Bound states in perturbation theory

Paul Hoyer*
Department of Physics, POB 64, FIN-00014 University of Helsinki, Finland†

A perturbative expansion for QED and QCD bound states is formulated in $A^0 = 0$ gauge. The constituents of each Fock state are bound by their instantaneous interaction. In QCD an $\mathcal{O}\left(\alpha_s^0\right)$ confining potential arises from a homogeneous solution of Gauss' constraint. The Cornell potential is obtained for quarkonia, and corresponding ones for higher Fock states, baryons and glueballs. Light mesons and glueballs lie on linear Regge trajectories and their daughters. A condensate of massless $J^{PC} = 0^{++}$ bound states allows to include spontaneous chiral symmetry breaking.

Principles of bound state perturbation theory

There is a widely held opinion that QED atoms and QCD hadrons differ fundamentally, the former being perturbative ($\alpha \ll 1$) and the latter non-perturbative ($\alpha_s \gtrsim 1$). I shall argue that this conclusion may be premature. Bound state perturbation theory differs essentially from the expansion of scattering amplitudes in terms of Feynman diagrams. Atoms are not as perturbative, and hadrons not as non-perturbative as sometimes thought.

Even "lowest order" bound state wave functions are non-polynomial in the perturbative parameter α. Wave functions have no unique expansion, since powers of α may be shifted between the lowest term and its "higher order" corrections. This was first recognized for the Bethe-Salpeter equation [1, 2]. For Positronium it allows to start from solutions of the Schrödinger equation with the classical potential $V(r) = -\alpha/r$ [3]. For hadrons the first term in a perturbative expansion needs to include color confinement and chiral symmetry breaking.

Bound states cannot be properly derived from Feynman diagrams, which assume non-interacting (*in* and *out*) states at asymptotic times. Formally, there is no overlap between free and bound states because bound states have finite size. Consequently, no Feynman diagram can have a bound state pole. Having quarks and gluons infinitely separated in the initial and final states is inconsistent with confinement in QCD.

Hadron dynamics is known to have atomic (perturbative) features. This is apparent for heavy quarkonia, which are well described by the Schrödinger equation with the Cornell potential [4, 5],

$$V(r) = V'r - \frac{4}{3}\frac{\alpha_s}{r} \quad \text{with} \quad V' \simeq 0.18 \text{ GeV}^2, \quad \alpha_s \simeq 0.39 \tag{1}$$

This potential was determined from fits to data and later found to agree with lattice QCD [6]. At face value, it indicates that the confinement scale V' arises already in the classical potential, with a gluon coupling α_s that is close to the perturbative $\alpha_s(m_\tau) \simeq 0.33$.

There are many ways to define bound state wave functions, and they are often gauge-dependent. Hence their perturbative expansion is not unique. On the other hand, bound state *masses* must have a unique expansion in powers of α and $\log\alpha$ since they are physical quantities (the Positronium expansion is compared with data in [7, 8]). In a proper perturbative approach each successive approximation of the bound state (wave function) should contribute to the mass at a corresponding, higher power of the coupling.

The hierarchy of bound state approximations is reflected in the Fock expansion of Positronium. The $|e^+e^-\rangle$ Fock state determines (through the Schrödinger equation) the binding energy at lowest order, $E_b = -\alpha^2 m_e/4$. Fock states with a transverse photon $|e^+e^-\gamma\rangle$, additional pairs $|e^+e^-e^+e^-\rangle$, *etc.* contribute to E_b at $\mathcal{O}\left(\alpha^4\right)$ and higher. This is possible because the $|e^+e^-\rangle$ state is bound by the instantaneous A^0 field, which is not a constituent. The instantaneous interaction similarly determines the properties of higher Fock states. Thus the Fock expansion defines a perturbative approach to bound states, with no ambiguity concerning the initial approximation.

A Fock approach may work also for hadrons. The quarkonium phenomenology based on (1) indicates that the state is dominated by the heavy quark pair. The reason is dynamical, since binding energies are much larger than the light quark and gluon masses. The quantum numbers of light hadrons with relativistically bound quarks likewise reflect their valence constituents, $|q\bar{q}\rangle$ and $|qqq\rangle$.

Having defined the perturbative framework we may ask whether it is compatible with color confinement. The confining potential must be due to the instantaneous interaction which binds the $|q\bar{q}\rangle$ quarkonia (see also [9]). The A^0 field is determined by its equation of motion (Gauss' law), which does not have the QCD scale. In the absence of loop contributions (which involve higher Fock states) the scale can arise only from a boundary condition. It turns out that there is a reason for choosing different boundary conditions for atoms and hadrons.

The $V(r) = -\alpha/r$ potential of the Positronium $|e^-(\boldsymbol{x}_1)e^+(\boldsymbol{x}_2)\rangle$ Fock state ($r = |\boldsymbol{x}_1 - \boldsymbol{x}_2|$) follows from Gauss' law, $-\boldsymbol{\nabla}^2 A^0(\boldsymbol{x}) = e\delta(\boldsymbol{x}-\boldsymbol{x}_1) - e\delta(\boldsymbol{x}-\boldsymbol{x}_2)$. The boundary condition $\lim_{|\boldsymbol{x}|\to\infty} A^0(\boldsymbol{x}) = 0$ eliminates long range interactions. The Fock state is invariant under global U(1) gauge transformations, which also leave A^0 invariant.

A color singlet hadron is similarly invariant under global SU(3) transformations, but the octet gluon field is not. Consequently a hadron's classical field $A_a^0(\boldsymbol{x}) = 0$ for all $\boldsymbol{x}$, and there is no need for a boundary condition as in QED. A $\left|q^C \bar{q}^C\right\rangle$ Fock state does generate a classical field for each quark color C, but it cancels in the sum over C. The quark q^C is bound by the field of the $\bar{q}^C$. That field is non-vanishing and depends on the boundary condition.

Homogeneous (sourceless) solutions of Gauss' law for the $\left|q^C \bar{q}^C\right\rangle$ Fock state have $A_a^0(\boldsymbol{x}) \neq 0$ at spatial infinity. Translation and rotation invariance restrict the sourceless field up to a single parameter Λ, which characterizes its $\boldsymbol{x}$-independent field energy density. The corresponding potential is linear for $|q\bar{q}\rangle$ (meson) and gg (glueball) states. The potential is confining also for $|qqq\rangle$ (baryons) and higher meson Fock states such as $|q\bar{q}g\rangle$.

Loop contributions are suppressed insofar as the lowest Fock state dominates at hadronic scales $Q \lesssim \Lambda$. In this energy range the coupling $\alpha_s(Q)$ is then independent of Q, which is consistent with its moderate value in (1).

In the following we illustrate our approach using QED atoms and then summarize the results obtained in QCD. A full description is given in [10].

Positronium in QED

We use temporal gauge ($A^0 = 0$) [11–15] since it reveals the hierarchy of Fock states. The electric fields $E^i = F^{i0} = -\partial_0 A^i$ are conjugate to the photon fields A^i ($i = 1, 2, 3$). Gauss' operator $G(\boldsymbol{x})$, defined by

$$G(t, \boldsymbol{x}) \equiv \frac{\delta \mathcal{S}}{\delta A^0(t, \boldsymbol{x})} = \partial_i E^i(t, \boldsymbol{x}) - e\psi^\dagger\psi(t, \boldsymbol{x}) \tag{2}$$

is time independent since it commutes with the Hamiltonian, $[G, \mathcal{H}] = 0$. Because A^0 is eliminated G does not vanish as an operator. However, $G(t, \boldsymbol{x})$ generates time-independent gauge transformations, which are unconstrained by the $A^0 = 0$ gauge condition. The gauge is completely fixed by imposing Gauss' law as a *constraint* on physical states [12],

$$G(t, \boldsymbol{x})\, |phys\rangle = 0 \tag{3}$$

The e^+e^- Fock component of Positronium may be expressed as (henceforth $t = 0$ is implicit),

$$\left|e^+e^-, M\right\rangle = \sum_{\alpha,\beta} \int d\boldsymbol{x}_1 d\boldsymbol{x}_2\, \bar{\psi}_\alpha(\boldsymbol{x}_1)\Phi_{\alpha\beta}(\boldsymbol{x}_1 - \boldsymbol{x}_2)\psi_\beta(\boldsymbol{x}_2)\,|0\rangle \equiv \int d\boldsymbol{x}_1 d\boldsymbol{x}_2\, \Phi(\boldsymbol{x}_1 - \boldsymbol{x}_2)\,|\boldsymbol{x}_1, \boldsymbol{x}_2\rangle \tag{4}$$

where $\psi(\boldsymbol{x})$ is the electron field and the wave function $\Phi(\boldsymbol{x}_1 - \boldsymbol{x}_2)$ is a 4×4 matrix in the Dirac indices α, β. The state (4) is invariant under space translations, as appropriate for the rest frame. In the non-relativistic limit only the electron and positron creation operators contribute ($b^\dagger$ in $\bar{\psi}$ and $d^\dagger$ in ψ), and the wave function Φ is simply related to the Schrödinger wave function [10].

Imposing the constraint (3) on the state $|\boldsymbol{x}_1, \boldsymbol{x}_2\rangle = \bar{\psi}_\alpha(\boldsymbol{x}_1)\psi_\beta(\boldsymbol{x}_2)\,|0\rangle$ determines the action of $\partial_i E^i(t, \boldsymbol{x})$ on this state,

$$\partial_i E_L^i(\boldsymbol{x})\,|\boldsymbol{x}_1, \boldsymbol{x}_2\rangle = e\psi^\dagger\psi(\boldsymbol{x})\,|\boldsymbol{x}_1, \boldsymbol{x}_2\rangle = e\big[\delta(\boldsymbol{x} - \boldsymbol{x}_1) - \delta(\boldsymbol{x} - \boldsymbol{x}_2)\big]\,|\boldsymbol{x}_1, \boldsymbol{x}_2\rangle \tag{5}$$

where the index L reminds that this constrains only the longitudinal component of $\boldsymbol{E}$. Requiring $\lim_{|\boldsymbol{x}|\to\infty} E_L^i(\boldsymbol{x}) = 0$,

$$E_L^i(\boldsymbol{x})\,|\boldsymbol{x}_1, \boldsymbol{x}_2\rangle = -\frac{e}{4\pi}\,\partial_i^x\Big(\frac{1}{|\boldsymbol{x} - \boldsymbol{x}_1|} - \frac{1}{|\boldsymbol{x} - \boldsymbol{x}_2|}\Big)\,|\boldsymbol{x}_1, \boldsymbol{x}_2\rangle \tag{6}$$

The QED Hamiltonian in temporal gauge is

$$\mathcal{H} = \int d\boldsymbol{x}\big[\tfrac{1}{2}(E_L^i E_L^i + E_T^i E_T^i) + \tfrac{1}{4}F^{ij}F^{ij} + \psi^\dagger(-i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} - e\boldsymbol{\alpha}\cdot\boldsymbol{A}_T + m\gamma^0)\psi\big] \tag{7}$$

The contribution of $\boldsymbol{E}_L^2$ is determined by (6). Partially integrating in $\boldsymbol{x}$ we get the classical potential[1],

$$\int d\boldsymbol{x}\, \tfrac{1}{2}\boldsymbol{E}_L^2(\boldsymbol{x})\,|\boldsymbol{x}_1,\boldsymbol{x}_2\rangle = -\frac{\alpha}{|\boldsymbol{x}_1-\boldsymbol{x}_2|}\,|\boldsymbol{x}_1,\boldsymbol{x}_2\rangle \equiv V(|\boldsymbol{x}_1-\boldsymbol{x}_2|)\,|\boldsymbol{x}_1,\boldsymbol{x}_2\rangle \tag{8}$$

Adding the contribution from the free fermion part of the Hamiltonian (7) and neglecting the higher Fock states created by $\psi^\dagger \boldsymbol{A}_T\psi$ we can impose stationarity on the Positronium state (4), $\mathcal{H}\,|e^+e^-,M\rangle = M\,|e^+e^-,M\rangle$. This gives the bound state equation for $\Phi(\boldsymbol{x})$,

$$\big[i\boldsymbol{\alpha}\cdot\overrightarrow{\boldsymbol{\nabla}} + m\gamma^0\big]\Phi(\boldsymbol{x}) + \Phi(\boldsymbol{x})\big[i\boldsymbol{\alpha}\cdot\overleftarrow{\boldsymbol{\nabla}} - m\gamma^0\big] = \big[M - V(|\boldsymbol{x}|)\big]\Phi(\boldsymbol{x}) \tag{9}$$

which in the non-relativistic limit reduces to the Schrödinger equation [10]. The Fock states $|e^+e^-\gamma\rangle$, $|e^+e^-e^+e^-\rangle\,,\ldots$ created by $\mathcal{H}$ contribute to M at higher orders in α, and their instantaneous potentials can be determined similarly as for $|e^+e^-\rangle$. Because Gauss' law is imposed as a constraint in temporal gauge the instantaneous interaction does not create new states. In Coulomb gauge ($\boldsymbol{\nabla}\cdot\boldsymbol{A}=0$) the operator A^0 can create an e^+e^- pair.

Hadrons in QCD

The $q\bar{q}$ Fock component of a meson state may be expressed similarly as for Positronium in (4),

$$|q\bar{q},M\rangle = \frac{1}{\sqrt{N_C}}\sum_{\alpha,\beta}\sum_{A,B}\int d\boldsymbol{x}_1 d\boldsymbol{x}_2\,\bar{\psi}_\alpha^A(\boldsymbol{x}_1)\delta^{AB}\Phi_{\alpha\beta}(\boldsymbol{x}_1-\boldsymbol{x}_2)\psi_\beta^B(\boldsymbol{x}_2)\,|0\rangle \equiv \int d\boldsymbol{x}_1 d\boldsymbol{x}_2\,\Phi(\boldsymbol{x}_1-\boldsymbol{x}_2)\,|\boldsymbol{x}_1,\boldsymbol{x}_2\rangle \tag{10}$$

The state is invariant under global gauge transformations since the wave function $\delta^{AB}\Phi_{\alpha\beta}(\boldsymbol{x})$ is diagonal in the quark colors A,B. In the temporal gauge of QCD ($A_a^0=0$) [12–15] Gauss constraint (3) has also a gluon contribution,

$$\partial_i E_{L,a}^i(\boldsymbol{x})\,|phys\rangle = g\big[-f_{abc}A_b^iE_c^i + \psi^\dagger T^a\psi(\boldsymbol{x})\big]\,|phys\rangle \tag{11}$$

The solution of this constraint for $\boldsymbol{E}_L$ is not unique due to contributions from large gluon fields (Gribov copies [16]). This does not concern the perturbative expansion in g which we consider here. At lowest order we may neglect the contribution of $\boldsymbol{E}_L$ on the rhs. of (11). For the $q\bar{q}$ Fock components defined in (10) (suppressing Dirac indices),

$$\partial_i E_{L,a}^i(\boldsymbol{x})\,|\boldsymbol{x}_1,\boldsymbol{x}_2\rangle = g\bar{\psi}_A(\boldsymbol{x}_1)T_{AB}^a\psi_B(\boldsymbol{x}_2)\big[\delta(\boldsymbol{x}-\boldsymbol{x}_1)-\delta(\boldsymbol{x}-\boldsymbol{x}_2)\big]\,|0\rangle \tag{12}$$

In QED the component $|\boldsymbol{x}_1,\boldsymbol{x}_2\rangle$ of Positronium gives rise to the dipole electric field (6). The meson state (10) on the other hand cannot create a classical, color octet electric field $\boldsymbol{E}_{L,a}(\boldsymbol{x})$ at *any* $\boldsymbol{x}$ since it is a color singlet under global gauge transformations. The expectation value of $\partial_i E_{L,a}^i(\boldsymbol{x})$ in the color C component of $|\boldsymbol{x}_1,\boldsymbol{x}_2\rangle$ is

$$\langle 0|\psi_\beta^{C\dagger}(\boldsymbol{x}_2)\gamma^0\psi_\alpha^C(\boldsymbol{x}_1)|\partial_i E_{L,a}^i(\boldsymbol{x})|\bar{\psi}_\alpha^C(\boldsymbol{x}_1)\psi_\beta^C(\boldsymbol{x}_2)\,|0\rangle \propto g\big[\delta(\boldsymbol{x}-\boldsymbol{x}_1)-\delta(\boldsymbol{x}-\boldsymbol{x}_2)\big]T_{CC}^a \tag{13}$$

An external observer does not see any color field since the sum over the quark colors C vanishes, $\mathrm{Tr}\,T^a=0$. On the other hand, the color C quark interacts only with the color C antiquark in the same Fock state, and thus experiences a non-vanishing $\boldsymbol{E}_L$. This motivates adding a homogeneous ($\partial_i E_{L,a}^i(\boldsymbol{x})=0$) term to the solution of (11),

$$E_{L,a}^i(\boldsymbol{x})\,|phys\rangle = -\partial_i^x\int d\boldsymbol{y}\Big[\kappa\,\boldsymbol{x}\cdot\boldsymbol{y} + \frac{g}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\Big]\mathcal{E}_a(\boldsymbol{y})\,|phys\rangle$$
$$\mathcal{E}_a(\boldsymbol{y}) = -f_{abc}A_b^iE_c^i(\boldsymbol{y}) + \psi^\dagger T^a\psi(\boldsymbol{y}) \tag{14}$$

with a normalization κ that is independent of $\boldsymbol{x}$ and $\boldsymbol{y}$. Since $\partial_i^x(\kappa\,\boldsymbol{x}\cdot\boldsymbol{y}) = \kappa\,y^i$ the field energy density of this term is independent of $\boldsymbol{x}$, as required by translation invariance for a sourceless contribution.

The QCD Hamiltonian in temporal gauge is

$$\mathcal{H} = \int d\boldsymbol{x}\big[\tfrac{1}{2}E_{L,a}^iE_{L,a}^i + \tfrac{1}{2}E_{T,a}^iE_{T,a}^i + \tfrac{1}{4}F_a^{ij}F_a^{ij} + \psi^\dagger(-i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + m\gamma^0 - g\boldsymbol{\alpha}\cdot\boldsymbol{A}_T^aT^a)\psi\big] \tag{15}$$

---

[1] The "self-energy" contributions proportional to $1/|\boldsymbol{x}_1-\boldsymbol{x}_1|$ and $1/|\boldsymbol{x}_2-\boldsymbol{x}_2|$ are independent of $\boldsymbol{x}_1,\boldsymbol{x}_2$ and may be subtracted.

where $F_{ij}^a = \partial_i A_j^a - \partial_j A_i^a - g f_{abc} A_i^b A_j^c$. From (14) the longitudinal electric field contributes

$$\mathcal{H}_V \equiv \tfrac{1}{2}\int d\boldsymbol{x}\, E_{a,L}^i E_{a,L}^i = \tfrac{1}{2}\int d\boldsymbol{x} \Big\{\partial_i^x \int d\boldsymbol{y}\Big[\kappa\,\boldsymbol{x}\cdot\boldsymbol{y} + \frac{g}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\Big]\mathcal{E}_a(\boldsymbol{y})\Big\}\Big\{\partial_i^x \int d\boldsymbol{z}\Big[\kappa\,\boldsymbol{x}\cdot\boldsymbol{z} + \frac{g}{4\pi|\boldsymbol{x}-\boldsymbol{z}|}\Big]\mathcal{E}_a(\boldsymbol{z})\Big\}$$

$$= \int d\boldsymbol{y} d\boldsymbol{z} \Big\{\boldsymbol{y}\cdot\boldsymbol{z}\Big[\tfrac{1}{2}\kappa^2 \textstyle\int d\boldsymbol{x} + g\kappa\Big] + \tfrac{1}{2}\dfrac{\alpha_s}{|\boldsymbol{y}-\boldsymbol{z}|}\Big\}\mathcal{E}_a(\boldsymbol{y})\mathcal{E}_a(\boldsymbol{z}) \equiv \mathcal{H}_V^{(0)} + \mathcal{H}_V^{(1)} \tag{16}$$

where the terms of $\mathcal{O}\left(g\kappa, g^2\right)$ were integrated by parts.

The states $|\boldsymbol{x}_1, \boldsymbol{x}_2\rangle = \sum_A \bar{\psi}_\alpha^A(\boldsymbol{x}_1)\psi_\beta^A(\boldsymbol{x}_2)\,|0\rangle$ are eigenstates of $\mathcal{H}_V$ since

$$\sum_a \mathcal{E}_a(\boldsymbol{y})\mathcal{E}_a(\boldsymbol{z})\,|\boldsymbol{x}_1, \boldsymbol{x}_2\rangle = C_F\big[\delta(\boldsymbol{y}-\boldsymbol{x}_1) - \delta(\boldsymbol{y}-\boldsymbol{x}_2)\big]\big[\delta(\boldsymbol{z}-\boldsymbol{x}_1) - \delta(\boldsymbol{z}-\boldsymbol{x}_2)\big]\,|\boldsymbol{x}_1, \boldsymbol{x}_2\rangle \tag{17}$$

where $C_F = (N^2-1)/2N = 4/3$ for $N = N_C = 3$. For $\mathcal{H}_V^{(0)}$ in (16) this gives,

$$\mathcal{H}_V^{(0)}\,|\boldsymbol{x}_1, \boldsymbol{x}_2\rangle = C_F\big[\tfrac{1}{2}\kappa^2 \textstyle\int d\boldsymbol{x} + g\kappa\big](\boldsymbol{x}_1 - \boldsymbol{x}_2)^2\,|\boldsymbol{x}_1, \boldsymbol{x}_2\rangle \tag{18}$$

The $\mathcal{O}\left(\kappa^2\right)$ contribution arises from the field energy and is proportional to the volume of space. It is irrelevant only if it is the same for all bound state components. This requires

$$\kappa = \frac{\Lambda^2}{g C_F}\frac{1}{|\boldsymbol{x}_1 - \boldsymbol{x}_2|} \tag{19}$$

We extract the factor gC_F from the definition of the universal scale Λ in order to simplify the coefficient of the confining potential. The $g\kappa$ term in (18) gives

$$V^{(0)}(|\boldsymbol{x}_1 - \boldsymbol{x}_2|) = g C_F \kappa\,(\boldsymbol{x}_1 - \boldsymbol{x}_2)^2 = \Lambda^2 |\boldsymbol{x}_1 - \boldsymbol{x}_2| \tag{20}$$

The gluon exchange potential arising from $\mathcal{H}_V^{(1)}$ in (16) is $V^{(1)} = -C_F\alpha_s/|\boldsymbol{x}_1 - \boldsymbol{x}_2|$. Neglecting higher Fock states the stationarity condition $\mathcal{H}\,|q\bar{q}, M\rangle = M\,|q\bar{q}, M\rangle$ gives the bound state equation (9) for the (color reduced) wave function in (10), with $V = V^{(0)} + V^{(1)}$. In the non-relativistic limit this agrees with the approach to quarkonia based on the Cornell potential (1) [4, 5].

At $\mathcal{O}\left(\alpha_s^0\right)$ only the linear potential (20) and the $\bar{\psi}\psi$ state (10) contribute, even for light (relativistic) quarks[2]. Fock states with transverse gluons such as $|q\bar{q}g\rangle$ are suppressed by factors of g. See [10] for a study of the states determined by the relativistic bound state equation (9).

The instantaneous potential is analogously derived for any state using (14). The field energy density, *i.e.*, the $\mathcal{O}\left(\kappa^2\right)$ term in the Hamiltonian $\mathcal{H}_V$ (16), must be the same for all states, making the scale Λ universal. Three examples [10]:

$$|gg\rangle = A_{a,T}^i(\boldsymbol{x}_1)\,A_{a,T}^j(\boldsymbol{x}_2)\,|0\rangle: \;\; V_{gg}(\boldsymbol{x}_1, \boldsymbol{x}_2) = \sqrt{\frac{N}{C_F}}\,\Lambda^2\,|\boldsymbol{x}_1 - \boldsymbol{x}_2| - N\,\frac{\alpha_s}{|\boldsymbol{x}_1 - \boldsymbol{x}_2|} \tag{21}$$

$$|qqq\rangle = \epsilon_{ABC}\psi_\alpha^{A\dagger}(\boldsymbol{x}_1)\,\psi_\beta^{B\dagger}(\boldsymbol{x}_2)\,\psi_\gamma^{C\dagger}(\boldsymbol{x}_1)\,|0\rangle: \;\; \text{With} \;\; d_{qqq}(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3) \equiv \frac{1}{\sqrt{2}}\sqrt{(\boldsymbol{x}_1 - \boldsymbol{x}_2)^2 + (\boldsymbol{x}_2 - \boldsymbol{x}_3)^2 + (\boldsymbol{x}_3 - \boldsymbol{x}_1)^2}\,,$$

$$V_{qqq}(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3) = \Lambda^2 d_{qqq}(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3) - \frac{2}{3}\,\alpha_s\Big(\frac{1}{|\boldsymbol{x}_1 - \boldsymbol{x}_2|} + \frac{1}{|\boldsymbol{x}_2 - \boldsymbol{x}_3|} + \frac{1}{|\boldsymbol{x}_3 - \boldsymbol{x}_1|}\Big) \tag{22}$$

$$|qgq\rangle = \bar{\psi}_A(\boldsymbol{x}_1)\,A_{b,T}^j(\boldsymbol{x}_g)T_{AB}^b\psi_B(\boldsymbol{x}_2)\,|0\rangle: \;\; \text{With} \;\; d_{qgq}(\boldsymbol{x}_1, \boldsymbol{x}_g, \boldsymbol{x}_2) \equiv \sqrt{\tfrac{1}{4}(N - 2/N)(\boldsymbol{x}_1 - \boldsymbol{x}_2)^2 + N(\boldsymbol{x}_g - \tfrac{1}{2}\boldsymbol{x}_1 - \tfrac{1}{2}\boldsymbol{x}_2)^2}\,,$$

$$V_{qgq} = \frac{\Lambda^2}{\sqrt{C_F}}\,d_{qgq}(\boldsymbol{x}_1, \boldsymbol{x}_g, \boldsymbol{x}_2) + \tfrac{1}{2}\,\alpha_s\Big[\frac{1}{N}\,\frac{1}{|\boldsymbol{x}_1 - \boldsymbol{x}_2|} - N\Big(\frac{1}{|\boldsymbol{x}_1 - \boldsymbol{x}_g|} + \frac{1}{|\boldsymbol{x}_2 - \boldsymbol{x}_g|}\Big)\Big] \tag{23}$$

In each case a bound state equation may be derived by adding the kinetic terms in the Hamiltonian, and the mixing with higher Fock components taken into account at higher orders of α_s. Let us illustrate this for $|gg\rangle$ ("glueball") states at $\mathcal{O}\left(\alpha_s^0\right)$.

---

[2] When the binding is relativistic the state (10) has virtual $q\bar{q}$ contributions, which may be thought of as due to Z-diagrams and have the features of sea quarks.

The full $\mathcal{O}\left(\alpha_s^0\right)$ Hamiltonian is $\mathcal{H}=\mathcal{H}_0+\mathcal{H}_V^{(0)}$, with $\mathcal{H}_V^{(0)}$ given in (16) and

$$\mathcal{H}_0=\int d\boldsymbol{x}\big[\tfrac{1}{2}E_{a,T}^iE_{a,T}^i+\tfrac{1}{2}A_{a,T}^i(-\delta_{ij}\boldsymbol{\nabla}^2+\partial_i\partial_j)A_{a,T}^j\big] \tag{24}$$

The canonical commutation relation $\big[E_a^i(t,\boldsymbol{x}),A_b^j(t,\boldsymbol{y})\big]=i\delta_{ab}\delta^{ij}\delta(\boldsymbol{x}-\boldsymbol{y})$ gives

$$\big[\mathcal{H}_0,A_{a,T}^i(\boldsymbol{x})\big]=iE_{a,T}^i(\boldsymbol{x}) \qquad\qquad \big[H_0,E_{a,T}^i(\boldsymbol{x})\big]=i\boldsymbol{\nabla}^2A_{a,T}^i(\boldsymbol{x}) \tag{25}$$

For $|gg\rangle$ to be an eigenstate of $\mathcal{H}_0$ it must have AA, AE , EA and EE components,

$$|gg\rangle\equiv\int d\boldsymbol{x}_1d\boldsymbol{x}_2\big[A_{a,T}^i(\boldsymbol{x}_1)A_{a,T}^j(\boldsymbol{x}_2)\Phi_{AA}^{ij}(\boldsymbol{x}_1-\boldsymbol{x}_2)+A_{a,T}^iE_{a,T}^j\Phi_{AE}^{ij}+E_{a,T}^iA_{a,T}^j\Phi_{EA}^{ij}+E_{a,T}^iE_{a,T}^j\Phi_{EE}^{ij}\big]|0\rangle \tag{26}$$

The free Hamiltonian $\mathcal{H}_0$ mixes the components. Suppressing the i,j indices (which are not changed by $\mathcal{H}_0+\mathcal{H}_V^{(0)}$),

$$\begin{aligned}\mathcal{H}_0\,|gg\rangle=i\int d\boldsymbol{x}_1d\boldsymbol{x}_2\,\Big\{&\big[E_a(\boldsymbol{x}_1)A_a(\boldsymbol{x}_2)+A_a(\boldsymbol{x}_1)E_a(\boldsymbol{x}_2)\big]\Phi_{AA}(\boldsymbol{x}_1-\boldsymbol{x}_2)+\big[E_aE_a+A_aA_a\boldsymbol{\nabla}^2\big]\Phi_{AE}\\&+\big[A_aA_a\boldsymbol{\nabla}^2+E_aE_a\big]\Phi_{EA}+\big[A_aE_a+E_aA_a\big]\boldsymbol{\nabla}^2\Phi_{EE}\Big\}|0\rangle\end{aligned} \tag{27}$$

The instantaneous Hamiltonian $\mathcal{H}_V^{(0)}$ gives all four components the same potential $V_{gg}^{(0)}$ as the AA term in (21). The eigenvalue condition $\mathcal{H}\,|gg\rangle=M\,|gg\rangle$ allows to express Φ_{AA},Φ_{AE} and Φ_{EA} in terms of Φ_{EE}, which should satisfy

$$\boldsymbol{\nabla}^2\Phi_{EE}(\boldsymbol{x})-\frac{V_g'}{M-V}\partial_r\Phi_{EE}(\boldsymbol{x})-\frac{V_g'}{r(M-V)}\Phi_{EE}(\boldsymbol{x})+\tfrac{1}{4}(M-V)^2\Phi_{EE}(\boldsymbol{x})=0 \tag{28}$$

where $V=V_g'\,|\boldsymbol{x}|$ with $V_g'=\sqrt{\frac{N}{C_F}}\,\Lambda^2=\frac{3}{2}\,\Lambda^2$ as in (21). The radial $r=|\boldsymbol{x}|$ and angular Ω variables may be separated for states with orbital angular momentum $\boldsymbol{L}^2=\ell(\ell+1)$ and $L^z=\lambda$, $\Phi_{EE}(\boldsymbol{x})=F(r)Y_{\ell\lambda}(\Omega)$. This gives the radial equation

$$F''(r)+\Big(\frac{2}{r}-\frac{V_g'}{M-V}\Big)F'(r)+\Big[\tfrac{1}{4}(M-V)^2-\frac{V_g'}{r(M-V)}-\frac{\ell(\ell+1)}{r^2}\Big]F(r)=0 \tag{29}$$

The glueball masses M are determined by requiring the wave function to be regular at $r=0$ and at $M-V(r)=0$, which ensures a probabilistic interpretation of Φ [10]. The glueball states lie on approximately linear Regge trajectories and their parallel daughters. Using $V'=\Lambda^2=0.18$ GeV^2 from the Cornell potential (1) the lightest glueball has $M\simeq 1.6$ GeV.

All bound states have corrections even at $\mathcal{O}\left(\alpha_s^0\right)$ stemming from real decays and hadron loops. The transition $a\to b+c$ is given by the matrix element $\langle b\,c\,|a\rangle$ which may be evaluated using the (zero width) meson and glueball states discussed above. For the parts of $|a\rangle$ that have a large r, and hence large $V(r)$, the transition has the features of color string breaking. Such transitions need to be iterated (expanded in powers of $1/N_C$) for unitarity to apply at the level of hadrons.

For massless ($m=0$) quarks the bound state equation (9) has chiral symmetry: If $\Phi(\boldsymbol{x})$ solves the equation then so does $\gamma_5\Phi(\boldsymbol{x})$. There are also massless bound states [17], since the condition that the wave function is regular at $r=0$ coincides with the condition at $V(r)=M$ when the potential is linear and $M=0$. The $J^{PC}=0^{++}$ "sigma" solution of (9) is analytic,

$$\frac{1}{N_\sigma}\Phi_\sigma(\boldsymbol{x})=J_0(\tfrac{1}{4}r^2)+i\,\boldsymbol{\alpha}\cdot\boldsymbol{x}\,\frac{1}{r}J_1(\tfrac{1}{4}r^2) \qquad\qquad (m=M=0) \tag{30}$$

where $J_{0,1}$ are Bessel functions and N_σ is a normalization constant. The corresponding massless 0^{-+} "pion" wave function is $\Phi_\pi(\boldsymbol{x})\propto\gamma_5\Phi_\sigma(\boldsymbol{x})$. These states have 4-momentum $P^\mu=0$ in all frames and are thus not physical. However, we may explicitly verify the standard mechanism for the spontaneous breaking of chiral symmetry. The sigma state can mix with the vacuum without breaking Poincaré invariance, forming a "chiral condensate",

$$|\chi\rangle\equiv\exp\Big[\int d\boldsymbol{x}_1\,d\boldsymbol{x}_2\,\bar{\psi}(\boldsymbol{x}_1)\,\Phi_\sigma(\boldsymbol{x}_1-\boldsymbol{x}_2)\,\psi(\boldsymbol{x}_2)\Big]\,|0\rangle \tag{31}$$

The scalar operator $\bar\psi(x)\psi(x)$ annihilates the sigma, $\langle 0|\bar\psi\psi\,|\sigma\rangle = \mathrm{Tr}\,[\gamma^0\Phi_\sigma(\mathbf{0})\gamma^0] = 4N_\sigma$. Hence chiral symmetry is broken by the chiral condensate,

$$\langle\bar\psi\psi\rangle = \langle\chi|\bar\psi(x)\psi(x)\,|\chi\rangle = 4N_\sigma\langle\chi|\chi\rangle \tag{32}$$

An infinitesimal chiral transformation $U_\chi(\beta)$ ($\beta \ll 1$) transforms the quark fields as

$$U_\chi(\beta)\,\bar\psi(\boldsymbol{x})\,U_\chi^\dagger(\beta) = \bar\psi(\boldsymbol{x})(1-i\beta\gamma_5) \qquad\qquad U_\chi(\beta)\,\psi(\boldsymbol{x})\,U_\chi^\dagger(\beta) = (1-i\beta\gamma_5)\psi(\boldsymbol{x}) \tag{33}$$

Since $\Phi_\pi = \gamma_5\Phi_\sigma = \frac{1}{2}\{\gamma_5,\Phi_\sigma\}$ (up to the relative normalization) we get,

$$U_\chi(\beta)\,|\chi\rangle = \exp\Big[\int d\boldsymbol{x}_1 d\boldsymbol{x}_2\,\bar\psi(\boldsymbol{x}_1)\big[\Phi_\sigma - i\beta\,\{\gamma_5,\Phi_\sigma\}\,\big]\psi(\boldsymbol{x}_2)\Big]\,|0\rangle = (1-2i\beta\,\hat\pi)\,|\chi\rangle \tag{34}$$

where $\hat\pi$ creates a pion, $|\pi\rangle = \hat\pi\,|0\rangle$. Thus a chiral transformation of $|\chi\rangle$ creates massless pions, as expected.

There is a massless ($M=0$) 0^{++} sigma state also for a non-vanishing quark mass, $m \neq 0$. The corresponding pion with $M \propto m^2$ satisfies the PCAC relation for small m [10], motivating further study.

Quantum field theory is subtle, and bound states especially so. The method presented here and in [10] needs further tests, and possibly modifications. Hopefully it can contribute to making bound state calculations more systematic and less of an art, as they are occasionally described.

My work on these topics over many years has been supported by discussions with colleagues, among them Jean-Paul Blaizot, Stan Brodsky, Dennis D. Dietrich, Matti Järvinen, Jörn Knoll and Stephane Peigné. During the preparation of this material I enjoyed visits to ECT* (Trento), Jlab (Newport News) and CP^3 (Odense). I am grateful for their hospitality, and to the Department of Physics at Helsinki University for my privileges of Professor Emeritus. An annual travel grant from the Magnus Ehrnrooth Foundation has been much appreciated.

* paul.hoyer@helsinki.fi
† http://www.helsinki.fi/~hoyer/

[1] William E. Caswell and G. Peter Lepage, "Reduction of the Bethe-Salpeter Equation to an Equivalent Schrodinger Equation, With Applications," Phys. Rev. **A18**, 810 (1978).
[2] G. Peter Lepage, *Two-body Bound States in Quantum Electrodynamics*, Ph.D. thesis, SLAC-R-0212 (1978).
[3] W. E. Caswell and G. P. Lepage, "Effective Lagrangians for Bound State Problems in QED, QCD, and Other Field Theories," Phys. Lett. **167B**, 437–442 (1986).
[4] E. Eichten, K. Gottfried, T. Kinoshita, K. D. Lane, and Tung-Mow Yan, "Charmonium: Comparison with Experiment," Phys. Rev. **D21**, 203 (1980).
[5] Estia Eichten, Stephen Godfrey, Hanna Mahlke, and Jonathan L. Rosner, "Quarkonia and their transitions," Rev. Mod. Phys. **80**, 1161–1193 (2008), arXiv:hep-ph/0701208 [hep-ph].
[6] Gunnar S. Bali, "QCD forces and heavy quark bound states," Phys. Rept. **343**, 1–136 (2001), arXiv:hep-ph/0001312 [hep-ph].
[7] Alexander A. Penin, "Testing Quantum Electrodynamics with Positronium State," *Proceedings, 12th DESY Workshop on Elementary Particle Physics: Loops and Legs in Quantum Field Theory (LL2014): Weimar, Germany, April 27-May 2, 2014*, PoS **LL2014**, 074 (2014).
[8] Gregory S. Adkins, "Higher order corrections to the hyperfine splitting of positronium," Hyperfine Interact. **233**, 59–66 (2015).
[9] Daniel Zwanziger, "No confinement without Coulomb confinement," Phys. Rev. Lett. **90**, 102001 (2003), arXiv:hep-lat/0209105 [hep-lat].
[10] Paul Hoyer, "Bound states and QCD," (2018), arXiv:1807.05598 [hep-ph].
[11] W. Heisenberg and W. Pauli, "On Quantum Field Theory. 2. (In German)," Z. Phys. **59**, 168–190 (1930).
[12] Jorge F. Willemsen, "Remarks on the Implementation of Gauss' Law in $A_0 = 0$ Gauge," Phys. Rev. **D17**, 574 (1978).
[13] J. D. Bjorken, "Elements of Quantum Chromodynamics," in *Quantum chromodynamics: Proceedings, 7th SLAC Summer Institute on Particle Physics (SSI 79), Stanford, Calif., 9-20 Jul 1979* (1979) p. 219.
[14] N. H. Christ and T. D. Lee, "Operator Ordering and Feynman Rules in Gauge Theories," Phys. Rev. **D22**, 939 (1980).
[15] George Leibbrandt, "Introduction to Noncovariant Gauges," Rev. Mod. Phys. **59**, 1067 (1987).
[16] V. N. Gribov, "Quantization of Nonabelian Gauge Theories," Nucl. Phys. **B139**, 1 (1978).
[17] Dennis D. Dietrich, Paul Hoyer, and Matti Järvinen, "Towards a Born term for hadrons," Phys. Rev. **D87**, 065021 (2013), arXiv:1212.4747 [hep-ph].