Microeconomic Theory

Lecture 3

Consumer with preferences

- From lecture 1: If $\succsim$ is rational, i.e. complete and rational, then $c^*(\cdot, \succsim)$ satisfies WA.

- Let $x(p, w) = c^*(B(p, w), \succsim)$, i.e. the outcome $x(p, w)$ maximizes preferences $\succsim$ in $B(p, w)$.

- Since $B(p, w) = B(\lambda p, \lambda w)$, for all $\lambda > 0$,.$x(p, w)$ is homogenous of degree 0.

- For Walras' Law, we need new assumptions.

Axiom 1 *Preferences* $\succsim$ *are* monotonic *if, for all* $x, y \in X$,

$$x_i > y_i, \text{ for all } i \text{ imply } x \succ y.$$

They are strongly monotonic *if, for all* $x, y \in X$,

$$x \geq y \text{ and } x \neq y \text{ imply } x \succ y.$$

- A weaker axiom with similar spirit:

Axiom 2 *Preferences* $\succsim$ *are locally* nonsatiated *if for all* $x \in X$ *and for all* $\delta > 0$, *there exists* $y \in X$ *such that*

$$\|y - x\| < \delta \text{ and } y \succ x.$$

- Local nonsatiation is implied by monotonicity but not vice versa.

- Local nonsatiation of $\succsim$ implies Walras' Law, thus monotonicity implies it.

- Assume from now on nonsatiated preferences $\succsim$ and unique $x(p, w)$. Then $x(\cdot, \cdot)$

  i is homogenous of degree 0.

  ii meets Walras' law.

- Hence, by lect. 2, $x(p, w)$ also satisfies the compensated law of demand, and the Slutsky matrix is negative semidefinite.

- To guarantee continuoity and uniqueness of $x(\cdot, \cdot)$ we make other structural assumptions:

Axiom 3 *Preferences* $\succsim$ *are* convex *if for all* $x, y, \in X$ *and for all* $t \in [0, 1]$,

$$x \succsim y \text{ implies } (tx + (1-t)y) \succsim y.$$

They are strictly convex *if for all* $x, y, \in X$ *and for all* $t \in (0, 1)$,

$$x \succsim y \text{ implies } (tx + (1-t)y) \succ y.$$

- Conevexity implies continuity of x, strict convexity its uniqueness.

- Recall the definition of quasi-concave functions:

Definition 4 *Let f be defined on the convex set $X \subset \mathbb{R}^n$. It is a* quasiconcave *function if and only if*

$$f(tx_1 + (1-t)x_2) \geq \min[f(x_1), f(x_2)]$$

for every $x_1, x_2 \in X$,and $0 \leq t \leq 1$.

Theorem 5 *Let $X \subseteq \mathbb{R}^n$ be a convex set. Function $f : X \to \mathbb{R}$ is quasiconcave if its upper contour sets $U(f, \alpha) = \{x : x \in X, f(x) \geq \alpha\}$ are convex sets for $\alpha \in \mathbb{R}$.*

- Recall the upper and lower contour sets $U(x, \succsim) = \{y \in X : y \succsim x\}$ and $L(x, \succsim) = \{y \in X : x \succsim y\}$.

Axiom 6 *Preferences $\succsim$ are* continuous *if, for all $x \in X$, the sets $U(x, \succsim)$ and $L(x, \succsim)$ are closed.*

Proposition 7 *Let rational preferences $\succsim$ be continuous. Then there is a continuous utility function u which represents those preferences. Moreover, if $\succsim$ is convex, then u is quasiconcave and convex valued.*

- Notice from here the connection between quasiconcavity of a representation and the convexity of the underlying preferences. Note also that a concave function is quasiconcave but not vice versa.

- Note also that single valuedness of $x(p,w)$ would follow from strict convexity of $\succsim$.

- Thus there is a continuous u such that $x\left(p,w\right)$ is the unique solution to the *utility maximization problem*

$$\max_{x\in B(p,w)} u\left(x\right).$$

or, equivalently,

$$\begin{aligned}&\max_{x\geq 0}\ u\left(x\right)\\ &\text{s.t. } p\cdot x\leq w\ .\end{aligned}$$

- Is it w.l.o.g. to assume $x(p,w)$ exists?

Proposition 8 *(Weierstrass) Let $f : X \to \mathbb{R}$ be a continuous function and X a compact set. Then f attains its maximum (minimum) on X.*

- Recall: $X \subset \mathbb{R}^n$ is compact if and only if it is closed and bounded.

- It is easy to see through counterexamples that continuity, boundedness and closedness are all required for the result.

- We have assumed $p \gg \mathbf{0}$, so that $B(p, w)$ is a compact convex set, and u is continuous, hence a utility maximizer does exist.

- How to characterize $x(p, w)$?

- Construct a Lagrangean

$$\mathcal{L}(x, \lambda : p, w) = u(x) - \lambda(p \cdot x - w),$$

where $\lambda \in \mathbb{R}$ is the Lagrange multiplier.

- Let $x(p, w) := x^*$ maximize the Lagrangean (assuming an interior solution $x^* > 0$). The first order conditions are

$$\begin{aligned} \frac{\partial u(x^*)}{\partial x_l} - \lambda p_l &= 0 \text{ for all } l = 1, \ldots L, \\ p \cdot x^* &= w. \end{aligned}$$

- Thus
$$\frac{\partial u\left(x^{*}\right)/\partial x_{l}}{\partial u\left(x^{*}\right)/\partial x_{k}}=\frac{p_{l}}{p_{k}}.$$

- The ratio p_l/p_k is the *marginal rate of substitution* between goods l and k at x^*.

- On the ther hand
$$\frac{\partial u\left(x^{*}\right)}{\partial p_{l}}=\lambda p_{l}$$

- Lagrange multiplier λ gives the marginal (shadow) value of relaxing the constraint, i.e. the marginal value of wealth w. To see this, recall that by the first order condition,

$$D_x \mathcal{L}(x^*, \lambda : p, w) = D_x u\left(x^*\right) - \lambda p = 0.$$

Recalling that $x^* = x\left(p, w\right)$, we obtain

$$\begin{aligned}
\frac{\partial}{\partial w}\mathcal{L}\left(x\left(p, w\right), \lambda : p, w\right) &= \frac{\partial}{\partial w}\left[u\left(x\left(p, w\right)\right) - \lambda\left(p \cdot x\left(p, w\right) - w\right)\right] \\
&= \frac{\partial x\left(p, w\right)}{\partial w}\left(D_x u\left(x\left(p, w\right)\right) - \lambda p\right) + \lambda \\
&= \lambda.
\end{aligned}$$

The second equality follows by the chain rule and the last one from the FOC of the Lagrangean. The important observation that only the direct effect of a parameter change matters (on the value function) is known as the *envelope theorem*.

- Define the *indirect utility function* by condition

$$v(p,w) = u(x(p,w)).$$

- What are the properties of $v(p,w)$ implied by the utility maximization problem?

- Conversely, if $v(\cdot,\cdot)$ is known, can we recover the utility function $u(\cdot)$, i.e. the preferences $\succsim$?

- Let u represent non-satiated and continuous preferences $\succsim$. Then $v(\cdot,\cdot)$ is:

  i homogenous of degree 0,

  ii strictly increasing in w, strictly decreasing in p_l for all l,

  iii continuous,

  iv quasiconvex if $\succsim$ is convex.

- Note that quasiconvexity is equal to $-v(p,w)$ being quasiconcave, or $\{(p,w) : v(p,w) \leq v'\}$ being a convex set for all $v' > 0$.

- If $\succsim$ is monotonic, then v need not be strictly increasing in w.

Proposition 9 *If $v(p, w)$ satisfies i-iv, then there exists a non-satiated, continuous, and quasiconcave $u(x)$ such that $v(p, w) = u(x(p, w))$.*

- We can solve such a $u(x)$ from the problem

$$\min_{p \in \mathbb{R}^L_{++}} \quad v(p, w)$$
$$\text{s.t.} \, p \cdot x = w.$$

Proposition 10 *(Roy's Identity) Let u represent nonsatiated, strictly convex, and continuous preferences $\succsim$. Given an indirect utility function $v(p,w)$, the Walrasian demand $x(p,w)$ can be recovered from*

$$x_l(p,w) = -\frac{\partial v(p,w)/\partial p_l}{\partial v(p,w)/\partial w}.$$

- To see this, note that

$$\begin{aligned}
\frac{\partial v(p,w)}{\partial p_l} &= \frac{\partial}{\partial p_l}\mathcal{L}(x(p,w),\lambda : p,w) \\
&= \frac{\partial}{\partial p_l}[u(x(p,w)) - \lambda(p\cdot x(p,w) - w)] \\
&= \frac{\partial x(p,w)}{\partial p_l}(D_x u(x(p,w)) - \lambda p) - \lambda x_l(p,w) \\
&= -\lambda x_l(p,w).
\end{aligned}$$

where the third equality follows by the chain rule and the last one from the FOC of the Lagrangean. As above,

$$\frac{\partial v\left(p, w\right)}{\partial w} = \lambda.$$

Duality

- The problem with the utility function and the indirect utility functions is that they are not observable, only x, p and w are. the important property called *duality* of the optimal choice transforms the problem into language of the observables, and hence allows us to make emprically testable preduictions.

- Given utility function $u(\cdot)$, denote by $h\left(p, \bar{u}\right)$ the choice that solves the expedinture minimizing problem given utility of at least u and prices p:

$$\min_{x \in \mathbb{R}^L_+} p \cdot x$$
$$\text{s.t. } u\left(x\right) \geq \bar{u}.$$

- Notice that even though the feasible set is not bounded, the problem has a solution when $p \in \mathbb{R}^L_{++}$.

- $h(p, \bar{u})$ is called the *Hicksian* or *compensated demand function*.

- Denote the value function under the minimizer by $e(p, \bar{u})$; the *expenditure function*.

$$e(p, \bar{u}) = p \cdot h(p, \bar{u}).$$

- The following observation is key to the development that follows.

Proposition 11 *Fix a price vector* $p \in \mathbb{R}^L_{++}$.

1. If $x^* = x(p, w)$, then $x^* = h(p, u(x^*)) = h(p, v(p, w))$.

2. If $x^* = h(p, \bar{u})$, then $x^* = x(p, p \cdot x^*) = x(p, e(p, \bar{u}))$.

- Summarizing:

$$x(p, w) = h(p, v(p, w)) \text{ and } h(p, \bar{u}) = x(p, e(p, \bar{u})).$$

- Obviously then also:

$$w = e(p, v(p, w)) \text{ and } u = v(p, e(p, \bar{u})).$$

Proposition 12 *Let u represent nonsatiated, strictly convex, and continuous preferences $\succsim$. Then, for all (p, w)*

$$h(p, \bar{u}) = D_p e(p, \bar{u}).$$

- To see this, observe that $e(p, \bar{u})$ is the value of the Lagrangean

  $$\mathcal{L}(x, \lambda : p, \bar{u}) = p \cdot x - \lambda[u(x) - \bar{u}]$$

  at the minimizer $x = h(p, \bar{u})$. That is, by FOC,

  $$\begin{aligned} p - \lambda D_x u(h(p, \bar{u})) &= 0, \\ u(h(p, \bar{u})) - \bar{u} &= 0. \end{aligned}$$

Thus, by the envelope theorem,

$$\begin{aligned} D_p e(p, \bar{u}) &= D_p \mathcal{L}(h(p, \bar{u}), \lambda : p, \bar{u}) \\ &= D_p\{p \cdot h(p, \bar{u}) - \lambda[u(h(p, \bar{u})) - \bar{u}]\} \\ &= h(p, \bar{u}) + [p - \lambda D_x u(h(p, \bar{u}))] \cdot D_p h(p, \bar{u}) \\ &= h(p, \bar{u}). \end{aligned}$$

- Hicksian demand $h(\cdot, \cdot)$ thus satisfies
  - adding up: $p \cdot h(p, \bar{u}) = w$,
  - homogeneity of degree 0 in p: $h(\alpha p, u) = h(p, \bar{u})$ for all p, u, and scalars $\alpha > 0$,
  - convexity: if $\succsim$ is convex, then $h(p, \bar{u})$ is a convex set; if $\succsim$ is strictly convex, then $h(p, \bar{u})$ is a function,

– matrix $D_p h\left(p, \bar{u}\right)$ negative semidefinite, *symmetric*, and satisfies $D_p h\left(p, \bar{u}\right) p = 0$ (by homogeneity).

- Expedinture function $e(\cdot, \cdot)$ satisfies:
  - $e\left(p, \bar{u}\right)$ is homogenous of degree 1 in p,
  - increasing in u and non-decreasing in p_l for all l,
  - *concave* in p,
  - continuous in $p, \bar{u}$.

- How to express $D_p h(p, \bar{u})$ in terms of $x(p, w)$?

- Recall :

$$h(p, \bar{u}) = x(p, e(p, \bar{u})).$$

Therefore (Slutsky Equation):

$$\begin{aligned}
& D_p h(p, \bar{u}) \\
= \; & D_p x(p, e(p, \bar{u})) + D_w x(p, e(p, \bar{u})) D_p e(p, \bar{u}) \\
= \; & D_p x(p, e(p, \bar{u})) + D_w x(p, e(p, \bar{u})) h(p, \bar{u})^T \\
= \; & D_p x(p, w) + D_w x(p, w) x(p, w)^T,
\end{aligned}$$

where we have set $w = e(p, \bar{u}) = e(p, v(p, w))$.

- The significance: $D_p h(p, \bar{u})$ and hence $D_p^2 e(p, \bar{u})$ can be computed from $x(p, w)$ which is observable and thereby potentially testable.

- Recall the properties of $S(p, w)$, the substitution matrix defined in the choice-based approach.

- $D_p h\,(p, \bar{u})$ is symmetric due to preference maximization. Conversely, if the observed $D_p h\,(p, \bar{u})$ not symmetric, then there are no (locally nonsatiated, cont.) preferences that rationalize them.

- The restrictions under the preference-based approach are stronger, so we obtain additional observable implications.

1. Homothetic Preferences

2. Quasilinear Preferences

3. Additively Separable Preferences

Definition 13 *A continuous rational preference relation* $\succsim$ *is* homothetic *if for all* $\alpha > 0$, *we have* $x \succsim y$ *if and only if* $\alpha x \succsim \alpha y$.

Exercise 14 *Show that for homothetic preferences,* $x(p, \alpha w) = \alpha x(p, w)$.

Engel curves are rays through origin.

Let $u(x)$ be a representatition of $\succsim$.

Exercise 15 *If $u(x)$ is linearly homogenous (i.e. $u(\alpha x) = \alpha u(x)$ for all x), then $\succsim$ is homothetic.*

Clearly all representations of $\succsim$ cannot be linearly homogenous.

Exercise 16 *If $\succsim$ is homothetic, then there exists a representation $u(x)$ that is linearly homogenous.*

Hint: Let $e = (1, ..., 1) \in \mathbb{R}^L_+$. Show that for each $x \in \mathbb{R}^L_+$, there is a unique $\lambda(x) \in \mathbb{R}$, such that $x \sim \lambda(x)e$. Show that if $\succsim$ is homothetic, then $\lambda(x)$ is linearly homogenous.

Denote now the first unit vector by $e_1 = (1, 0, ..., 0)$.

Definition 17 *A continuous rational preference relation* $\succsim$ *on* $\mathbb{R}\times\mathbb{R}^{L-1}_+$ *is* quasilinear *with respect to good* 1 *(numeraire) if* $x \sim y$ *implies that* $(x + \alpha e_1) \sim (y + \alpha e_1)$ *for all* $\alpha \in \mathbb{R}$ *and* $x + \alpha e_1 \succ x$ *for all* $\alpha > 0$.

Here the Engel curves are lines parallel to x_1 axis.

Exercise 18 *A rational preference relation* $\succsim$ *on* $\mathbb{R} \times \mathbb{R}^{L-1}_+$ *is quasilinear with respect to good 1 if and only if it admits a utility representation of the form* $u(x) = x_1 + \phi(x_2, ..., x_L)$. *(Show only the if part)*

Utility functions of the form

$$u(x) = \sum_{l=1}^{L} u_l(x_l)$$

are called additively separable. It can be shown that the following restriction on preferences is equivalent to additive separability:

Definition 19 *A continuous rational preference relation* $\succsim$ *is* additively separable *if for all* $M \subset \{1, .., L\}$ *we have*

$$(x_M, y_{-M}) \succsim \left(x'_M, y_{-M}\right) \text{ if and only if } \left(x_M, y'_{-M}\right) \succsim \left(x'_M, y'_{-M}\right)$$

for all $y_{-M}, y'_{-M} \in \mathbb{R}_+^{n(-M)}$, *where* $n(-M)$ *is the number of elements not in* M.

Exercise 20 *Show that whenever a utility representation is additively separable, then the above property holds. (The converse is hard.)*