107,** 1671-83.

[11] Krishna, V. (2002), *Auction Theory*, Academic Press, London.

[12] Krishna, V. and Morgan, J. (1997), An Analysis of the War of Attrition and the All-Pay Auction, *Journal of Economic Theory* **72,** 343-62.

[13] Maynard Smith, J. (1974), The Theory of Games and Animal Conflict, *Journal of Theoretical Biology* **47**, 209-21.

[14] Milgrom, P. and Weber, R. (1982), A Theory of Auctions and Competitive Bidding, *Econometrica* **50,** 1089-122.

[15] Moldovanu, B. and Sela, A. (2001), The Optimal Allocation of Prizes in Contests, *American Economic Review* **91**, 542.

[16] Moulin, H. (1986), *Game Theory for the Social Sciences*, New York UP, New York.

[17] Myerson, R. (1981), Optimal Auction Design, *Mathematics of Operation Research* **6,** 58-73.

[18] Tullock, G. (1980). Efficient Rent-seeking, in J.M. Buchanan, R.D. Tollison, and G. Tullock, (eds.) *Toward a Theory of the Rent-seeking Society*. College Station, Texas: Texas A&M University Press

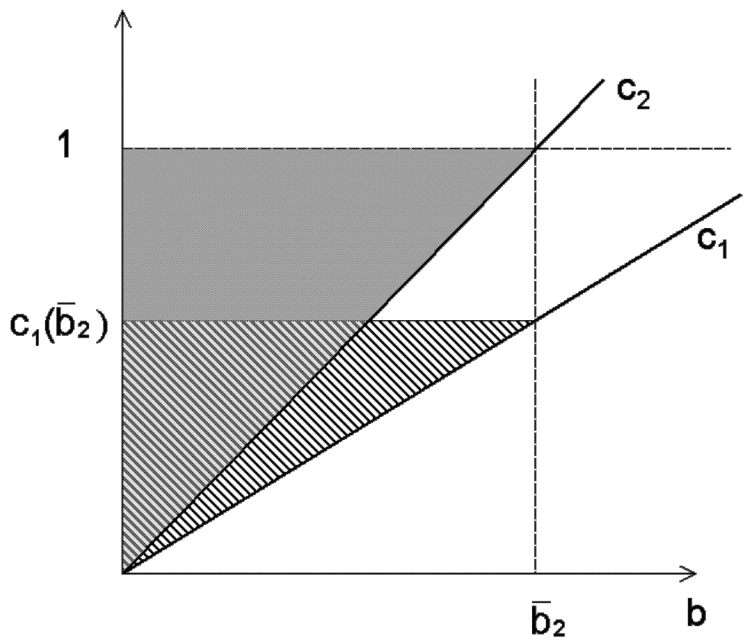

Figure 1

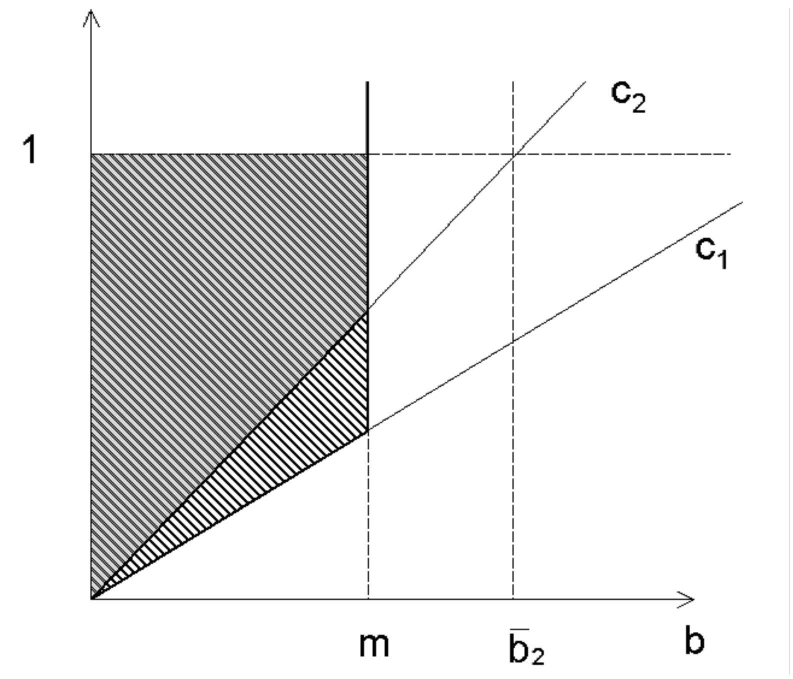

Figure 2

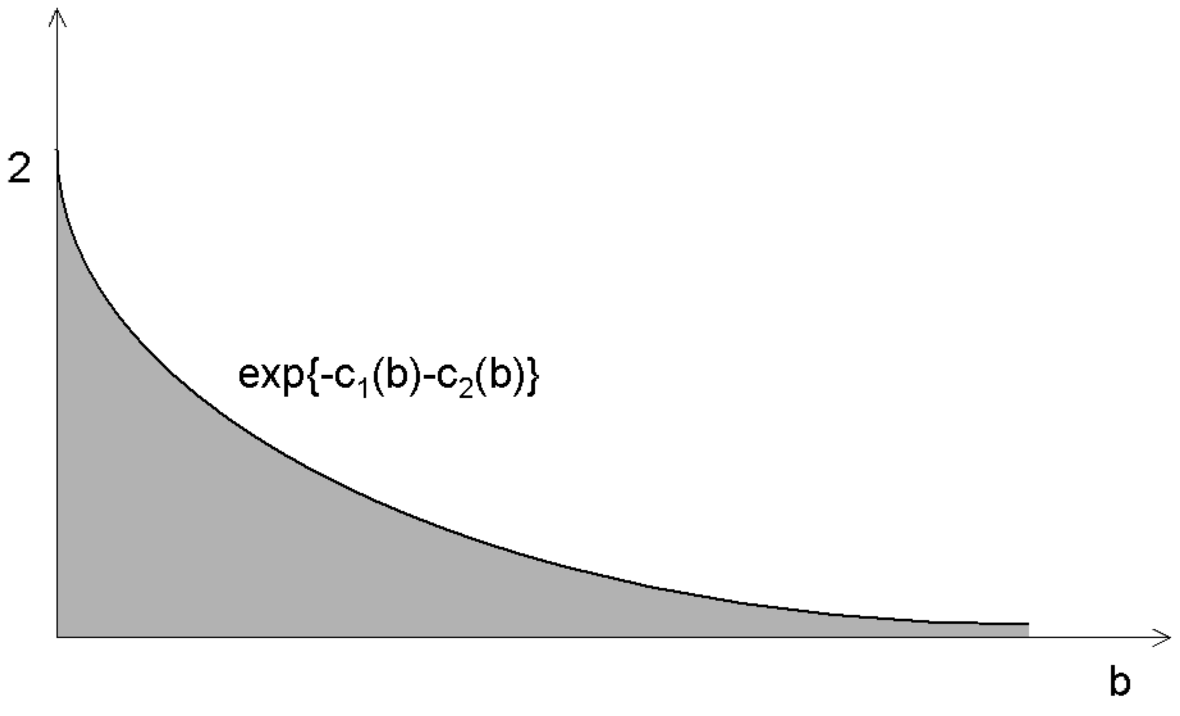

Figure 3

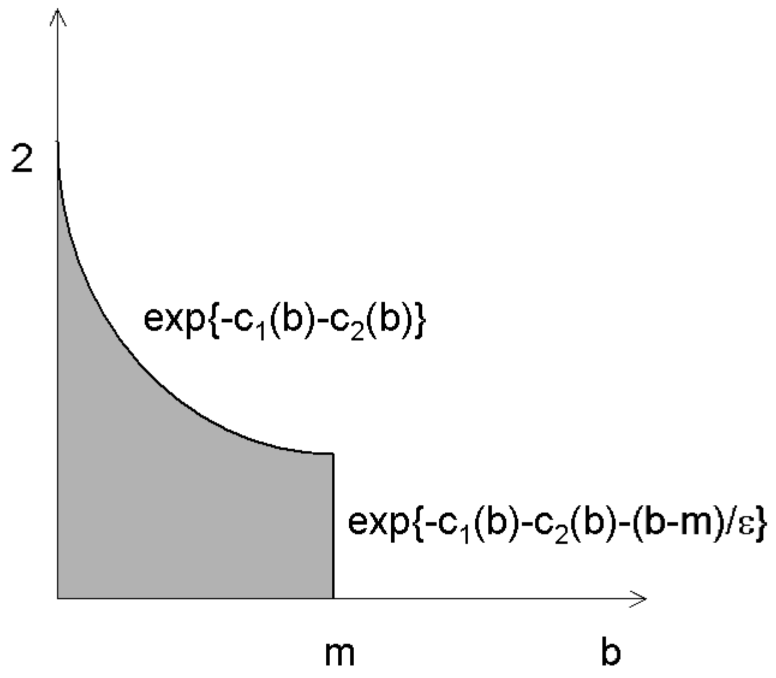

Figure 4