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Bargaining with many players: A limit result ☆

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ABSTRACT

We provide a simple characterization of the stationary subgame perfect equilibrium of an alternating offers bargaining game when the number of players increases without a limit. Core convergence literature is emulated by increasing the number of players by replication. The limit allocation is interpreted in terms of Walrasian market for being the first proposer.

1. Introduction

We study an n-player alternating offers bargaining game where the players try to agree on a division of a pie. Time proceeds in discrete periods to infinity, player 1 starts the game, and the proposer in any period is the player who first rejected the offer of the previous period. We are interested in what happens when the number of players increases. Our way of increasing the population parallels the core convergence literature as we replicate the situation so that while the number of players is increased the size of the pie increases proportionally: each replica of players brings in a new pie to the pool of shareable pies. This could reflect matters e.g. when similar nations group together as a federation.

Having a large set of players is attractive since in the limit almost all players act as responders; only one player enjoys the first proposer advantage and hence, as the number of replicas becomes large, the solution becomes almost distortion-free. We show that in the limit the unique stationary subgame perfect equilibrium has a simple characterization in terms of a single replica's preferences. Finally, the resulting single replica outcome has an attractive Walrasian interpretation: the unique equilibrium in a market where the first proposer right is sold to a *single* replica of bargainers induces the same outcome.

The primitive of our model are the time preferences á la Fishburn and Rubinstein (1982). This approach does not make assumptions concerning the concavity of utility functions.[1] Under similar assumptions, Kultti and Vartiainen (2007) show that the stationary equilibrium outcome converges to the Nash-bargaining solution when the length of the *time period* goes to zero. Since no additional assumptions are made on the utility representations, this is an extension of Binmore, Rubinstein and Wolinsky (1986). We now show that the limit outcome under replication (but fixed time interval) converges to a well defined solution also, but different from the Nash solution.[2]

2. The model

A pie of size $X>0$ is to be divided among the set $N=\{1, 2, \ldots n\}$ of players. The set of divisions of the pie is

$$S^n(X) = \{x \in \mathbb{R}^n : \textstyle\sum_{i=1}^n x_i \le X,\ x_i \ge 0,\ \text{for all } i\}.$$

Let us write $x=(x_1,\ldots,x_n)$ and $x_{-i}=(x_1,\ldots, x_{i-1}, x_{i+1},\ldots, x_n)$.

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[1] Our assumptions about preferences are weaker than, for instance, in Kirshna and Serrano (1996). As their (unique) equilibrium is stationary, our results can be interpreted as an extension of theirs.

[2] One could also study what happens when the size of the cake is kept fixed and the number of players is increased. Then there is convergence to the Nash-bargaining solution because the utility frontier becomes practically linear. The same reasoning applies when the size of the cake is increased while keeping the number of players fixed.

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The players' preferences over divisions and timing constitute the primitive of the model. The pie can be divided at any point of time $T=\{0, 1, 2,\ldots\}$. Let division $0=(0,\ldots,0)$ serve as the reference point, and let (complete, transitive) preferences over $S\times T$ satisfy, for all $x, y \in S$, for all $i \in N$ and for all $s, t \in T$, the following properties (Fishburn and Rubinstein, 1982; Osborne and Rubinstein, 1990, Ch. 4):

A1. $(x, t) \succeq_i (0, 0)$
A2. $(x, t) \succeq_i (y, t)$ if and only if $x_i \ge y_i$.
A3. If $s>t$, then $(x, t) \succeq_i (x, s)$, with strict preference if $x_i>0$.
A4. If $(x^k, t^k) \succeq_i (y^k, s^k)$ for all $k=1, \ldots$, with limits $(x^k, t^k) \to (x, t)$ and $(y^k, s^k) \to (y, s)$ then $(x, t) \succeq_i (y, s)$.
A5. $(x, t) \succeq_i (y, t+1)$ if and only if $(x, 0) \succeq_i (y, 1)$, for any $t \in T$.

A1–A5 hold throughout the paper. By A2, the Pareto-optimal divisions at any date are given by

$$P^n(X) = \{x \in S^n(X) : \textstyle\sum_{i=1}^n x_i = X\}.$$

For each i there is a function v_i: $[0, X] \to [0, X]$, defining the present consumption value of x_i in date 1:

$$(y, 0) \sim_i (x, 1) \text{ if } v_i(x_i) = y_i, \text{ for all } x, y \in S^n(X). \tag{1}$$

Fishburn and Rubinstein (1982) show that given A1–A5, $v_i(\cdot)$ is continuous and increasing on $[0, X]$.

We assume that the loss of delay increases in the share of the pie.

A6. $x_i - v_i(x_i)$ is strictly increasing and differentiable.

That is,

$$\frac{dv_i^{-1}(x_i)}{dx_i} = \frac{1}{v_i'(x_i)} > 1, \text{ for all } x_i \ge 0. \tag{2}$$

This property will be used when we prove the existence of a stationary equilibrium.

3. The game

Given N and X, we focus on a unanimity bargaining game $\Gamma^N(X)$ defined as follows: At any stage $t \in \{0, 1, 2, \ldots\}$,

- Player $i(t) \in$ N makes an offer $x \in S^n(X)$. Players $j \neq i(t)$ accept or reject the offer in the ascending order of their index.[3]
- If all $j \neq i(t)$ accept, then x is implemented. If j is the first who rejects, then j becomes $i(t+1)$.
- $i(0)=1$.

We focus on the stationary subgame perfect equilibria, simply equilibria or SPE in the sequel, of the game, where:

1. Each $i \in N$ makes the same proposal $x(i)$ whenever he proposes.
2. Each i's acceptance decision in period t depends only on x_i that is offered to him in that period.

We now characterize equilibria (see Krishna and Serrano, 1996).

Proposition 1. *x is a stationary equilibrium outcome of $\Gamma^N(X)$ if and only if $x=(\underline{x}_1+d, \underline{x}_2, \ldots, \underline{x}_n)$, for the $\underline{x}$ such that*

$$\underline{x}_i = v_i\left(\underline{x}_i + d\right), \text{ for all } i \in N, \tag{3}$$

$$\sum_{i=1}^{n} \underline{x}_i = X - d \tag{4}$$

Proof. Only if: In a stationary SPE the game ends in finite time. Assume that it never ends. Then each player receives zero. This means that in all subgames each player must get zero. Otherwise there would be a subgame where some offer $y=(y_1, \ldots, y_n)$ is accepted. Because of stationarity this offer is accepted in every subgame. In particular, player 1 can deviate in the first period and offer $y=(y_1,\ldots,y_n)$. This is a profitable deviation and constitutes a contradiction with the assumption that there is a stationary SPE where the game never ends.

Assume next that there is a stationary SPE where an offer $x(i)$ by some player $i \in \{1, 2,\ldots,n\}$, is not accepted immediately. Denote by $z(i)$ the equilibrium outcome in a subgame that starts with an offer $x(i)$ of player i. But now player i could offer $z(i)$ instead of $x(i)$; everyone else would accept the offer as in the stationary equilibrium acceptance depends only on the offer.

Thus, in any equilibrium, $i(t)$'s offer $x(i(t))=(x_j(i(t)))_{j\in N}$ is accepted at stage $t \in \{0, 1, 2, \ldots\}$. In stationary equilibrium the time index t can be relaxed from $x(i(t))$. An offer x by i is accepted by all $j \neq i$ if

$$x_j(i) \ge v_j\left(x_j(j)\right), \text{ for all } j \neq i. \tag{5}$$

Player i's equilibrium offer $x(i)$ maximizes his payoff with respect to constraint (5) and the resource constraint. By A3, all constraints in (5) and the resource constraint must bind. That is,

$$x_j(i) = v_j\left(x_j(j)\right), \text{ for all } j \neq i, \tag{6}$$

and

$$\sum_{i=1}^{n} x_i(j) = X, \text{ for all } j. \tag{7}$$

Since player i's acceptance decision is not dependent on the name of the proposer, there is $\underline{x}_i>0$ such that $x_i(j)=\underline{x}_i$ for all $j \neq i$. By Eq. (6), $x_j(i)<x_j(j)$ for all j. Hence there is $d>0$ such that

$$\textstyle\sum_{i=1}^n \underline{x}_i = X - d. \tag{8}$$

By Eqs. (6) and (8), $\underline{x}$ and d do meet Eqs. (3) and (4). Since 1 is the first proposer, the resulting outcome is $x(1)=(\underline{x}_1+d, \underline{x}_2,\ldots, \underline{x}_n)$.

If: Let $\underline{x}$ and d meet Eqs. (3) and (4). Construct the following stationary strategy: Player i always offers $\underline{x}_{-i}$ and does not accept less than $\underline{x}_i$. Player i's offer y is accepted by all $j \neq i$ only if

$$y_j \ge v_j\left(X - \textstyle\sum_{k \neq j} \underline{x}_k\right) = v_j\left(\underline{x}_j + d\right), \text{ for all } j \neq i. \tag{9}$$

Since v_j is increasing, and since

$$\underline{x}_j = v_j\left(\underline{x}_j + d\right), \text{ for all } j \neq i,$$

i's payoff maximizing offer to each j is $\underline{x}_j$. □

Thus, to find a stationary equilibrium it is sufficient to find $\underline{x}$ and d that meet Eqs. (3) and (4).

By Eq. (2), $v_i^{-1}(x_i) - x_i$ is a continuous and monotonically increasing function. Thus, the function $e_i(\cdot)$ such that

$$e_i(x_i) := v_i^{-1}(x_i) - x_i, \text{ for any } x_i \ge 0, \tag{10}$$

is continuous and monotonically increasing.

Define $\bar{e}_i \in (0, \infty]$ by

$$\sup_{x_i \ge 0} e_i(x_i) := \bar{e}_i.$$

Since $e_i(\cdot)$ is continuous and monotonically increasing, also its inverse

$$x_i(e) := e_i^{-1}(e), \text{ for all } e \in [0, \bar{e}_i],$$

is continuous and monotonically increasing in its domain $[0, \bar{e}_i]$. Condition (10) can now be stated in the form

$$x_i(e) = v_i(x_i(e) + e), \text{ for all } e \in [0, \bar{e}_i]. \tag{11}$$

[3] The order in which players' response to a proposal does not affect the results.

Proposition 2. *There is a unique stationary equilibrium of $\Gamma^N(X)$.*

Proof. By A1 and A3, $x_i(0)=0$. Since, for all i, $x_i^{-1}(\cdot)$ is a monotonically increasing function on $\mathbb{R}_+$ having its supremum at $\bar{e}$, it follows that $\lim_{e\to\bar{e}_i} x_i(e)=\infty$. Thus, since $\sum_{i=1}^n x_i(e)+e$ is a continuous function of e on $[0, \bar{e}_i]$ ranging from 0 to ∞, there is, by the Intermediate Value Theorem, a unique $d>0$ such that

$$\sum_{i=1}^n x_i(d) = X - d.$$

By Eq. (11)0, the pair $(x(d), d)$ meets Eqs. (3) and (4). □

For later purposes, we now identify a property of the players preferences. Let $X=1$.

Lemma 1. *There is unique y^* and $d^*>0$ such that*

$$y_i^* = v_i(y_i^* + d^*), \text{ for all } i = 1, .., n, \tag{12}$$

$$\sum_{i=1}^n y_i^* = 1. \tag{13}$$

Proof. Let $e_i(\cdot)$ be defined as in Eq. (10). Since $e_i(\cdot)=x_i^{-1}(\cdot)$ is a monotonically increasing function from $\mathbb{R}_+$ to$[0, \bar{e}_i]$ with $e_i(0)=0$, there is a unique $d^* \in (0, \bar{e}_i]$ such that $\sum_{i=1}^n x_i(d^*)=1$. By (11), $x_i(d^*)=v_i(x_i(d^*)+d^*)$ is uniquely defined, for all i. Let $x(d^*)=y^*$. □

By Lemma 1 and Lemma 1, the following corollary is immediate.

Corollary 1. *$(y_1^*+d^*, y_2^*,\ldots, y_n^*)$ as defined in Eqs. (12) and (13) forms the unique equilibrium outcome of $\Gamma^N(1+d^*)$.*

4. The limit result

We now increase the size of the problem by replicating a one-pie, n-player problem k times. That is, in a k-replicated problem we allow each *replica* of n players to bring a pie of size 1 to the pool of shareable pies, and the resulting set of $k\cdot n$ players bargain over the resulting pie of size k according to the procedure specified in the previous section.

Formally, let $N=\{1,\ldots, n\}$ be a set of original agents, and relabel them by $\{11, 12, \ldots, 1n\}$. Let the k times replicated – or *k-replicated* – set of agents be $\{11,\ldots,1n, 21,\ldots,2n,\ldots k1,\ldots,kn\}$. That is, the k-replicated problem contains k agents of type $i \in N$, each with the preferences of i. Attaching the player li the index $h(li)=n\cdot(l-1)+i$, we may order players $11, \ldots, kn$ according to their h-indices $\{h(11),\ldots, h(kn)\}=\{1,\ldots n\cdot k\}$. Using this indexation of the players, we specify a game $\Gamma^{\{1,\ldots,n\cdot k\}}(k)$, for any $k=1, 2, \ldots$. Then Propositions 1 and 2 are valid for any k-replicated problem.[4]

By Proposition 1, the equilibrium of the k-replicated problem is characterized by $\underline{x}(k)\in S^{k\cdot n}(k)$ and $d(k)>0$ meeting Eqs. (3) and (4). By symmetry, the following result is immediate:

Lemma 2. *$\underline{x}_{li}(k)=\underline{x}_{(l+1)i}(k)$, for all $i\in N$, for all $l\in\{1,\ldots,k\}$, for all $k=1,2,\ldots$.*

Because of Lemma 2, it is sufficient to focus on $\underline{x}_{1\cdot}(k)=(\underline{x}_{11}(k),\ldots,\underline{x}_{1n}(k))$. We may rewrite Eq. (4), for all $k\in\{1, 2,\ldots\}$,

$$d(k) = k\left(1 - \sum_{i=1}^n \underline{x}_{1i}(k)\right) \geq 0. \tag{14}$$

Let $\{\underline{x}(k)\}_{k=1}^\infty$ be a sequence of points meeting Eqs. (3) and (4) for the respective k-replicated problems, for all k.

Lemma 3. *Sequence $\{\underline{x}_{1\cdot}(k)\}_{k=1}^\infty$ is bounded.*

Proof. If $\{\underline{x}_{1\cdot}(k)\}_{k=1}^\infty$ is not bounded, there is a subsequence $\{\underline{x}_{1\cdot}(k_t)\}_{t=1}^\infty$ and j such that $\underline{x}_{1j}(k_t)\to\infty$. But given $\underline{x}_{1i}\geq 0$ for all i, this would violate the budget constraint (14). □

By Lemma 3, $\{\underline{x}_{1\cdot}(k_t)\}_{k=1}^\infty$ has a convergent subsequence.

Lemma 4. *Let $\{\underline{x}_{1\cdot}(k_t)\}_{t=1}^\infty$ be a convergent subsequence of $\{\underline{x}_{1\cdot}(k_t)\}_{k=1}^\infty$ such that $\underline{x}_{1\cdot}(k_t)\to y$. Then $\sum_{i=1}^n y_i=1$.*

Proof. By (2), and the continuity of v_i,

$$\begin{aligned} y_i &= lim_t\, \underline{x}_{1i}(k_t) \\ &= lim_t\, v_i(\underline{x}_{1i}(k_t) + d(k_t)) \\ &= v_i(lim_t\, \underline{x}_{1i}(k_t) + lim_t\, d(k_t)) \\ &= v_i(y_i + lim_t\, d(k_t)). \end{aligned}$$

By Eq. (10), $d(k_t)\to e_i(y_i)$. By Eq. (14),

$$\sum_{i=1}^n \underline{x}_{1i}(k_t) = 1 - \frac{d(k_t)}{k_t}.$$

Given that $d(k_t)\to e_i(y_i)$, we have $\sum_{i=1}^n \underline{x}_{1i}(k)\to 1$. □

Now we give a characterization of the unique convergence point of $\underline{x}(k)$ on the Pareto frontier. More generally, the efficient n-vector y^* specifies how the gains of each generation are distributed among the members of the generation when the economy grows large. This is our main result.

Proposition 3. *$\underline{x}_{1\cdot}(k)$ converges to y^* as specified in Eqs. (12) and (13) when k tends to infinity.*

Proof. Since, by Lemma 3, sequence $\{\underline{x}_{1\cdot}(k)\}_{k=1}^\infty$ is bounded, it suffices to show that every convergent subsequence of it converges to y^*. Let subsequence $\{\underline{x}_{1\cdot}(k_t)\}_{t=1}^\infty$ converge to y. By Lemma 4, $\sum_{i=1}^n y_i=1$. There is a $d>0$ such that $y_i=v_i(y_i+d)$ for all $i=1,\ldots,n$. By Lemma 1, $y=y^*$ and $d=d^*$. □

Thus, by Lemma 2, *all* sequences $\{\underline{x}_{l\cdot}(k)\}$, for $l=0,1,\ldots$, converge to $y^*=(y_1^*,\ldots,y_n^*)$ which is characterized by the data of the original n players as in Eqs. (12) and (13).

Fig. 1 depicts how the limit outcome of a single replica is specified in the $n=2$ case. For any $d>0$, identify the function $v_1(1+d-x_2)=x_1$ on $x_2\in[0,1]$, and the function $v_2(1+d-x_1)=x_2$ on $x_1\in[0,1]$. The unique intersection (y_1, y_2) of the two functions satisfies

$$\begin{aligned} v_1((1+d-y_2) &= y_1, \\ v_2(1+d-y_1) &= y_2. \end{aligned}$$

Then d^* chosen such that the intersection (y_1^*, y_2^*) satisfies $y_1^*+y_2^*=1$. Given such d^*,

$$\begin{aligned} v_1(1+d^*-y_2^*) &= v_1(y_1^*+d^*) = y_1^*, \\ v_2(1+d^*-y_1^*) &= v_2(y_2^*+d^*) = y_2^*. \end{aligned}$$

Thus (y_1^*, y_2^*) and d^* satisfy the conditions inn Eqs. (3) and (4) of a two player game with $X=1+d^*$.

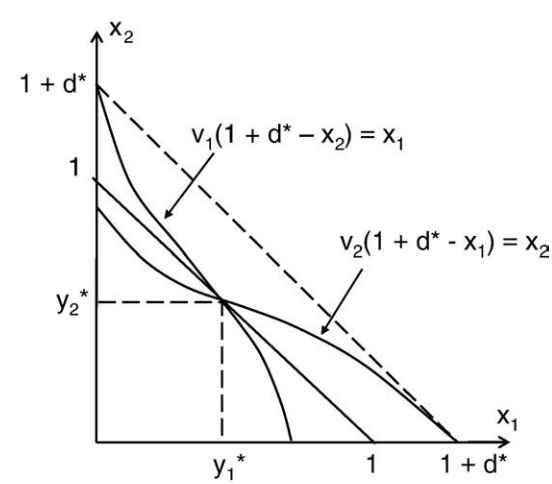

Fig. 1.

[4] Any indexation of the players would do.

5. Market for the first-proposer right

To conclude, we give a "Walrasian" interpretation to the characterized limit outcome y^*. Being the first proposer in the bargaining game is valuable. Consider a market where an arbitrator sells the right to be the first proposer in a bargaining game to one of the n bargainers and, once one player i, has paid price p for the right, adds p to the pool of resources over which bargaining takes places. That is, given the original size 1 of the pie, the player i becomes the first proposer in the bargaining game $\Gamma^n(1+p)$.

We study Walrasian markets for the first proposer right. We concentrate on *perfectly competitive* equilibria. Inspired by Makowski and Ostroy (2001), we see perfect competition as a situation where each agent gets exactly what he brings to the economy.[5] By Makowski and Ostroy (2001), this holds if the agents are *price takers* and the demand is *perfectly elastic*: given price p there are buyers that are *indifferent* between buying or not buying. That is, in our set up the price p is such that for at least two buyers the payoff from buying the right with price p is equal to the payoff of letting the other buyer buy the right at price p.

We claim that d^* is the unique such price and y^* is the resulting allocation of the pie.

Proposition 4. $e(y^*)$ *is the unique perfectly competitive Walrasian price under which the players are indifferent, and* y^* *is the resulting allocation of the pie, for* d^* *and* y^* *as specified in Lemma 1.*

Proof. Let $z_j(1+p)$ be what a non-proposer j gets in the game $\Gamma^n(1+p)$. Then the proposer i's share is $1+p-\sum_{j\neq i} z_j(1+p)$. Since only one player, say i again, eventually becomes the first proposer, there is a $j\neq i$ who is indifferent between buying or not. Thus j's payoff from buying the proposing right under p equals the buying cost p and the alternative cost $z_i(1+p)$, i.e.,

$$1+p-\sum_{k\neq j} z_k(1+p) = p + z_j(1+p).$$

Then $1=\sum_{i=1}^{n} z_i(1+p)$. By Eqs. (3) and (4) (with $X=1+p$),

$$z_i(1+p) = v_i(z_i(1+p)+p), \text{ for all } i=1,\ldots,n.$$

By Lemma 1, $z_i(1+p)=y_i^*$ for all i, and $p=d^*$. □

By Proposition 3, the unique outcome y^* of the Walrasian market for the first-proposing right can be thought as the expected outcome of bargaining when the number of bargainers grows large and the probability of a particular player having the right be the first proposer becomes negligible. Having a large set of players is attractive since the resulting bargaining outcome reflects strong *average* fairness: all but one generation distribute their resources without a first mover distortion. Thus the simple market game with small number of players can be used to simulate the distortion-free many-player bargaining outcome.

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[5] This is called *full appropriation*.