Dynamical models for paleo-climatic time series: statistics and noise induced transitions

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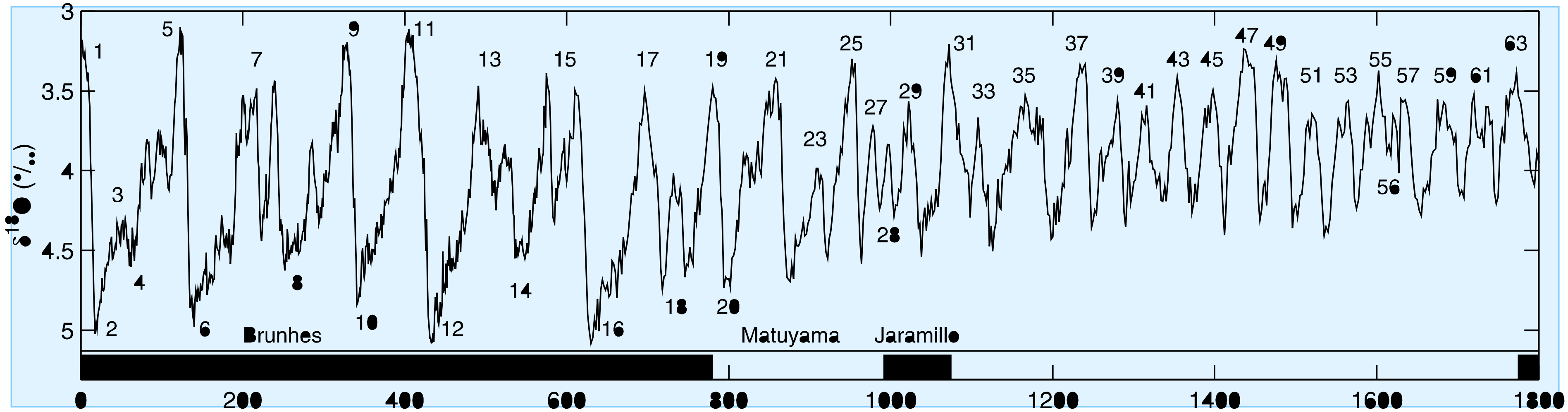
1. Paleo-climatic time series



Lisiecki, Raymo, Paleoceanography 2005 variation of concentration of ^{18}O to ^{16}O from deep sea core measurements obtained at 57 different sites (for example Brunhes, Matuyama, Jaramillo):

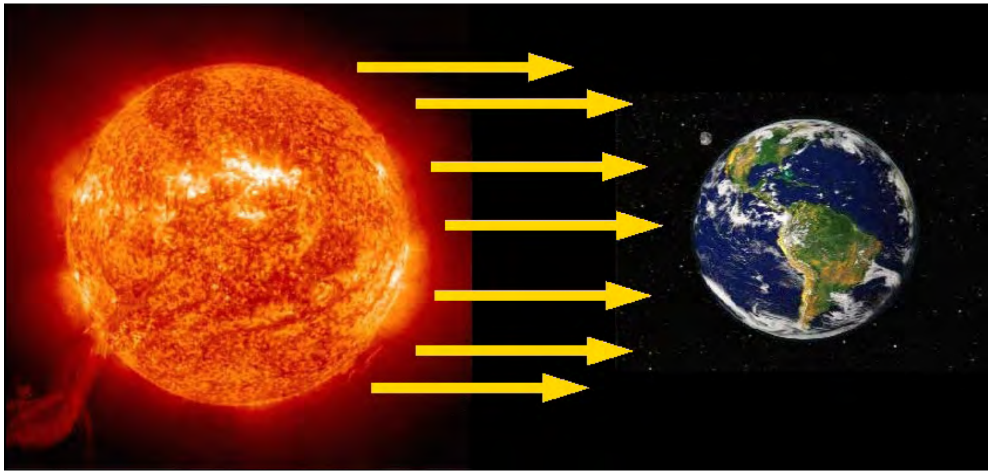
time series of global average temperature

basic property:

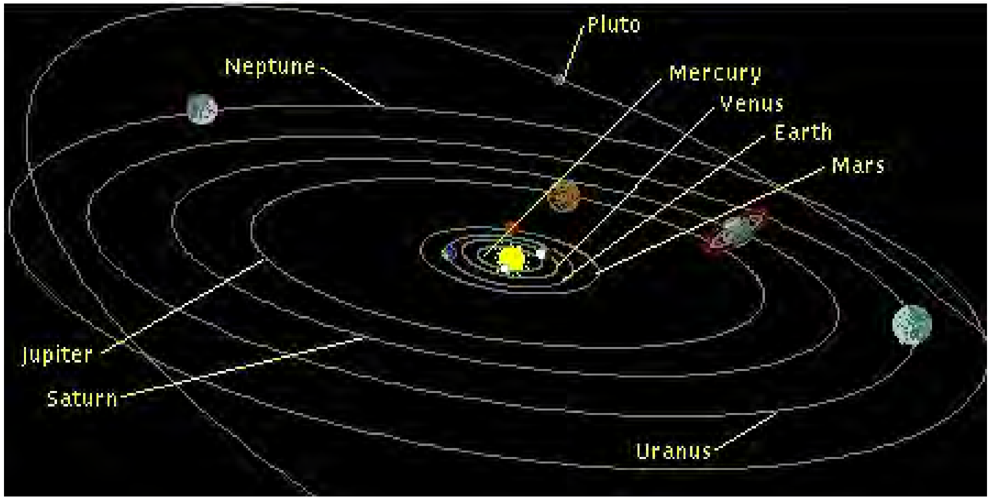
- from 0 to -1 Myr periodicity $\sim$ 100 000 y
- from -1 Myr to -1.8 Myr periodicity $\sim$ 44 000 y

Question: where does periodicity come from?

2. Astronomical factors



Earth receives its energy from the sun.
solar constant $Q \approx 1366 W/m^2$



Trajectories of planets are ellipses (Kepler).

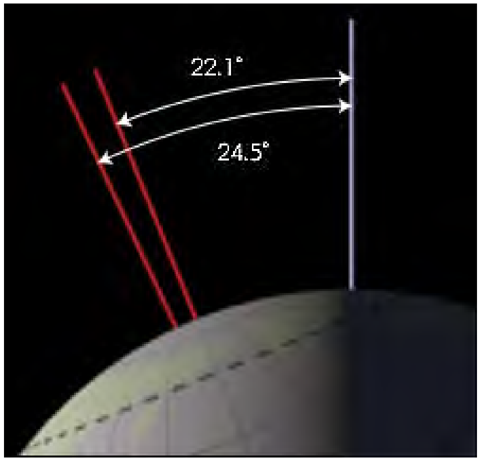
absorption of solar energy varies seasonally.

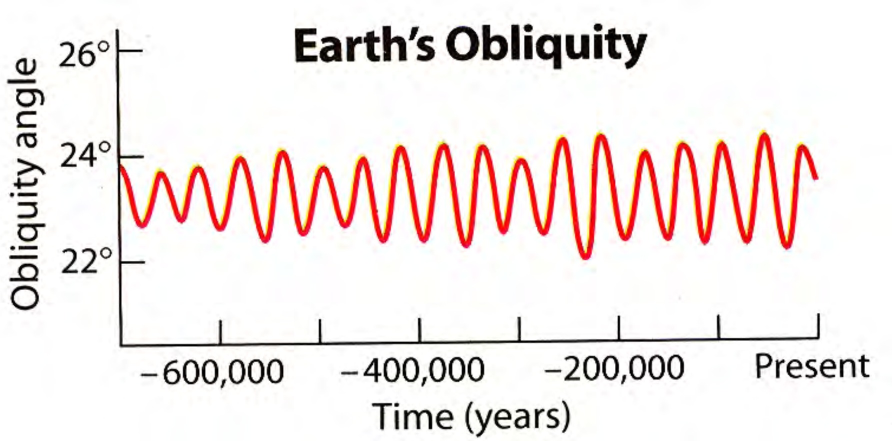
Question: Are there variations with periods of $\approx$ 100 000, 40 000 years?

3. Milankovich cycles

Milankovich (1920): astronomically caused perturbations of earth's trajectory generate three basic cycles: axial precession, axial tilt, eccentricity

axial tilt (41 000 y)



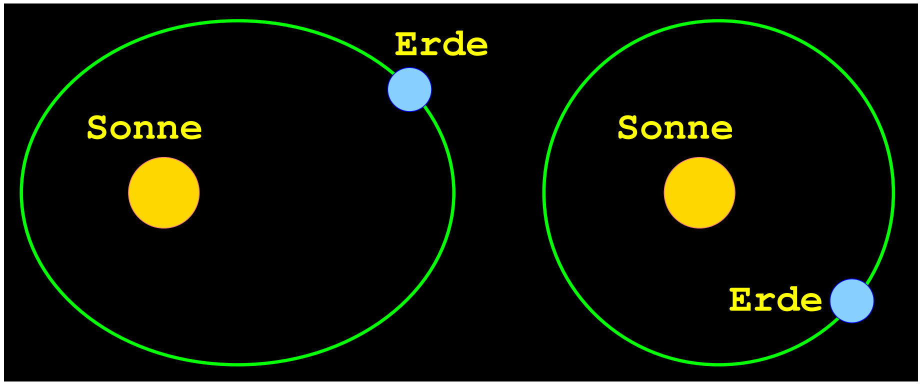


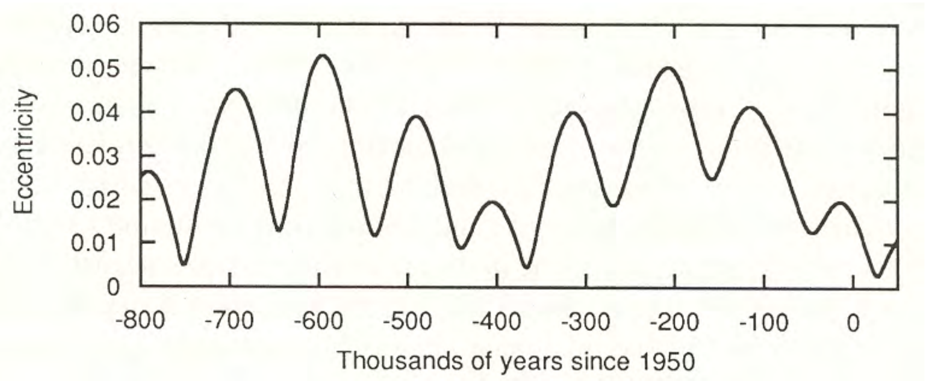
periodic wobbling of earth's rotation axis: 2,4 degrees of variation of axial tilt with respect to plane of rotation; periodicity: 41 000 y; inclination today: 23,4 degrees.

increasing tilt: increasing amplitude of seasonal insolation, in summer increased insolation, in winter less

3. Milankovich cycles

eccentricity (100 000 y)





eccentricity: measures deviation of ellipse from circle; periodic variation between 0,005 and 0,058; reasons: interaction of gravitational fields of Jupiter and Saturn; periodicity: 100 000 y; eccentricity today: 0,017.

orbital mechanics: extreme eccentricity means seasons at aphelion (far from sun) last longer.

4. The paradigm: energy balance models

(Lit: C., G. Nicolis ’80, Benzi, Parisi, Sutera, Vulpiani ’80; McNamara, Wiesenfeld ’89;...)

Aim: analytical explanation of basic features of glacial cycles in earth’s history

Model basis of Energy-Balance Models (EBM):

balance between incoming and outgoing radiative energy;

incoming (solar) radiation varies in periods of 10^5 years:
variation explained by Milankovich (eccentricity) cycle

outgoing radiative energy: earth considered as a black body radiating energy in proportion to the fourth power of temperature

earth considered as point in space; T_t: global averaged temperature at time t

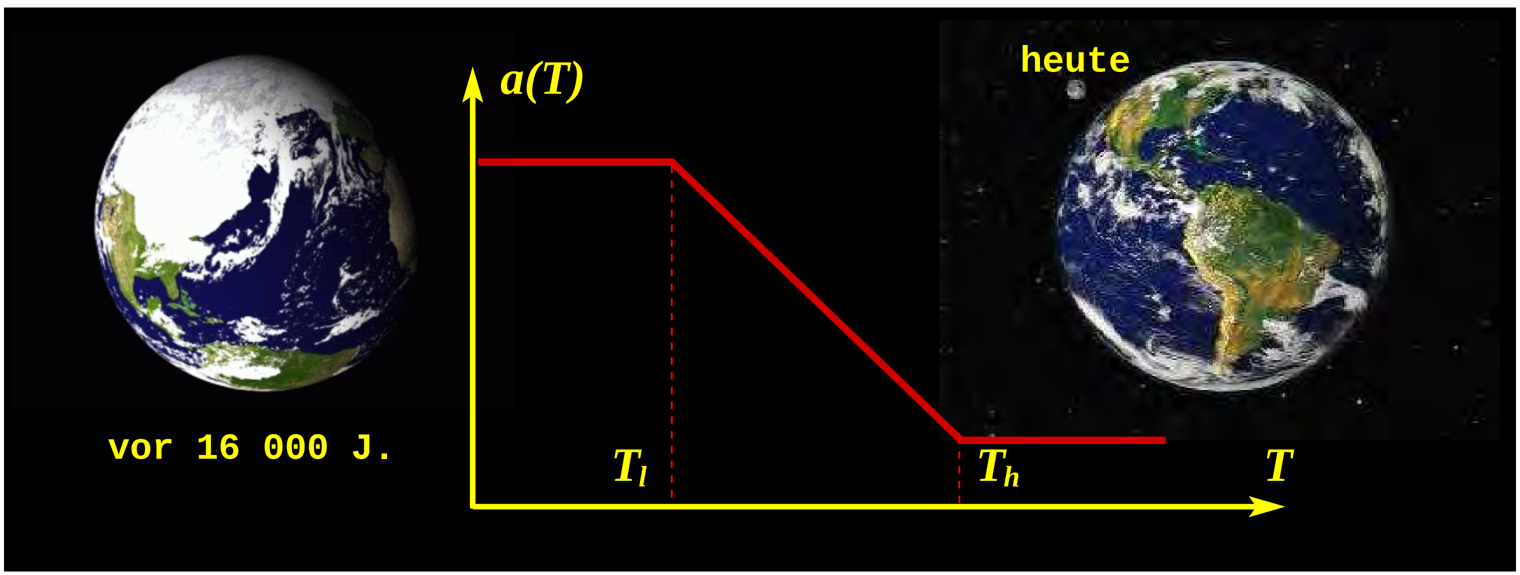
4. The paradigm: energy balance models

power of incoming radiation S^{in}:

$$S^{in}(t) = Q(t)(1 - a(T(t)))$$

- $Q(t) = Q_0(1 + A \sin \Omega t)$ — solar constant, modulated by Milankovich eccentricity, $\Omega = \frac{2\pi}{100\,000}\,[\frac{1}{J}]$, A small.

- albedo $a(T) = \frac{\text{reflected radiation}}{\text{incoming radiation}}$



e.g. fresh snow 0.75-0.95, grassland 0.12-0.30, forests 0.05-0.20

4. The paradigm: energy balance models

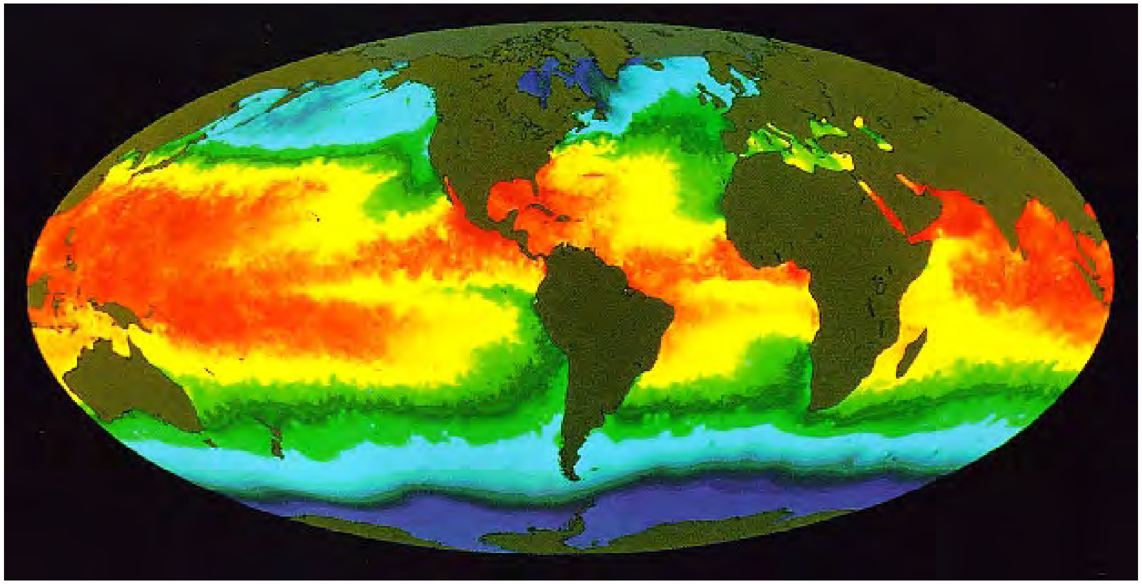
Earth considered as ideal black body

Stefan–Boltzmann law for power of black body radiation

power of outgoing radiation S_{out}:

$$S^{out}(t) = \sigma T(t)^4$$

$\sigma = 5.669 \cdot 10^{-8} \frac{W}{m^2 K^4}$ — Stefan–Boltzmann constant



4. The paradigm: energy balance models

Energy Balance (EB)

energy change = power incoming radiation - power outgoing radiation

first law of thermodynamics

energy change $= c \cdot$ temperature change

energy change at $t = S^{in}(t) - S^{out}(t)$,

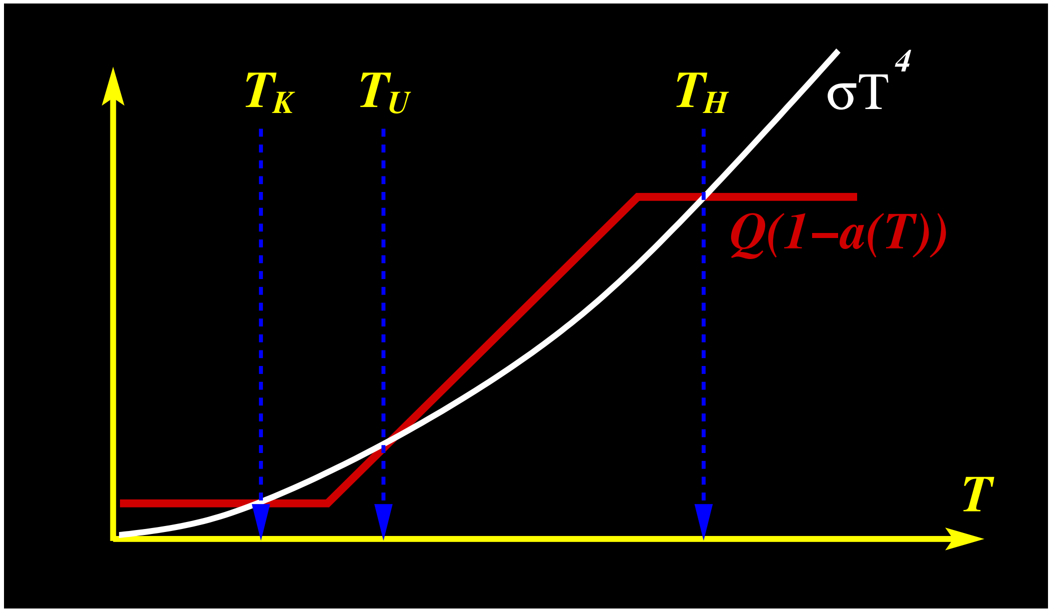
$$c\frac{dT(t)}{dt} = Q(t)(1 - a(T(t))) - \sigma T(t)^4$$

Henderson-Sellers, Mc Guffie:

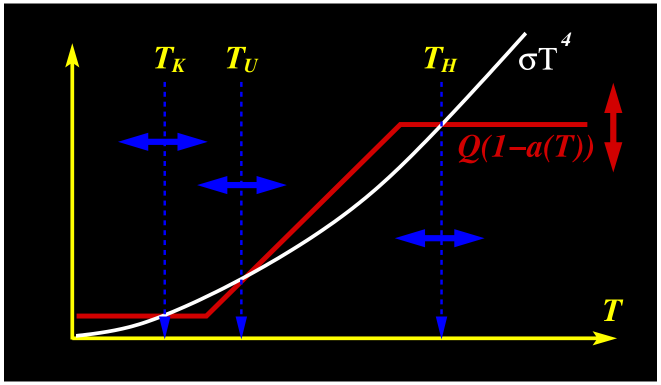
These models have been instrumental in increasing our understanding of the climate system and in the development of new parameterizations and methods of evaluating sensitivity for more complex and realistic models.

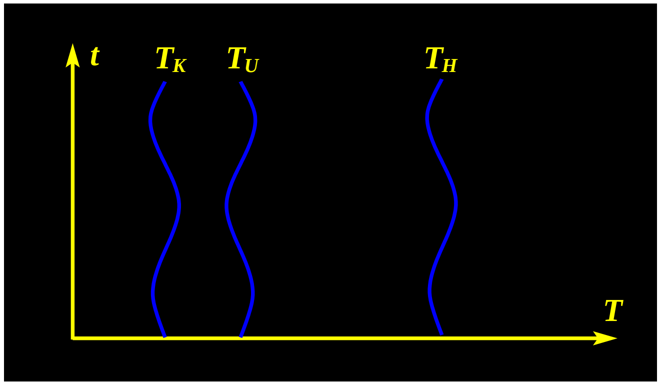
4. The paradigm (climate background)

warm age
ice age } equilibria T_K, T_H of equation, solution of $\frac{dT(t)}{dt} = 0$



4. The paradigm: energy balance models





Problems: A small

- temperature shifts too small
- relaxation times too long
- no transitions between T_K, T_H

Idea: (Nicolis, Benzi et al.)
add noise term to EBM:

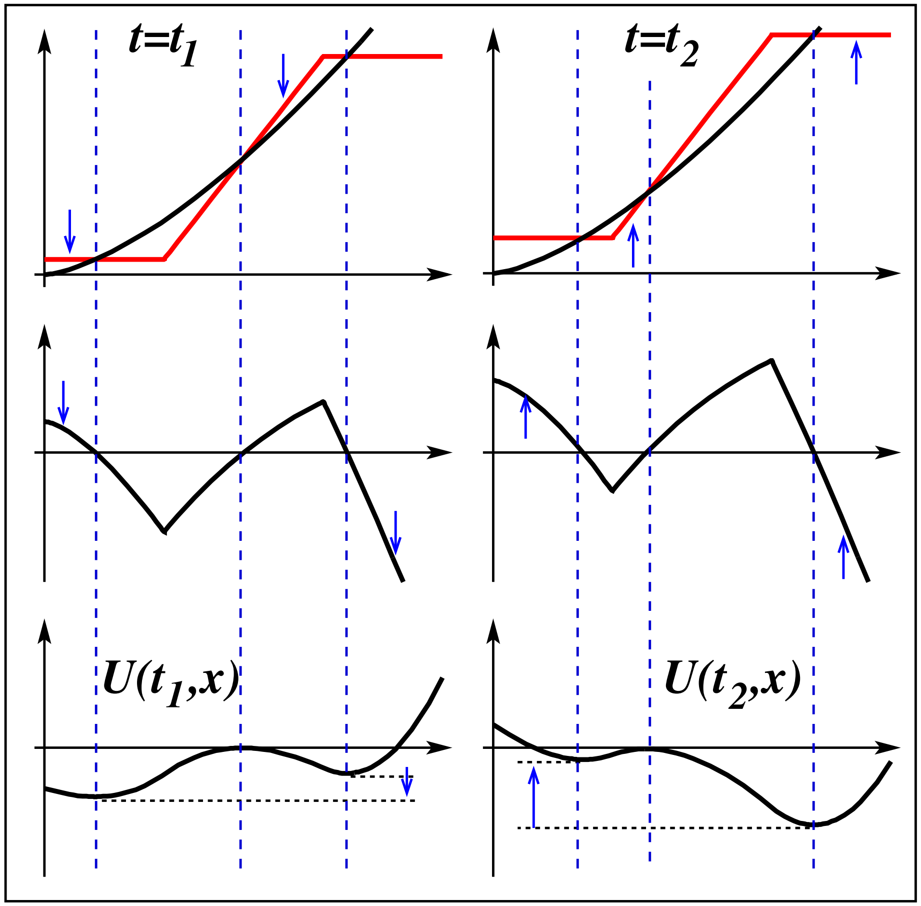
$$c\frac{dT(t)}{dt} = Q(t)(1 - a(T(t))) - \sigma T(t)^4 + \sqrt{\varepsilon}\dot{W}(t)$$

W: 1-dim. Brownian motion

- transitions possible
- relaxation times more realistic

4. The paradigm: energy balance models

General situation



$$-U'(t,x) = f(t,x) = Q(t)(1 - a(x)) - \sigma x^4, \quad x \leftrightarrow T$$

5. Weakly periodic dynamical systems with noise

Consider system as motion of overdamped physical particle in potential landscape given by weakly periodic potential function (with 2 wells, 1 saddle)

$$\tilde{U}(t,x), \quad t \geq 0, x \in \mathbb{R}^d,$$

$\tilde{U}(kT+t,\cdot) = \tilde{U}(t,\cdot)$, period T very large

white noise of intensity ε, W d–dimensional standard Wiener process

$Y(t) \leftarrow T(t)$; motion of particle given by SDE

$$dY^\varepsilon(t) = -\nabla \tilde{U}(t, Y^\varepsilon(t))dt + \sqrt{\varepsilon} dW(t).$$

To investigate interplay of T and ε: rescale time $\quad t \to \frac{t}{T}$, use $U(t,x) = \tilde{U}(\frac{t}{T}, x), t \geq 0, x \in \mathbb{R}^d$; U has period 1

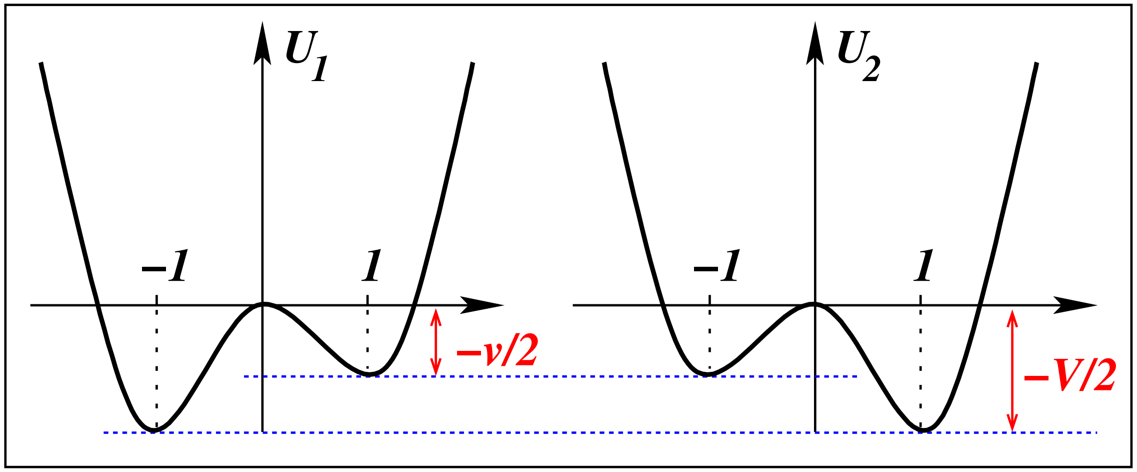
General system:

$$dX^\varepsilon(t) = -\nabla U(t, X^\varepsilon(t))dt + \sqrt{\varepsilon} dW(t).$$

5. Weakly periodic dynamical systems with noise

Use the following simple caricature of the potential function

$$U(t,x) = \begin{cases} U_1(x), & t \in [0, \frac{1}{2}[, \\ U_2(x) = U_1(-x), & t \in [\frac{1}{2}, 1[, \end{cases} \quad \text{periodic}$$

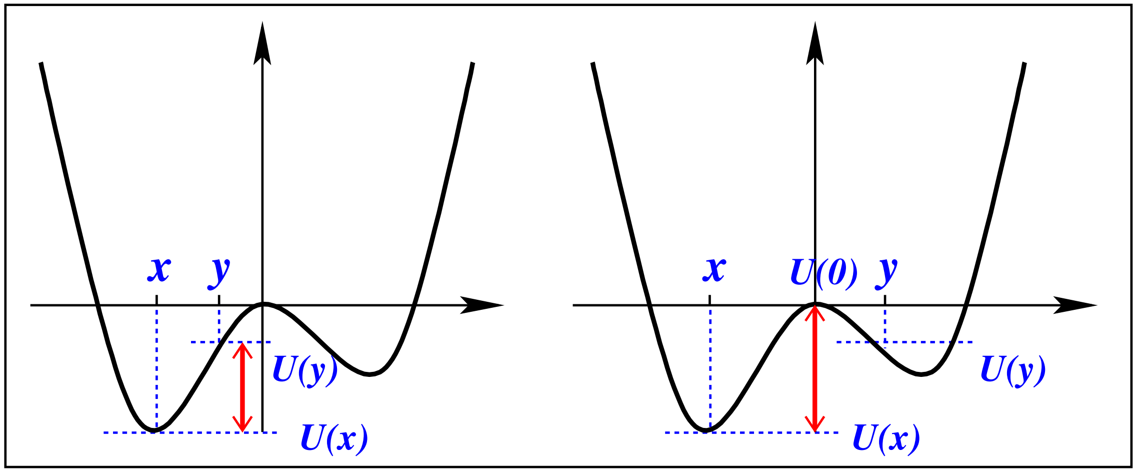


Problem of stochastic resonance:

given T large, find $\varepsilon = \varepsilon(T)$ for which tuning of (X^ε) is optimal

6. Gaussian transition times

action functional $\Rightarrow$ pseudopotential $V(x,y) = 2[\text{work } x \to y]$,



$$V(x,y) = 2[U(y) - U(x)]^+ \quad V(x,y) = 2[U(0) - U(x)]^+$$

$\tau_y^\varepsilon = \inf\{t \geq 0 : X_t^\varepsilon = y\}$ transition time

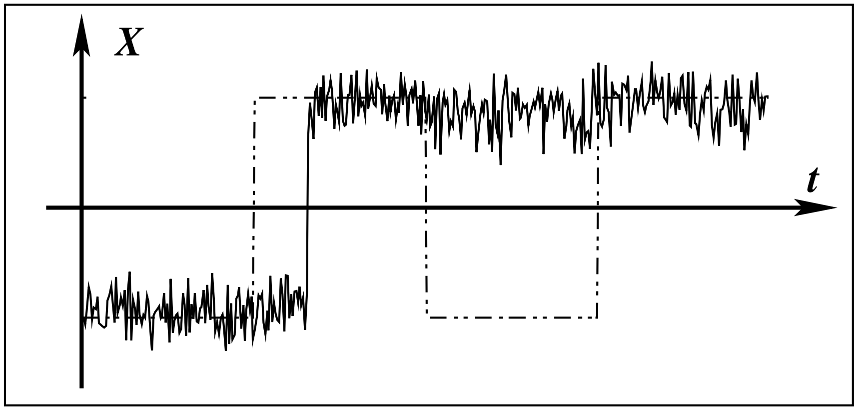
Thm 1 (Freidlin, transition law)
for all δ

$$\mathbf{P}_x \left[\exp\left(\frac{V(x,y) - \delta}{\varepsilon^2}\right) \leq \tau_y^\varepsilon \leq \exp\left(\frac{V(x,y) + \delta}{\varepsilon^2}\right) \right] \to_{\varepsilon \to 0} 1$$

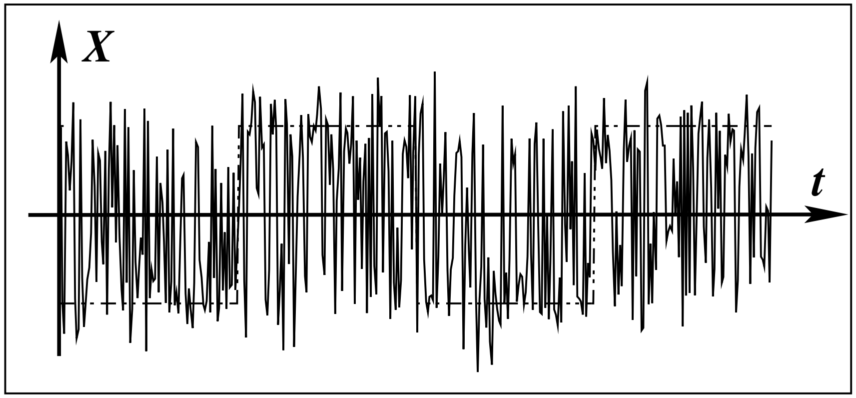
interprets Kramers-Eyring law: $E_x(\tau_y^\varepsilon) \sim \exp(\frac{V(x,y)}{\varepsilon^2})$

7. The heuristics of optimal tuning

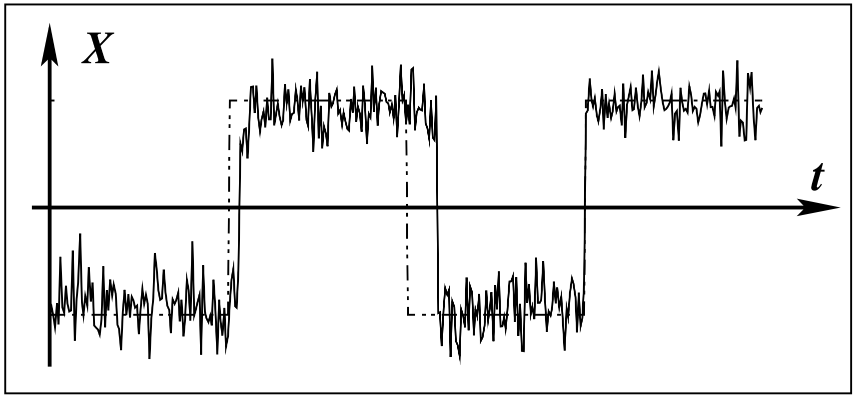
ε small



ε big



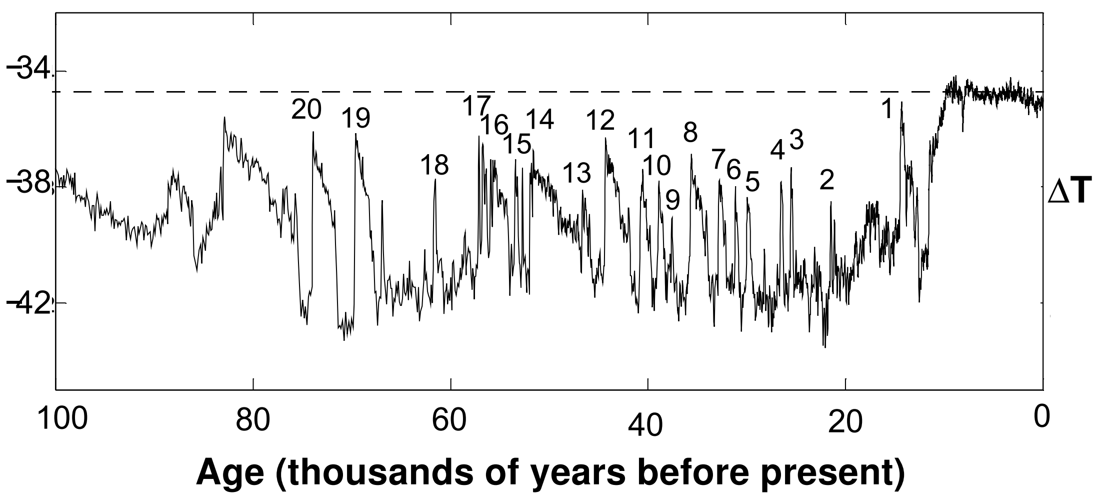
ε good



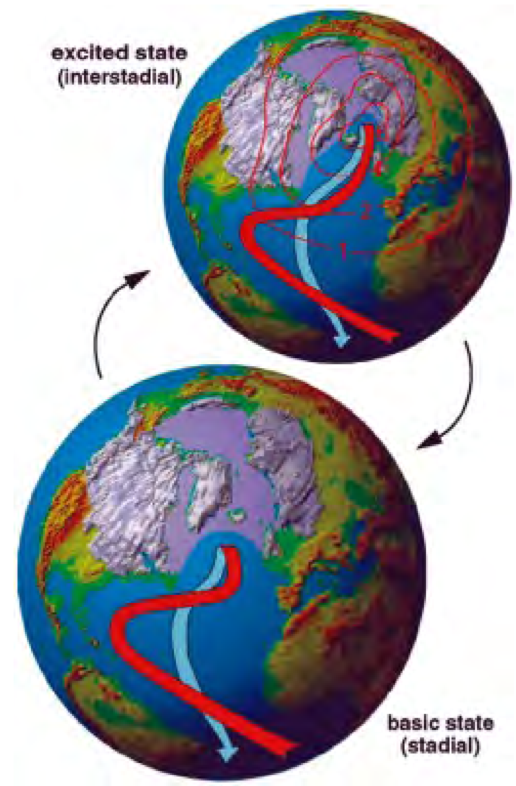
8. Dansgaard-Oeschger events

temperature indicators: ^{18}O, ^{16}O, methane, calcium etc.

GRIP ice core data: 20 abrupt changes in climate of Greenland during last ice age (-91 000 to -11 000 y) (D/O events).



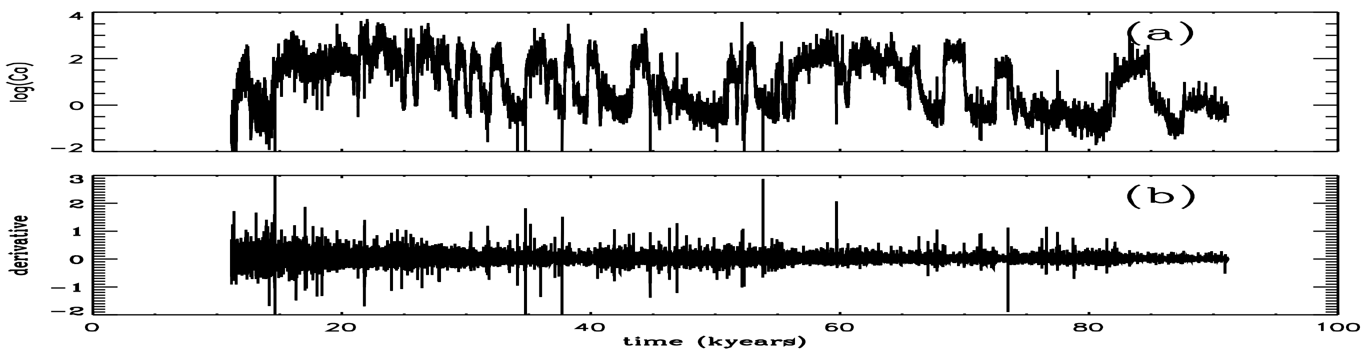
- rapid warming by 5-10°C within one decade
- subsequent slower cooling within a few centuries
- fast return to stable cold ground state



simulations: Ganopolsky/Rahmstorf, Potsdam Institute for Climate Impact Research

9. Dansgaard-Oeschger events. Statistical analysis

Calcium signal from GRIP: about 80 000 samples for 80 000 y



typical waiting time between D/O events: 1000 – 2000 y,
waiting times between D/O events: multiples of ∼1470 years.

What triggers the transitions?

modeling by Langevin equation:

$$dX(t) = -U'(t, X(t))dt + \text{NOISE}$$

U — multi well potential, wells correspond to climate states

P. Ditlevsen (*Geophys. Res. Lett. 1999*): power spectrum analysis of time series:
NOISE contains strong α-stable component with $\alpha \approx 1.75$.

10. p-Variation as test statistic

Which model of noise fits best with time series: estimate, test parameter

Ditlevsen's analysis: power spectrum of residua of time series

Problem: Stationarity?

Aim: better test statistics than peaks of power spectrum.

Model assumption: with some U interpret data as

$$X^\varepsilon(t) = x - \int_0^t U'(X^\varepsilon(s-))ds + \varepsilon L(t) = Y^\varepsilon(t) + L^\varepsilon(t)$$

L Lévy process containing α-stable component with unknown α, Y^ε of bounded variation; estimate, test α

Idea: p-variation characteristic for fluctuation behavior of noise processes.

$$V_t^{p,n}(X) = \sum_{i=1}^{[nt]} |X(\frac{i}{n}) - X(\frac{i-1}{n})|^p, \quad V_t^p = \lim_{n\to\infty} V_t^{p,n}$$

11. α-stable Lévy Processes

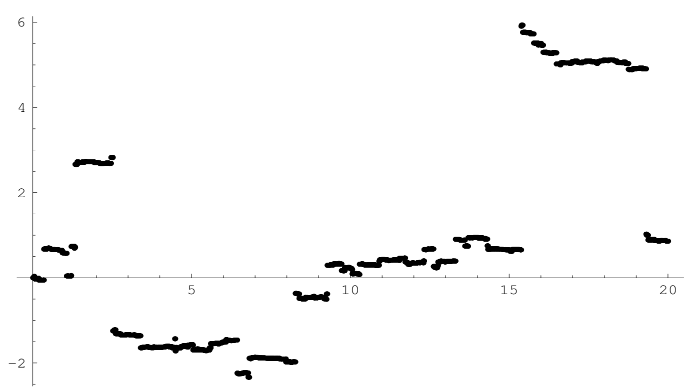
L Lévy process with characteristics (d, γ, ν) iff

$$E(\exp(iuL(t))) = \exp(t(-\frac{1}{2}du^2 + i\gamma u + \int_{\mathbf{R}} [e^{iuy} - 1 - iuy1_{\{|y|\leq 1\}}]\nu(dy))), \quad u \in \mathbf{R}, t \geq 0,$$

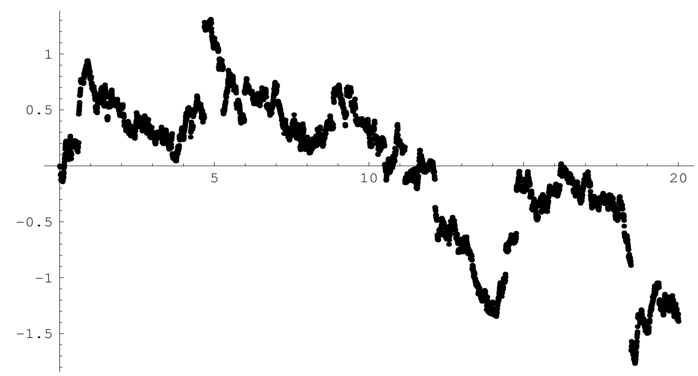
ν measure on Borel sets in $\mathbf{R}$ with $\nu(\{0\}) = 0$, $\int_{\mathbf{R}} [|y|^2 \wedge 1]\nu(dy) < \infty$.

L α-stable symmetric Lévy process if

$$E(\exp(iuL(t))) = \exp(-c(\alpha)t|u|^\alpha), \quad \nu(dy) = \frac{1}{|y|^{\alpha+1}}dy, \quad u, y \in \mathbf{R}.$$



$\alpha = 0.75$



$\alpha = 1.75$

12. p-Variation and the Blumenthal-Getoor Index

L α-stable process with jump measure ν; then p-variation identified by

Blumenthal-Getoor index

$$\beta_L = \inf\{s \geq 0 : \int_{\{|y|\leq 1\}} |y|^s \nu(dy) < \infty\}$$

$$\gamma_L = \inf\{p > 0 : V_1^p(L) < \infty\}$$

Thm 2
L symmetric α-stable. Then

$$\gamma_L = \beta_L = \alpha.$$

Problem: How to read $\gamma_L = \alpha$ off the sequence $(V_t^{p,n}(L))_{n\in\mathbf{N}}$?

Calls for results about the asymptotic behavior of the sequence.

13. The case $\alpha < 2$

(Lit: Corcuera, Nualart, Wörner '07; case $p < \alpha$ for LLN type, $p < \frac{\alpha}{2}$ for CLT type)

Problem: $p < \frac{\alpha}{2} < 1$ not satisfactory for paleo-climatic data! Beyond $\frac{\alpha}{2}$ no CLT type result available, no asymptotic normality, but asymptotically of different type.

Thm 3 (LLT type)
L α-stable with $\alpha \in]0,2[$. Then

$$(V_t^{p,n}(L) - B_t^n(\alpha,p))_{t\geq 0} \to \tilde{L}$$

weakly with respect to the Skorokhod metric, and an independent $\frac{\alpha}{p}$-stable process $\tilde{L}$. Here

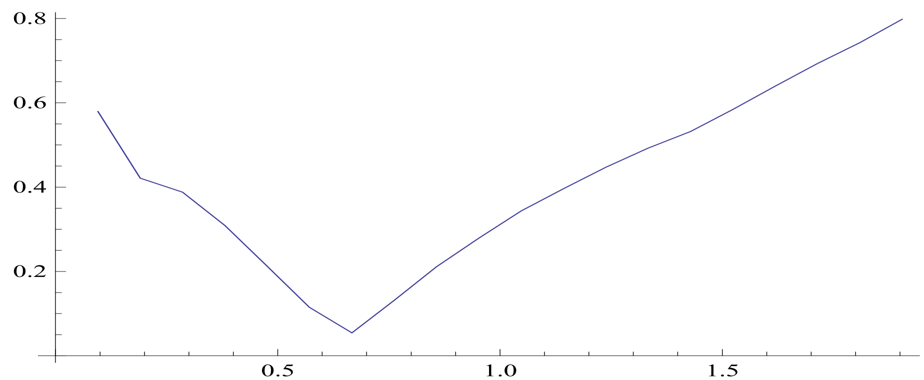
$$B_t^n(\alpha,p) = \begin{cases} n^{1-\frac{p}{\alpha}} t E(|L(1)|^p), & \frac{\alpha}{2} < p < \alpha, \\ nt^2 E(\sin((nt)^{-1}|L(1)|^p)), & p = \alpha, \\ 0, & \alpha < p. \end{cases}$$

Same result with $L + Y$ instead of L if Y is of finite p-variation and $\frac{\alpha}{2} < p < 1$ or $p > \alpha$.

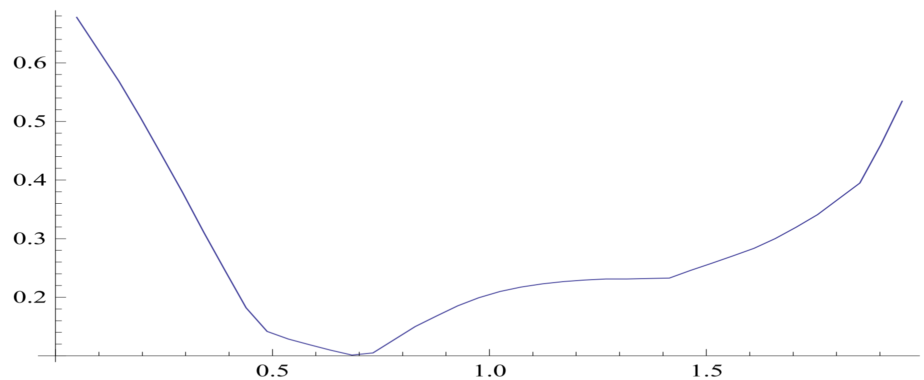
14. Test for α with real and simulated data

Thm 4 law of $V^{2p,n}(X)$ converges to $\frac{1}{2}$-stable law if data of time series X have α-stable residuals, $\alpha = p$

Kolmogorov-Smirnov statistics: distance between empirical law of $V^{2p,n}(X)$ and $\frac{1}{2}$-stable law, as a function of p; minimum of curve: right α



simulated time series of a 0.6-stable Levy process, $n = 200$



real time series from the Greenland ice, $n = 200$

15. The dynamics: simple system with Levy noise

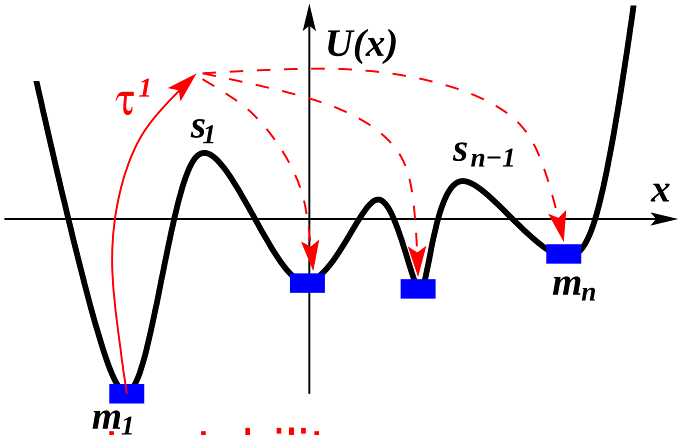
consider SDE driven by α-stable Lévy noise of small intensity

$$X^\varepsilon(t) = x - \int_0^t U'(X^\varepsilon(s-))\, ds + \varepsilon L(t), \quad \varepsilon \downarrow 0.$$

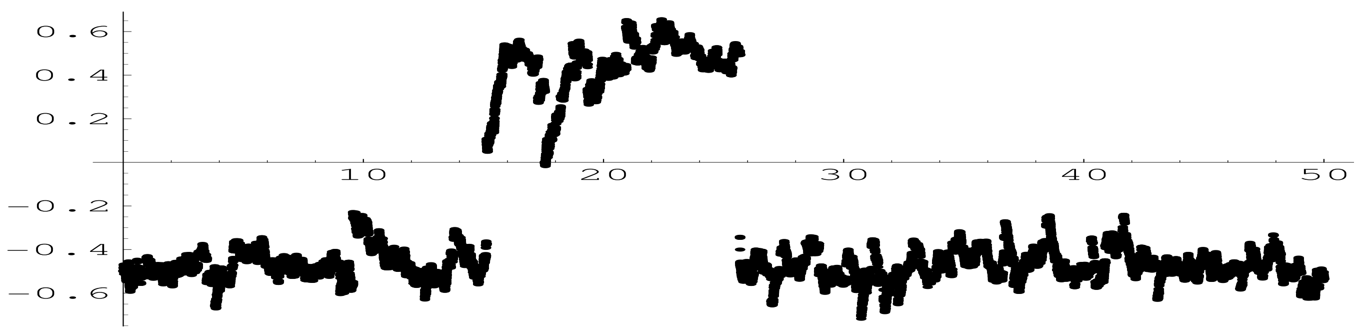
- L is α-stable symmetric Lévy process, $\alpha \in (0,2)$

multi well potential U

- n local minima m_i
- $n-1$ local maxima s_i
- $U''(m_i) > 0$, $U''(s_i) < 0$

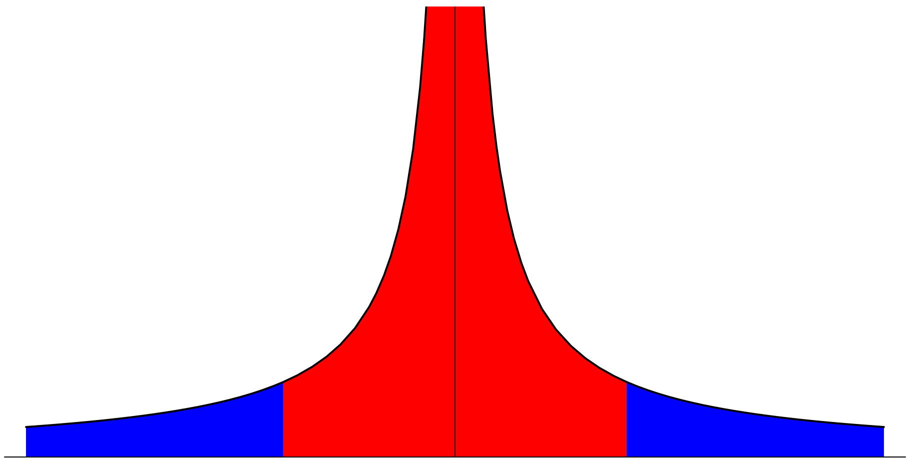


aim: investigate exit and transition rates, meta-stability.



16. The dynamics: probabilistic approach of exit times

$$L(t) = \xi^\varepsilon(t) + \eta^\varepsilon(t)$$



$$\nu_\xi^\varepsilon = \nu\big|_{[-\frac{1}{\sqrt{\varepsilon}},\frac{1}{\sqrt{\varepsilon}}]}, \qquad \nu_\eta^\varepsilon = \nu\big|_{[-\frac{1}{\sqrt{\varepsilon}},\frac{1}{\sqrt{\varepsilon}}]^c}$$

$$\nu_\xi^\varepsilon(\mathbb{R}) = \infty \qquad \nu_\eta^\varepsilon(\mathbb{R}) = \frac{2}{\alpha}\varepsilon^{\alpha/2} = \beta_\varepsilon$$

$\varepsilon\xi^\varepsilon$ sum of ε·BM and small jump ($\leq \sqrt{\varepsilon}$) process

$\varepsilon\eta^\varepsilon$ big jump ($\geq \sqrt{\varepsilon}$) compound Poisson process

big jumps at τ_k, inter-jump time T_k with exponential law

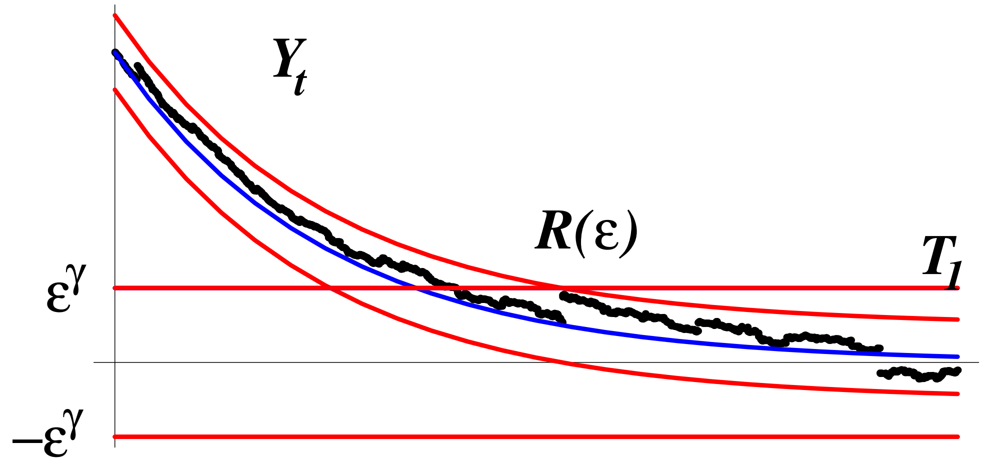
$$E(T_k) = (\beta^\varepsilon)^{-1} = \frac{\alpha}{2}\varepsilon^{-\alpha/2}$$

17. The dynamics: the small and large jump parts

U with stable state 0, exit from $[-b, a]$ for $a, b > 0$

between big jumps X^ε is Y perturbed by $\varepsilon\xi^\varepsilon$

$$X^\varepsilon(t) = x - \int_0^t U'(X^\varepsilon(s-))\,ds + \varepsilon\xi^\varepsilon(t), \quad t \in [0, T_1), \quad Y(t) = x - \int_0^t U'(Y(s))\,ds$$



deviation $$\mathbf{P}\left(\sup_{[0,T_1)} |X^\varepsilon(t) - Y(t)| \ge \frac{\varepsilon^\gamma}{2}\right) \le \mathbf{P}\left(\sup_{[0,T_1)} |\varepsilon\xi^\varepsilon(t)| \ge \frac{\varepsilon^\gamma}{C}\right) \le e^{-1/\varepsilon^\delta}$$

relaxation $$T(x,\varepsilon) = \int_{\varepsilon^\gamma/2}^x \frac{dy}{|U'(y)|} \approx \int_\delta^x \frac{dy}{|U'(y)|} + \int_{\varepsilon^\gamma/2}^\delta \frac{dy}{My}$$

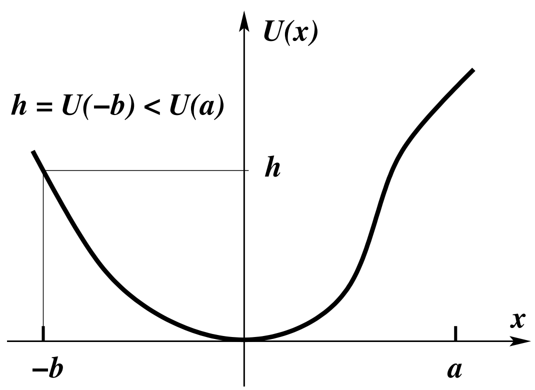
$$\approx \text{Const} + \frac{\gamma}{M}|\ln\varepsilon| \le R(\varepsilon) = \mathcal{O}(|\ln\varepsilon|)$$

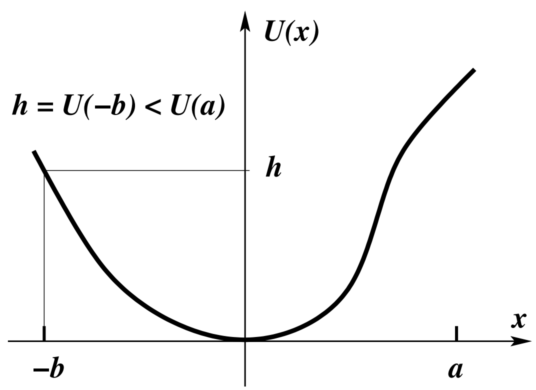
asymptotically, big jumps coincide with exits

18. the dynamics: Gaussian vs. Lévy

$$\hat{\sigma} = \inf\{t \geq 0 \,:\, \hat{X}^\varepsilon(t) \notin [-b, a]\}$$

$$\sigma = \inf\{t \geq 0 \,:\, X^\varepsilon(t) \notin [-b, a]\}$$





$$\hat{X}^\varepsilon(t) = x - \int_0^t U'(\hat{X}^\varepsilon(t))\,ds + \varepsilon W(t)$$

$$X^\varepsilon(t) = x - \int_0^t U'(X^\varepsilon(s-))\,ds + \varepsilon L(t)$$

Thm 5 (Freidlin-Wentzell):

$$\mathbf{P}_x(e^{(2h-\delta)/\varepsilon^2} < \hat{\sigma} < e^{(2h+\delta)/\varepsilon^2}) \to 1$$

Thm 6

$$\mathbf{P}_x(\tfrac{1}{\varepsilon^{\alpha-\delta}} < \sigma < \tfrac{1}{\varepsilon^{\alpha+\delta}}) \to 1$$

Kramers' law ('40, Williams, Bovier et al.):

$$\mathbf{E}_x\hat{\sigma} \approx \frac{\varepsilon\sqrt{\pi}}{|U'(-b)|\sqrt{U''(0)}} e^{2h/\varepsilon^2}$$

$$\mathbf{E}_x\sigma \approx \frac{1}{\varepsilon^\alpha}\Big(\int_{\mathbb{R}\setminus[-b,a]} \frac{dy}{|y|^{1+\alpha}}\Big)^{-1}$$

Exponential law (Day, Bovier et al.)

$$\mathbf{P}_x(\tfrac{\hat{\sigma}}{\mathbf{E}_x\hat{\sigma}} > u) \sim \exp(-u)$$

$$\mathbf{P}_x(\tfrac{\sigma}{\mathbf{E}_x\sigma} > u) \sim \exp(-u)$$