From QED atoms to QCD hadrons

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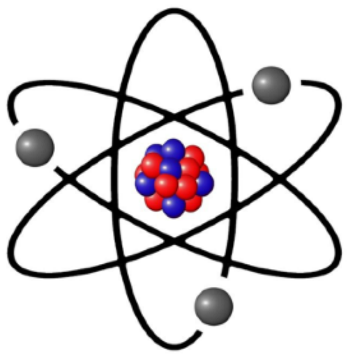
Scattering vs. Bound States in QFT



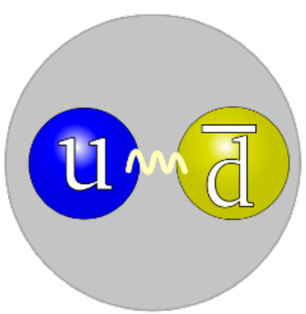
S-matrix (textbooks): Interaction Picture, start from H_0

Bound states through an infinite sum of Feynman diagrams

Regenerates the classical potential $V(r) = -\alpha/r$

Physics: Constituents propagate in their mutual fields

Consider: Poincaré covariance



Instantaneous (in time) interactions in relativistic dynamics

The QCD scale Λ is compatible with the action

Get confining potential $\propto \Lambda$ of $O(\alpha_s^0)$

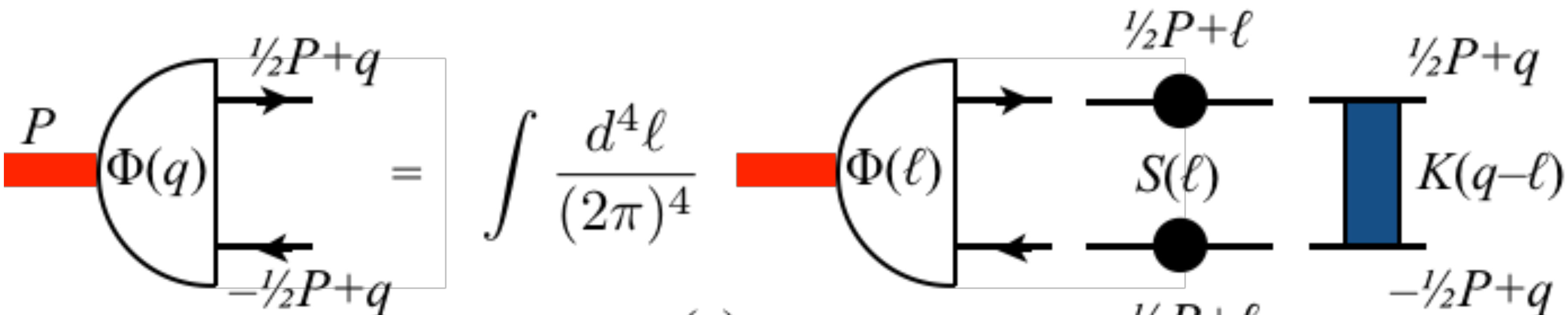
QCD dynamics at $\alpha_s=0$ has parton-hadron duality

PH 2101.06721 and 2304.11903

Brief review of bound state PT (1)

1951 **A Relativistic Equation for Bound-State Problems**

E. E. SALPETER AND H. A. BETHE
Newman Laboratory of Nuclear Studies, Cornell University, Ithaca, New York
(Received August 24, 1951)



(a)

Allows to calculate relativistic corrections to atomic binding energies

Very complicated: No analytic solution at "lowest order"

Brief review of bound state PT (2)

1978 **Reduction of the Bethe-Salpeter equation to an equivalent Schrödinger equation, with applications**

William E. Caswell
Stanford Linear Accelerator Center, Stanford University, Stanford, California 94305 and Department of Physics, Brown University, Providence, Rhode Island 02912

G. Peter Lepage
Stanford Linear Accelerator Center, Stanford University, Stanford, California 94305

(Received 10 February 1978)

We propose a new relativistic two-body formalism which reduces to a nonrelativistic Schrödinger theory for a single effective particle. The formalism is equal in rigor to that of Bethe and Salpeter, and considerably simpler to apply. We illustrate its use by computing $O(\alpha^6)$ terms in the ground-state splitting of muonium and positronium involving infinite Coulomb exchange.

"Lowest order" bound state wave functions are of $O(\alpha^\infty)$

This allows to reorder the perturbative series.

The choice of state to expand around determines its higher order corrections

Brief review of bound state PT (3)

1986: NRQED   Today's method of choice for QED

Volume 167B, number 4   PHYSICS LETTERS   20 February 1986

EFFECTIVE LAGRANGIANS FOR BOUND STATE PROBLEMS IN QED, QCD, AND OTHER FIELD THEORIES

W.E. CASWELL   G.P. LEPAGE

A renormalization group strategy for the study of bound states in field theory is developed Our analysis is completely different from conventional analyses, based upon the Bethe–Salpeter equation, and it is far simpler This is illustrated in state-of-the-art calculations for the ground state splittings in muonium and positronium

Expand around the Schrödinger atom

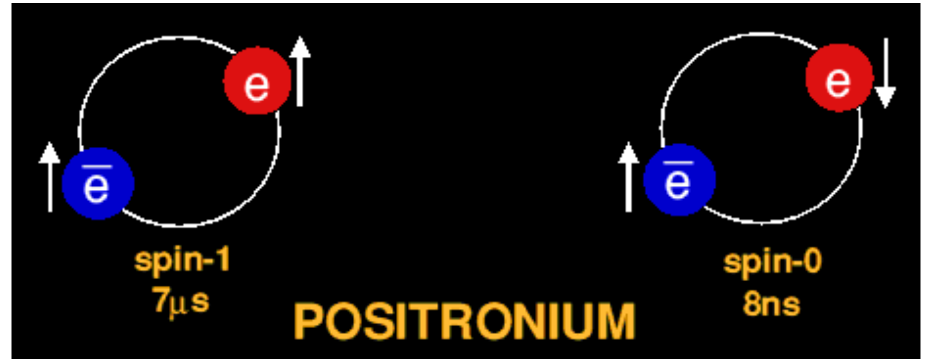
Calculate the corrections using the effective, non-relativistic field theory of NRQED

Expansion in powers of p_e/m_e

$$
\begin{aligned}
L_{\rm eff} = & -\tfrac{1}{2}(E^2 - B^2) + \psi_{\rm e}^\dagger(\mathrm{i}\partial_t - e\phi + \mathbf{D}^2/2m)\psi_{\rm e} \\
& + \psi_{\rm e}^\dagger[c_1\mathbf{D}^4/8m^3 + c_2(e/2m)\boldsymbol{\sigma}\cdot \boldsymbol{B} \\
& + c_3(e/8m^2)\,\nabla\cdot\boldsymbol{E} + c_4(e/8m^2)\{\mathrm{i}\mathbf{D}\cdot\boldsymbol{E}\times\boldsymbol{\sigma}\}]\,\psi_{\rm e} \\
& + \psi_{\rm e}^\dagger[d_1(e/8m^3)\{\mathbf{D}^2,\boldsymbol{\sigma}\cdot\boldsymbol{B}\}]\,\psi_{\rm e} \\
& - (d_2/m_{\rm e}m_\mu)(\psi_{\rm e}^\dagger\boldsymbol{\sigma}\psi_{\rm e})\cdot(\psi_\mu^\dagger\boldsymbol{\sigma}\psi_\mu) + \ldots,
\end{aligned}
$$

Brief review of bound state PT (4)

G. S. Adkins,
Hyperfine Interact. **233** (2015) 59



Hyperfine splitting in Positronium

$$\begin{aligned}\Delta\nu_{QED} \;=\; & m_e\alpha^4\left\{\frac{7}{12}-\frac{\alpha}{\pi}\left(\frac{8}{9}+\frac{\ln 2}{2}\right)\right.\\ & +\frac{\alpha^2}{\pi^2}\left[-\frac{5}{24}\pi^2\ln\alpha+\frac{1367}{648}-\frac{5197}{3456}\pi^2+\left(\frac{221}{144}\pi^2+\frac{1}{2}\right)\ln 2-\frac{53}{32}\zeta(3)\right]\\ & \left.-\frac{7\alpha^3}{8\pi}\ln^2\alpha+\frac{\alpha^3}{\pi}\ln\alpha\left(\frac{17}{3}\ln 2-\frac{217}{90}\right)+\mathcal{O}\left(\alpha^3\right)\right\} = 203.39169(41)\ \mathrm{GHz}\end{aligned}$$

depends on log(α)

$\Delta\nu_{EXP} = 203.394 \pm .002$ GHz

The perturbative series for measurable quantities is unique

QED agrees with data.

The instantaneous potential of QED

Is an instantaneous interaction compatible with a local action?

Or is it only a non-relativistic approximation?

How do we describe constituents propagating in their mutual field?

Avoid free propagation (Feynman diagrams)

Gauge theories have instantaneous interactions:
Although their action is local, the gauge may be fixed non-locally

The lack of $\partial_0 A^0$ and $\nabla\cdot\boldsymbol{A}$ in $F_{\mu\nu}F^{\mu\nu}$ means that A^0 and $\boldsymbol{A}_L$ do not propagate

The values of A^0 and $\boldsymbol{A}_L$ are determined by the choice of gauge

Covariant gauge fixing: $\mathcal{L}_{GF} = (\partial_\mu A^\mu)^2$ adds the missing derivatives

This hides the instantaneous potential, obscures bound state dynamics.

Coulomb gauge ($\nabla\cdot\boldsymbol{A} = 0$) Keeps explicit rotational invariance

$-\boldsymbol{\nabla}^2 A^0(t,\boldsymbol{x}) = e\,\psi^\dagger(t,\boldsymbol{x})\psi(t,\boldsymbol{x})$ Operator EoM (Gauss' law)

Canonical quantization complicated by constraints: $S \neq S(\partial_0 A^0)$

Temporal gauge in QED: $A^0(t,\boldsymbol{x}) = 0$

Canonical quantisation is straightforward: $A^0 = \partial_0 A^0 = 0$

$$\left[E^i(t,\boldsymbol{x}), A^j(t,\boldsymbol{y})\right] = i\delta^{ij}\delta(\boldsymbol{x}-\boldsymbol{y})$$

$E^i = -\partial_0 A^i$ Electric field

$A^0(t,\boldsymbol{x}) = 0$ is preserved under time-independent gauge transformations.

These are generated by the operator of "Gauss' law": Willemsen (1978)

$$\frac{\delta \mathcal{S}_{QED}}{\delta A^0(x)} = \partial_i E^i(x) - e\psi^\dagger\psi(x)$$

Does not vanish as an operator since $A^0 = 0$

Physical states must be invariant under all gauge transformations:

$$\frac{\delta \mathcal{S}_{QED}}{\delta A^0(x)}|phys\rangle = 0$$

Determines $\nabla\cdot\boldsymbol{E}_L$ from the charges in each state

Temporal gauge in QCD: $A_a^0 = 0$

Canonical quantization without constraints.
Gauss law is not an EoM

Willemsen (1978)

$$\frac{\delta \mathcal{S}_{QCD}}{\delta A_a^0} = \boldsymbol{\nabla}\cdot\boldsymbol{E}_a + g f_{abc}\boldsymbol{A}_b\cdot\boldsymbol{E}_c - g\psi^\dagger T_a \psi$$

Generates time-independent gauge transformations

$$\frac{\delta \mathcal{S}_{QCD}}{\delta A_a^0(x)}\,|phys\rangle = 0$$

Ensures the full gauge invariance of physical states.
Defines $\nabla\cdot\boldsymbol{E}_L$ for each $|phys\rangle$

Include a homogeneous solution for $\boldsymbol{E}_{L,a}$: $\nabla\cdot\boldsymbol{E}_{L,a}(\boldsymbol{x}) = 0$

Introduces the QCD scale from a boundary condition

Maintaining translation and rotation symmetry imposes tight constraints

Color singlet states required for translation invariance

Including a homogeneous solution for E_L

$$\boldsymbol{E}_L^a(\boldsymbol{x})\,|phys\rangle = -\boldsymbol{\nabla}_x \int d\boldsymbol{y}\Big[\kappa\,\boldsymbol{x}\cdot\boldsymbol{y} + \frac{g}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\Big]\mathcal{E}^a(\boldsymbol{y})\,|phys\rangle$$

where $\mathcal{E}_a(\boldsymbol{y}) = -f_{abc}A_b^i E_c^i(\boldsymbol{y}) + \psi^\dagger T^a \psi(\boldsymbol{y})$

$$\mathcal{H}_V \equiv \tfrac{1}{2}\int d\boldsymbol{x} \sum_a \boldsymbol{E}_L^a \cdot \boldsymbol{E}_L^a$$

$$= \int d\boldsymbol{y}d\boldsymbol{z}\Big\{\boldsymbol{y}\cdot\boldsymbol{z}\Big[\tfrac{1}{2}\kappa^2\int d\boldsymbol{x} + g\kappa\Big] + \tfrac{1}{2}\frac{\alpha_s}{|\boldsymbol{y}-\boldsymbol{z}|}\Big\}\mathcal{E}_a(\boldsymbol{y})\mathcal{E}_a(\boldsymbol{z})$$

The field energy ∝ volume of space is irrelevant only if it is universal.

This relates the normalisation ϰ for all physical states,

leaving a universal scale $\Lambda = O(\alpha_s{}^0)$ as the single parameter.

$q\bar{q}$ Fock state potential

$$|q(\boldsymbol{x}_1)\bar{q}(\boldsymbol{x}_2)\rangle \equiv \sum_A \bar{\psi}^A(\boldsymbol{x}_1)\,\psi^A(\boldsymbol{x}_2)\,|0\rangle$$ globally color singlet

$$\mathcal{H}_V \equiv \tfrac{1}{2}\int d\boldsymbol{x}\sum_a \boldsymbol{E}_L^a\cdot\boldsymbol{E}_L^a$$

$$\mathcal{H}_V\,|q\bar{q}\rangle = V_{q\bar{q}}\,|q\bar{q}\rangle$$

$$V_{q\bar{q}}(\boldsymbol{x}_1,\boldsymbol{x}_2) = \Lambda^2|\boldsymbol{x}_1-\boldsymbol{x}_2| - C_F\frac{\alpha_s}{|\boldsymbol{x}_1-\boldsymbol{x}_2|}$$ Cornell potential

This potential is valid also for relativistic $q\bar{q}$ Fock states, in any frame

The universal vacuum energy density is $E_\Lambda = \dfrac{\Lambda^4}{2g^2C_F}$

Baryon Fock state potential

Baryon: $$|q(\boldsymbol{x}_1)q(\boldsymbol{x}_2)q(\boldsymbol{x}_3)\rangle \equiv \sum_{A,B,C} \epsilon_{ABC}\psi_A^\dagger(\boldsymbol{x}_1)\,\psi_B^\dagger(\boldsymbol{x}_2)\,\psi_C^\dagger(\boldsymbol{x}_3)\,|0\rangle$$

$$V_{qqq}(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3) = \Lambda^2 d_{qqq}(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3) - \frac{2}{3}\,\alpha_s\Big(\frac{1}{|\boldsymbol{x}_1-\boldsymbol{x}_2|}+\frac{1}{|\boldsymbol{x}_2-\boldsymbol{x}_3|}+\frac{1}{|\boldsymbol{x}_3-\boldsymbol{x}_1|}\Big)$$

$$d_{qqq}(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3) \equiv \frac{1}{\sqrt{2}}\sqrt{(\boldsymbol{x}_1-\boldsymbol{x}_2)^2+(\boldsymbol{x}_2-\boldsymbol{x}_3)^2+(\boldsymbol{x}_3-\boldsymbol{x}_1)^2}$$

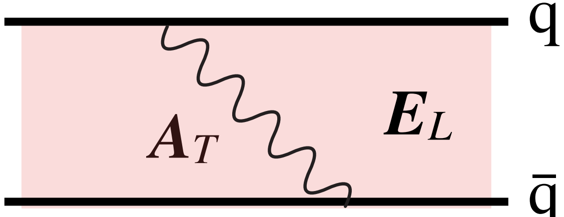
When two of the quarks coincide the potential reduces to the $q\bar{q}$ potential:

$$V_{qqq}(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_2) = \Lambda^2|\boldsymbol{x}_1-\boldsymbol{x}_2| - \frac{4}{3}\,\frac{\alpha_s}{|\boldsymbol{x}_1-\boldsymbol{x}_2|} = V_{q\bar{q}}(\boldsymbol{x}_1,\boldsymbol{x}_2)$$

Analogous potentials are obtained for any globally color singlet quark and gluon Fock state, such as $q\bar{q}g$ and gg.

The qg$\bar{\text{q}}$ potential

A q$\bar{\text{q}}$ state, with the exchange of a transverse gluon:



$$|q(\boldsymbol{x}_1)g(\boldsymbol{x}_g)\bar{q}(\boldsymbol{x}_2)\rangle \equiv \sum_{A,B,b} \bar{\psi}_A(\boldsymbol{x}_1)\, A_b^j(\boldsymbol{x}_g) T_{AB}^b \psi_B(\boldsymbol{x}_2)\,|0\rangle$$

$$V_{qgq}^{(0)}(\boldsymbol{x}_1, \boldsymbol{x}_g, \boldsymbol{x}_2) = \frac{\Lambda^2}{\sqrt{C_F}}\, d_{qgq}(\boldsymbol{x}_1, \boldsymbol{x}_g, \boldsymbol{x}_2) \qquad \text{(universal } \Lambda\text{)}$$

$$d_{qgq}(\boldsymbol{x}_1, \boldsymbol{x}_g, \boldsymbol{x}_2) \equiv \sqrt{\tfrac{1}{4}(N - 2/N)(\boldsymbol{x}_1 - \boldsymbol{x}_2)^2 + N(\boldsymbol{x}_g - \tfrac{1}{2}\boldsymbol{x}_1 - \tfrac{1}{2}\boldsymbol{x}_2)^2}$$

$$V_{qgq}^{(1)}(\boldsymbol{x}_1, \boldsymbol{x}_g, \boldsymbol{x}_2) = \tfrac{1}{2}\,\alpha_s \Big[\frac{1}{N}\,\frac{1}{|\boldsymbol{x}_1 - \boldsymbol{x}_2|} - N\Big(\frac{1}{|\boldsymbol{x}_1 - \boldsymbol{x}_g|} + \frac{1}{|\boldsymbol{x}_2 - \boldsymbol{x}_g|}\Big)\Big]$$

When q and g coincide:

$$V_{qgq}^{(0)}(\boldsymbol{x}_1 = \boldsymbol{x}_g, \boldsymbol{x}_2) = \Lambda^2|\boldsymbol{x}_1 - \boldsymbol{x}_2| = V_{q\bar{q}}^{(0)}$$

$$V_{qgq}^{(1)}(\boldsymbol{x}_1 = \boldsymbol{x}_g, \boldsymbol{x}_2) = V_{q\bar{q}}^{(1)}$$

The gg potential

A "glueball" component: $$|g(\boldsymbol{x}_1)g(\boldsymbol{x}_2)\rangle \equiv \sum_a A_a^i(\boldsymbol{x}_1)\, A_a^j(\boldsymbol{x}_2)\,|0\rangle$$

has the potential $$V_{gg} = \sqrt{\frac{N}{C_F}}\,\Lambda^2\,|\boldsymbol{x}_1 - \boldsymbol{x}_2| - N\,\frac{\alpha_s}{|\boldsymbol{x}_1 - \boldsymbol{x}_2|}$$

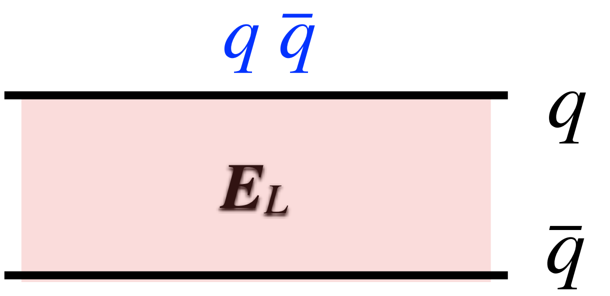
This agrees with the qgq̄ potential where the quarks coincide:

$$V_{gg}(\boldsymbol{x}, \boldsymbol{x}_g) = V_{qg\bar{q}}(\boldsymbol{x}, \boldsymbol{x}_g, \boldsymbol{x})$$

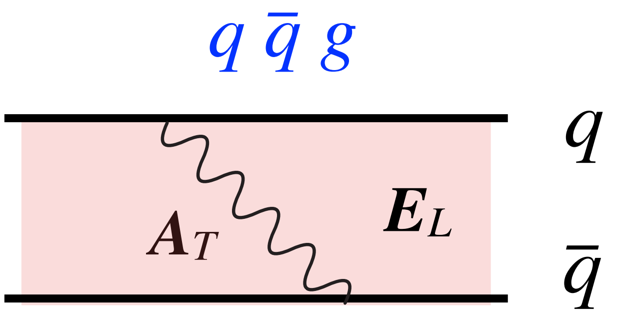
It is straightforward to work out the instantaneous potential for any Fock state.

Bound Fock expansion for mesons in $A^0=0$ gauge

The perturbative expansion in α_s starts from the $|q\bar{q}\rangle$ Fock state, bound by the $O(\alpha_s^0)$ instantaneous potential $V_{q\bar{q}}$:



$O(\alpha_s)$ corrections include states with transverse gluons and quark pairs, determined perturbatively by $H_{QCD}|q\bar{q}\rangle$



Each Fock component of the bound state includes its $O(\alpha_s^0)$ instantaneous potential.

This Fock expansion is valid in any frame, and is formally exact at $O(\alpha_s^\infty)$.

$\mathcal{O}\left(\alpha_s^0\right)$ $q\bar{q}$ bound states

An $\mathcal{O}\left(\alpha_s^0\right)$ meson state with $\boldsymbol{P} = 0$ and wave function Φ:

$$|M\rangle = \sum_{A,B;\alpha,\beta} \int d\boldsymbol{x}_1 d\boldsymbol{x}_2\, \bar{\psi}_\alpha^A(t=0,\boldsymbol{x}_1)\delta^{AB}\Phi_{\alpha\beta}(\boldsymbol{x}_1-\boldsymbol{x}_2)\psi_\beta^B(t=0,\boldsymbol{x}_2)\,|0\rangle$$

The (rest frame) bound state condition $H\,|M\rangle = M\,|M\rangle$ gives

$$\left[i\gamma^0\boldsymbol{\gamma}\cdot\overrightarrow{\boldsymbol{\nabla}} + m\gamma^0\right]\Phi(\boldsymbol{x}) + \Phi(\boldsymbol{x})\left[i\gamma^0\boldsymbol{\gamma}\cdot\overleftarrow{\boldsymbol{\nabla}} - m\gamma^0\right] = \left[M - V(|\boldsymbol{x}|)\right]\Phi(\boldsymbol{x})$$

where $\boldsymbol{x} \equiv \boldsymbol{x}_1 - \boldsymbol{x}_2$ and $V(\boldsymbol{x}) = \Lambda^2|\boldsymbol{x}|$ at $\mathcal{O}\left(\alpha_s^0\right)$

In the non-relativistic limit ($m \gg \Lambda$) this reduces to the Schrödinger equation.

$\Longrightarrow$ The quarkonium phenomenology with the Cornell potential.

Example: $-\eta_P = \eta_C = (-1)^j$ states at $O(\alpha_s^0)$

$$\Phi_{-+}(\boldsymbol{x}) = \Big[\frac{2}{M-V}(i\boldsymbol{\alpha}\cdot\vec{\boldsymbol{\nabla}} + m\gamma^0) + 1\Big]\gamma_5\, F_1(r) Y_{j\lambda}(\hat{\boldsymbol{x}})$$

Radial equation: $F_1'' + \Big(\frac{2}{r} + \frac{V'}{M-V}\Big)F_1' + \Big[\tfrac{1}{4}(M-V)^2 - m^2 - \frac{j(j+1)}{r^2}\Big]F_1 = 0$

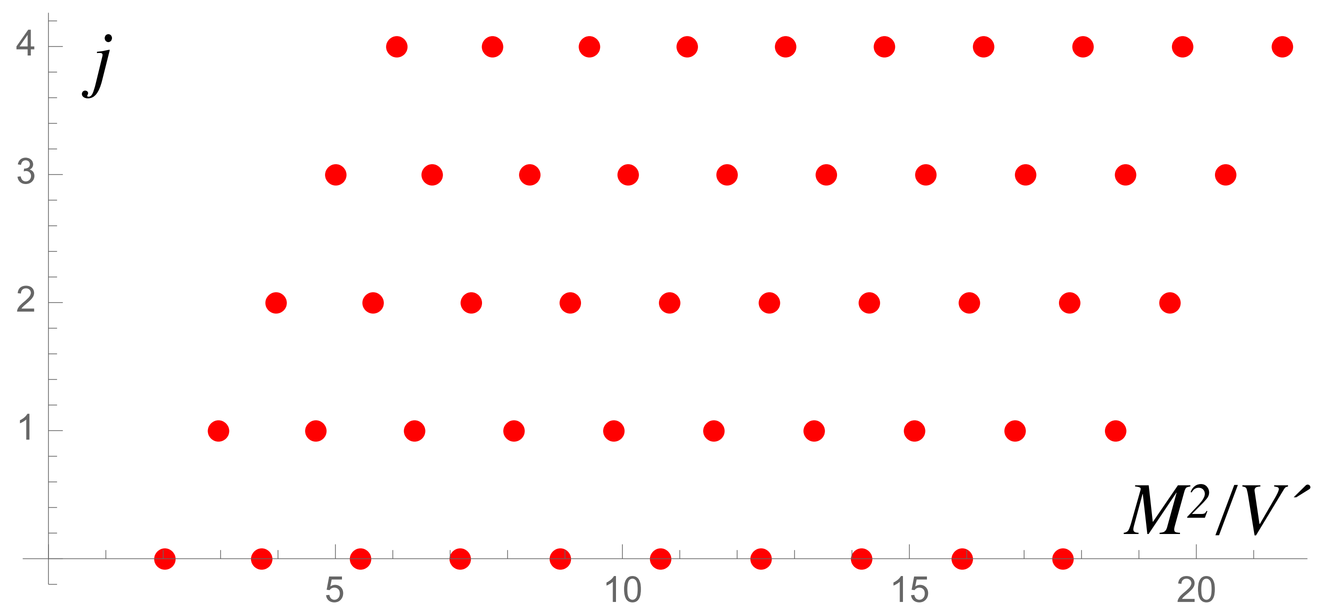
Regularity of the wave function determines the bound state masses M

Mass spectrum:

Linear Regge trajectories with daughters

Spectrum similar to dual models

$m = 0$



Bound states with momentum P

Bound state with CM momentum $\boldsymbol{P}$:

$$|M,P\rangle = \frac{1}{\sqrt{N_c}}\sum_{A,B}\int d\boldsymbol{x}_1 d\boldsymbol{x}_2\,\bar{\psi}^A(\boldsymbol{x}_1)e^{i\boldsymbol{P}\cdot(\boldsymbol{x}_1+\boldsymbol{x}_2)/2}\delta^{AB}\Phi^{(\boldsymbol{P})}(\boldsymbol{x}_1-\boldsymbol{x}_2)\psi^B(\boldsymbol{x}_2)\,|0\rangle$$

Wave function satisfies bound state equation:

$$i\boldsymbol{\nabla}\cdot\big\{\boldsymbol{\alpha},\Phi^{(\boldsymbol{P})}(\boldsymbol{x})\big\} - \tfrac{1}{2}\boldsymbol{P}\cdot\big[\boldsymbol{\alpha},\Phi^{(\boldsymbol{P})}(\boldsymbol{x})\big] + m\big[\gamma^0,\Phi^{(\boldsymbol{P})}(\boldsymbol{x})\big] = \big[E-V(\boldsymbol{x})\big]\Phi^{(\boldsymbol{P})}(\boldsymbol{x})$$

The energy eigenvalues satisfy (for a linear potential only): $E=\sqrt{M^2+\boldsymbol{P}^2}$

An infinitesimal boost ξ in the *z*-direction transforms the wave function as

$$\partial_\xi\Phi^{(\boldsymbol{P})} = \frac{z}{E-V}\,\boldsymbol{P}\cdot\boldsymbol{\nabla}\Phi^{(\boldsymbol{P})} - \frac{z}{2(E-V)}\big[\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}V,\Phi^{(\boldsymbol{P})}\big] - \tfrac{1}{2}\big[\alpha_3,\Phi^{(\boldsymbol{P})}\big]$$

Frame dependence of EM form factors

In a perturbative expansion each order in α_s, including $O(\alpha_s^0)$, must have exact Poincaré covariance.

Check with electromagnetic form factor for any states *A*, *B*:

$$F^{\mu}_{AB}(y) = \langle B, \boldsymbol{P}_B | j^\mu(y) \, | A, \boldsymbol{P}_A \rangle = e^{i(P_B - P_A)\cdot y} \langle B, \boldsymbol{P}_B | j^\mu(0) \, | A, \boldsymbol{P}_A \rangle$$

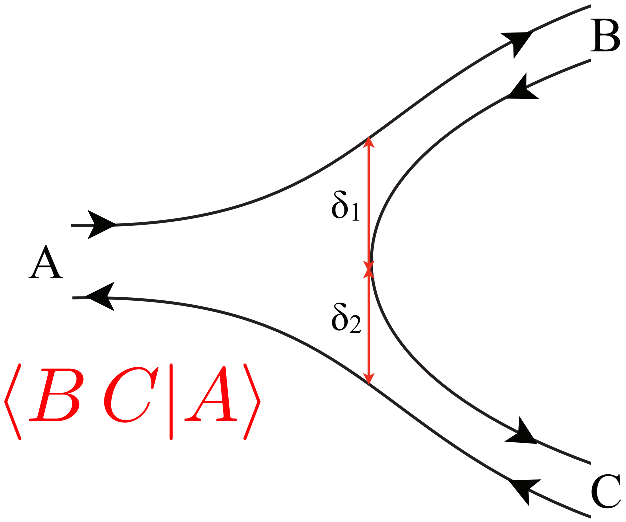
$$F^{\mu}_{AB}(q) = \int d^4y \, e^{-iq\cdot y} \, F^{\mu}_{AB}(y) \equiv (2\pi)^4 \delta^4(P_B - P_A - q) G^{\mu}_{AB}(q)$$

$$G^{\mu}_{AB} = \int d\boldsymbol{x} \, e^{i(\boldsymbol{P}_B - \boldsymbol{P}_A)\cdot \boldsymbol{x}/2} \, \text{Tr} \left\{ \Phi^\dagger_B(\boldsymbol{x}) \gamma^\mu \gamma^0 \Phi_A(\boldsymbol{x}) \right\}$$

With $\partial_\xi E = P$, $\partial_\xi P = E$ and $\partial_\xi \Phi$ as above, the form factor G^μ indeed transforms as a 4-vector.

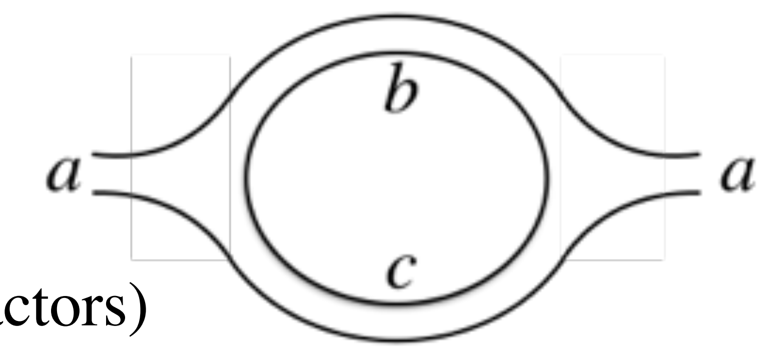
QFT dynamics at $\mathcal{O}\left(\alpha_s^0\right)$

String breaking: Quark pairs created in $V(r)$:



Bound state overlap determined by their wf's: $\langle B\,C|A\rangle$

Hadron loops: Required by unitarity



Dihadron component of hadron wf (cf. form factors)

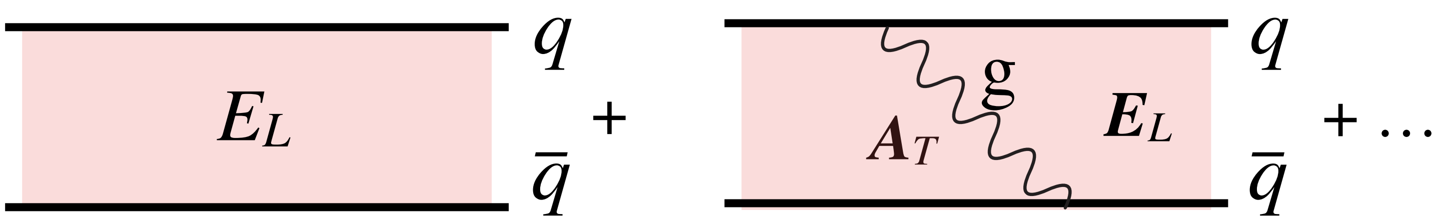
A new α_s^0 world defined by QCD

Summary

Bound states in QED and QCD should be defined systematically, as for the perturbative S-matrix.

In temporal gauge ($A^0 = 0$) the constituents instantaneously determine $\nabla\cdot\boldsymbol{E}_L$

A Bound Fock expansion: Formally exact when summed to all orders in α



Including a homogeneous solution for $\boldsymbol{E}_L$ gives confinement in QCD

The $O(\alpha_s^0)$ “Born term” provides a promising hadron dynamics