A perturbative expansion for bound states

Seminar at CPHT, Ecole Polytechnique
13 December 2022

Paul Hoyer, University of Helsinki

Hadrons and atoms have unexpected similarities

Can the first-principles bound state methods of QED be adapted to QCD?

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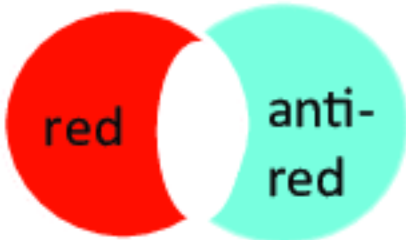
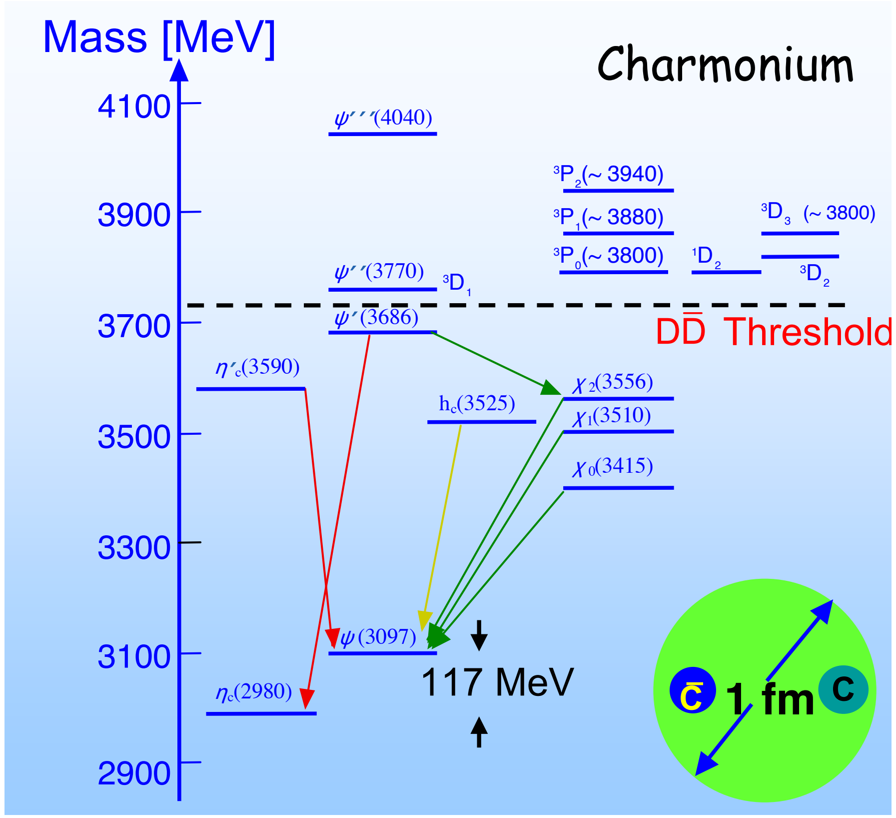
This is a real possibility!

I. Motivations

Non-relativistic bound states

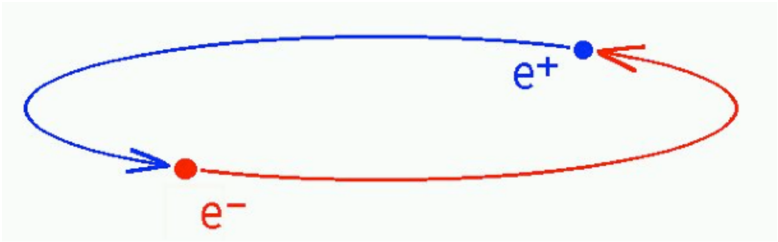
QCD: $b\bar{b}$, $c\bar{c}$ quarkonia

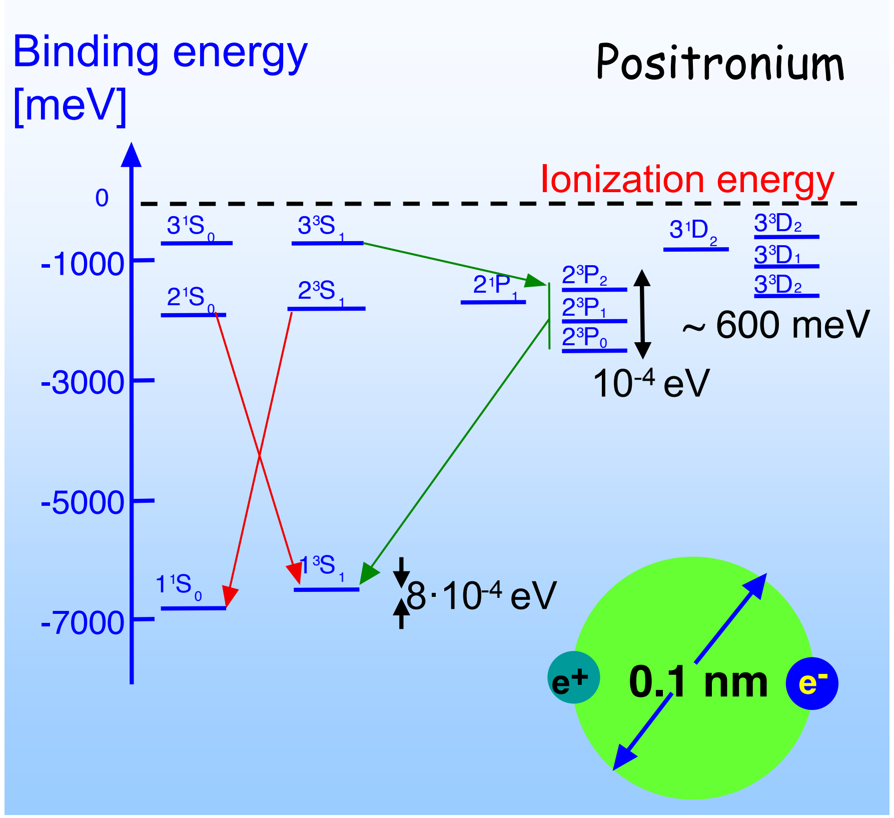




$$V(r) = V'r - \frac{4}{3}\frac{\alpha_s}{r}$$

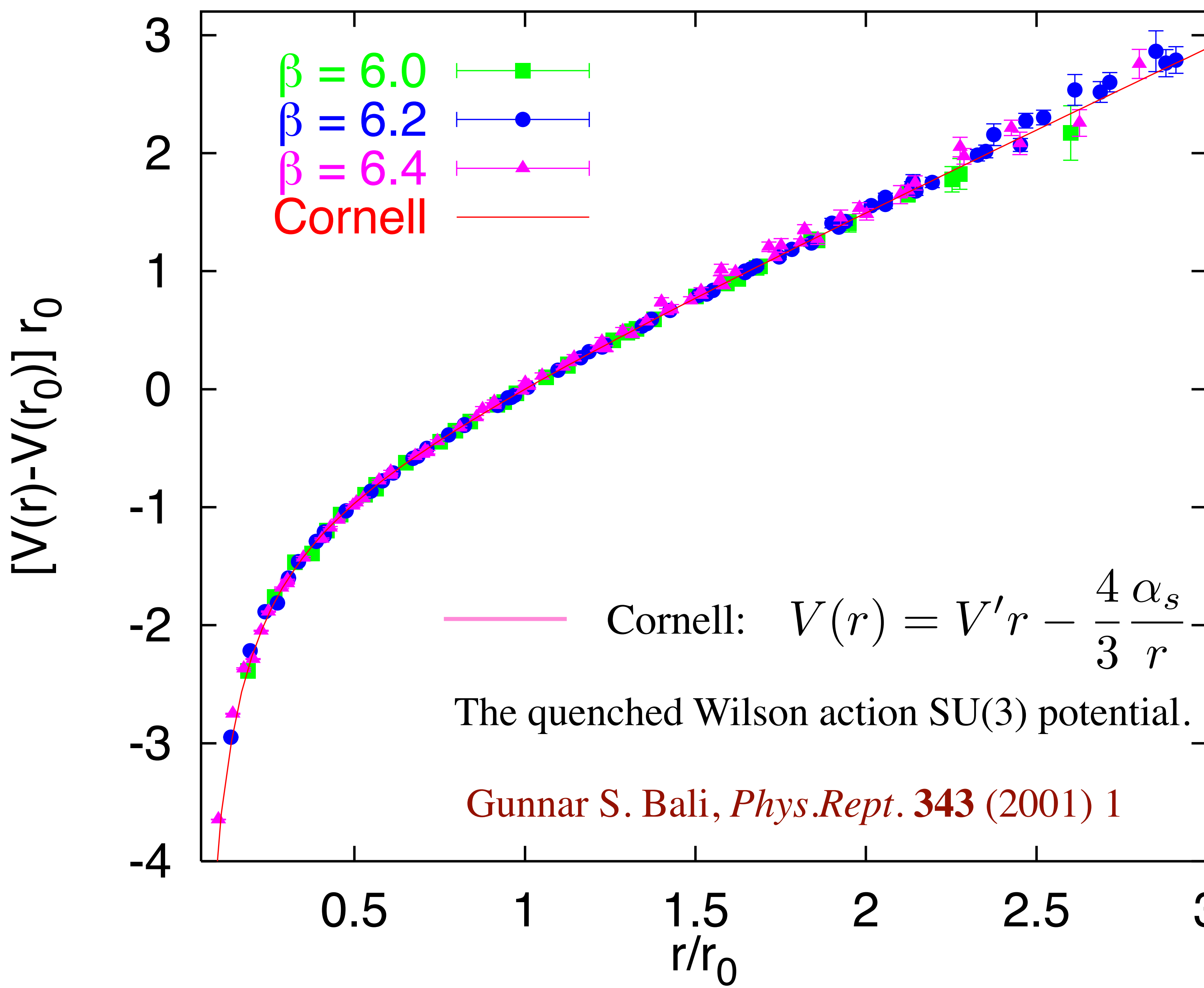
QED: e^+e^- atoms





$$V(r) = -\frac{\alpha}{r}$$

Lattice QCD agrees with the Cornell potential



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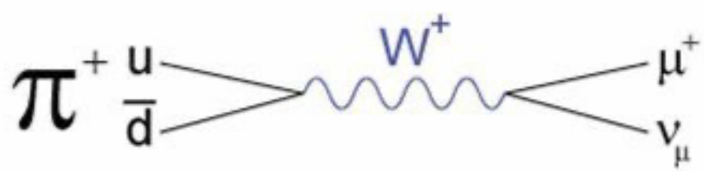
Current quark Fock states

Mesons have a sizeable current $q\bar{q}$ Fock component

E.g., pion decay

Stan Brodsky



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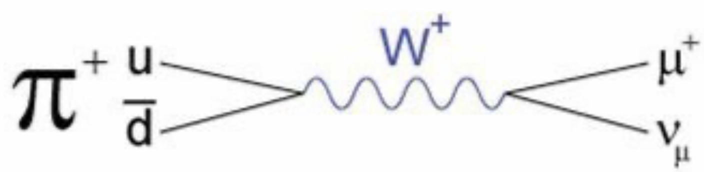

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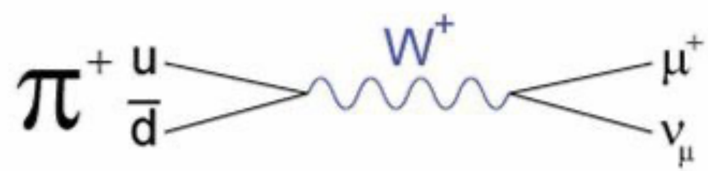

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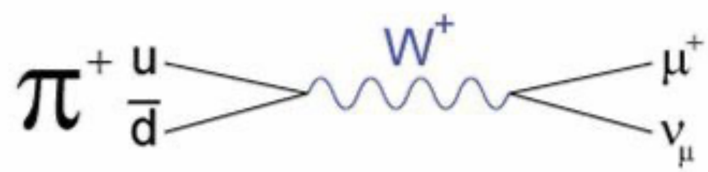

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Why do resonances have narrow widths: Γ << M ?

α_s is small (perturbative)

Strongly bound Positronium in QED_2 (D = 1+1)

In QED_2 the spectrum can be determined both for weak ($e/m << 1$) and strong ($e/m >> 1$) coupling

S. Coleman, Annals Phys. **101** (1976) 239

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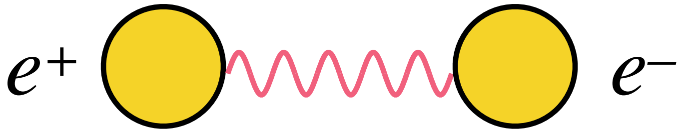
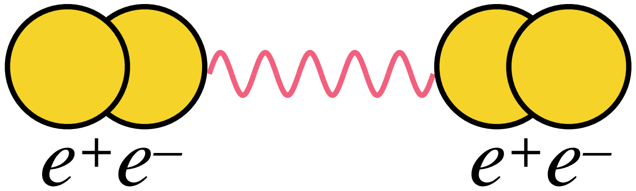


Bound states of weakly interacting fermions

$e/m >> 1$



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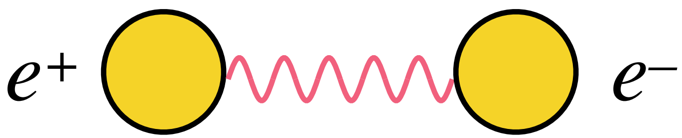

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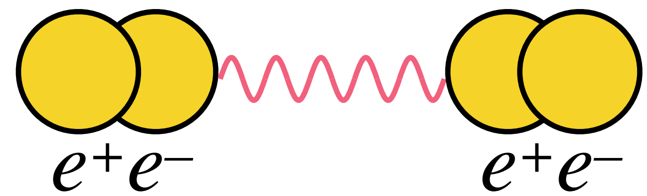

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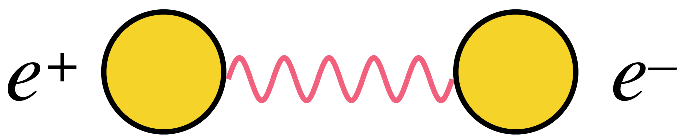

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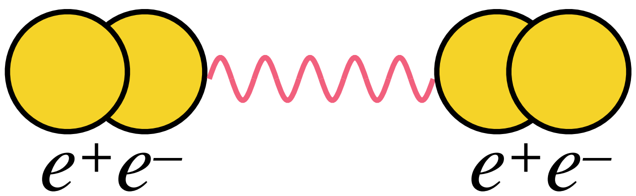

Bound states of weakly interacting bosons

For $e/m \to \infty$ QED_2 describes a non-interacting, pointlike boson field.

Paradox: The hadron spectrum suggests weakly bound valence quarks, yet the light quarks are strongly bound (relativistic).

PQED for atoms is impressive

Example: Hyperfine splitting in Positronium

G. S. Adkins,
Hyperfine Interact. **233** (2015) 59

$$\Delta\nu_{QED} = m_e\alpha^4 \left\{ \frac{7}{12} - \frac{\alpha}{\pi}\left(\frac{8}{9} + \frac{\ln 2}{2}\right) + \frac{\alpha^2}{\pi^2}\left[-\frac{5}{24}\pi^2 \ln\alpha + \frac{1367}{648} - \frac{5197}{3456}\pi^2 + \left(\frac{221}{144}\pi^2 + \frac{1}{2}\right)\ln 2 - \frac{53}{32}\zeta(3)\right] - \frac{7\alpha^3}{8\pi}\ln^2\alpha + \frac{\alpha^3}{\pi}\ln\alpha\left(\frac{17}{3}\ln 2 - \frac{217}{90}\right) + \mathcal{O}\left(\alpha^3\right)\right\} = 203.39169(41)\ \text{GHz}$$

$\Delta\nu_{EXP}$ = 203.394± .002 GHz

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$\Delta\nu_{EXP}$ = 203.394± .002 GHz

Yet the principles of PQED for bound states remain obscure:

"Bound state theory is non-perturbative, but it is possible to develop expressions in increasing orders of α. There is an art in developing a theoretical expression in this manner"

Bodwin, Yennie and Gregorio, Rev. Mod. Phys. **57** (1985) 723

Atoms from the QED action

The Schrödinger equation is postulated in Introductory Quantum Mechanics.

In QFT it should be derived from S_{QED},

– yet this is not done in textbooks.

C.f.: $\sqrt{M^2+P^2} \simeq M + P^2/2M$

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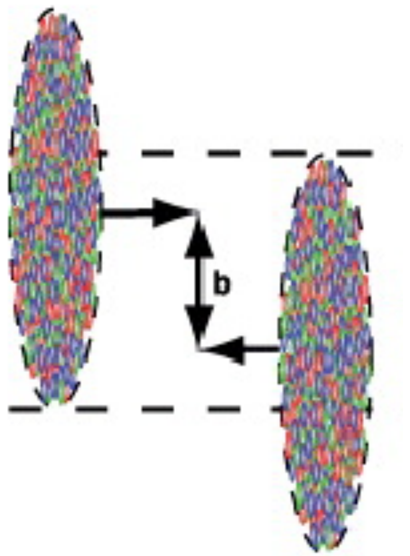
Moving bound states are often depicted as ellipses due to Lorentz contraction

(How) is the classical relativistic concept of contraction realised in QFT?



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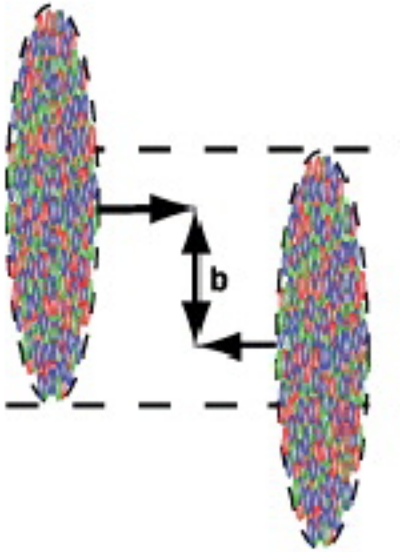

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Poincaré symmetry:

What is the wave function of Positronium in motion?

A strategy for bound states

Assume: The similarities of atoms and hadrons are not “accidental”

Consider the principles for atoms in QED

Try to apply the QED methods to QCD

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How to get strong binding with a small coupling

How can the confinement scale Λ_{QCD} arise?

A strategy for bound states

Assume: The similarities of atoms and hadrons are not “accidental”

Consider the principles for atoms in QED

Try to apply the QED methods to QCD

How to get strong binding with a small coupling

How can the confinement scale Λ_{QCD} arise?

Even failure is instructive:

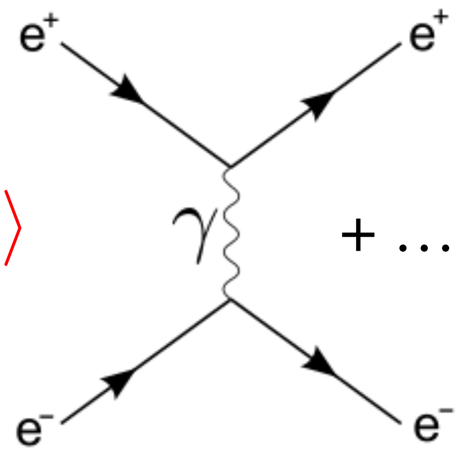
Bound state principles: Art → Theory

Poincaré invariance for atoms

II. Method

Perturbative expansion: Scattering vs. bound states

$e^+e^- \to e^+e^-$

$|in\rangle$  + ... $|out\rangle$

Scattering amplitudes are expanded around free states

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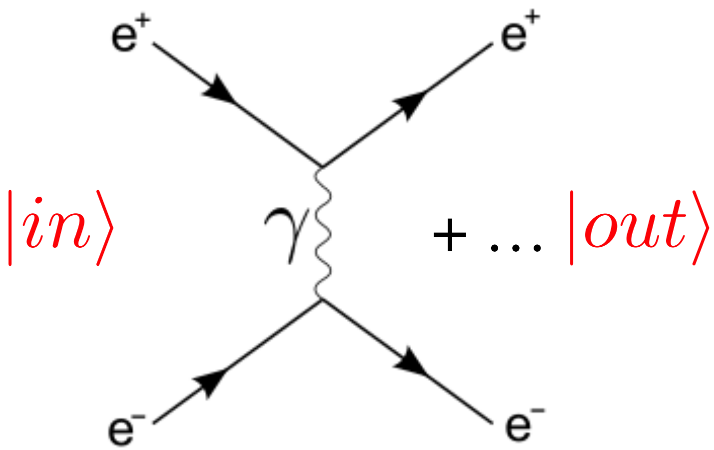

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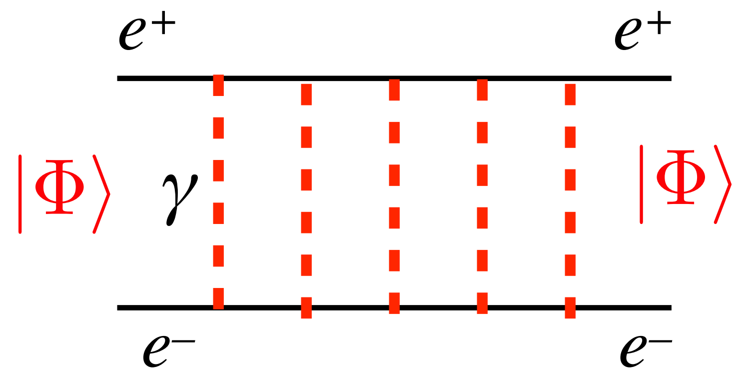

Atoms are expanded around an initial bound state

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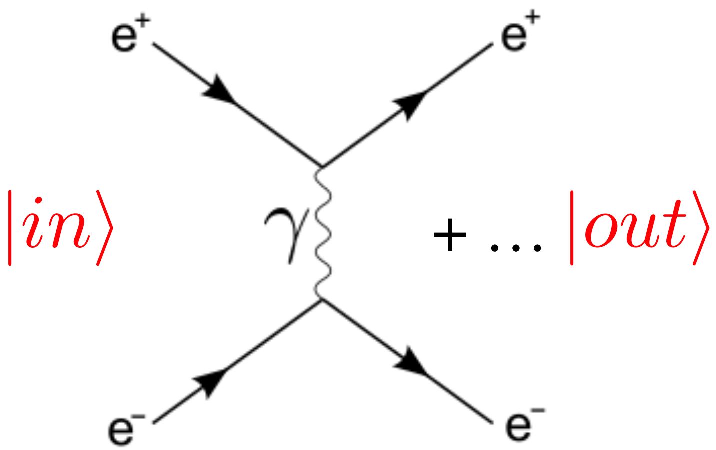

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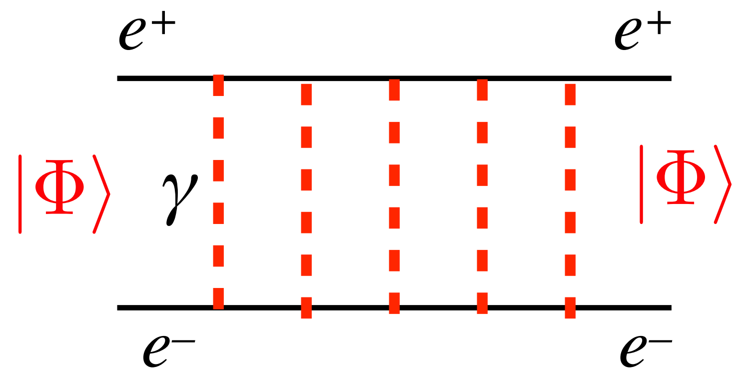

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Schrödinger wave functions for atoms $\Phi_S(\alpha)$ are exponential in α

Their power corrections $\Phi(\alpha)(1 + c_1\alpha + c_2\alpha^2 \ldots)$ depend on $\Phi(\alpha)$.

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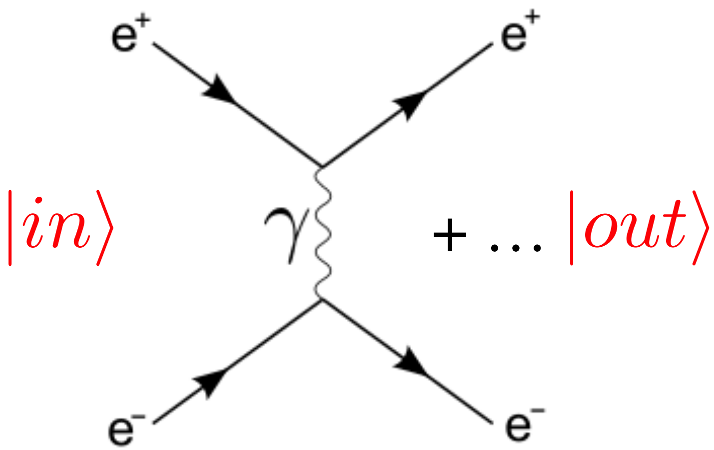

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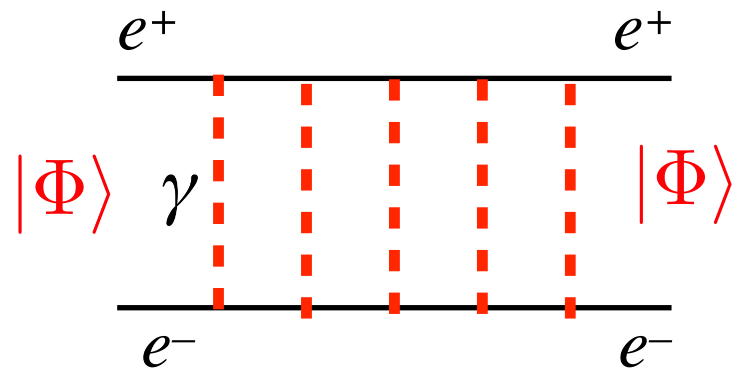

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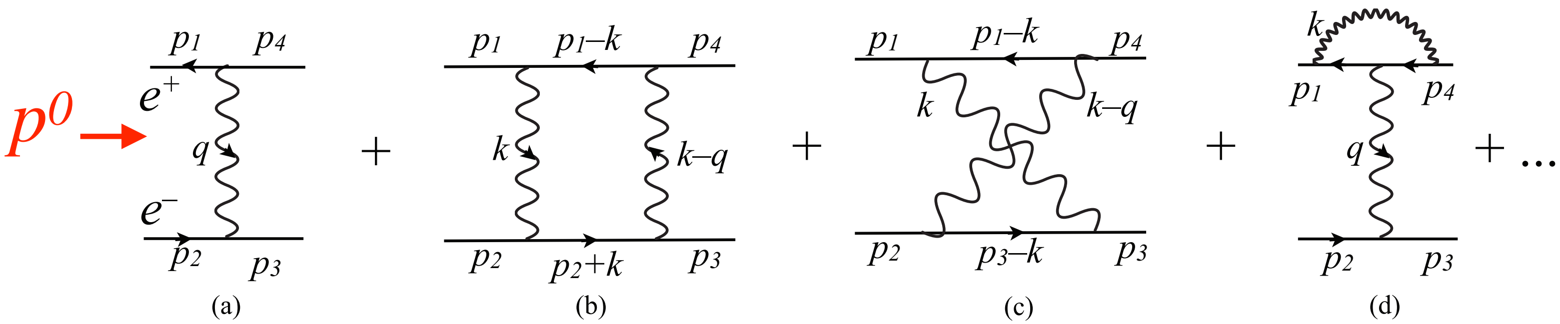
The perturbative expansion for wave functions is not unique, it depends on the choice of initial state.

Caswell & Lepage (1975)

The Schrödinger equation from Feynman diagrams

$$e^+e^- \to e^+e^-$$



$$= \frac{|\varphi_{e^+e^-}|^2}{p^0 - E + i\varepsilon} + \ldots$$

Bound state poles in $e^+e^- \to e^+e^-$ arise only through a divergence of the perturbative sum

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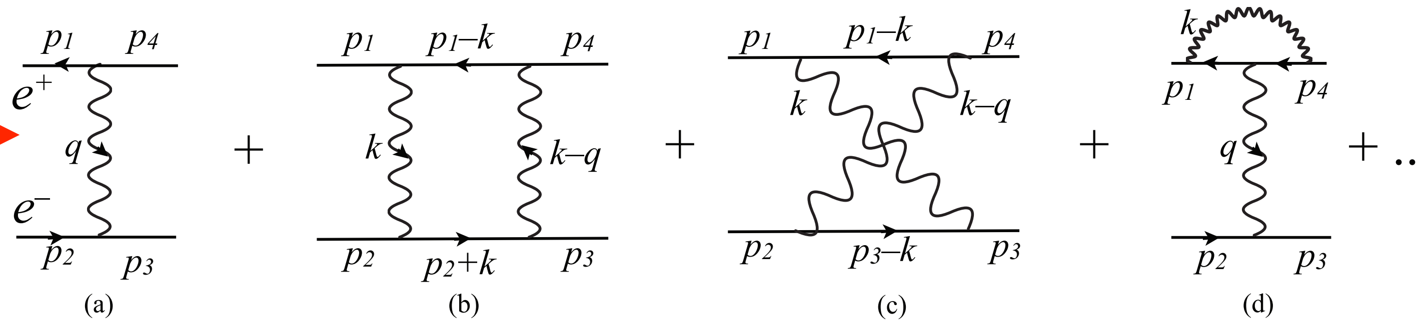

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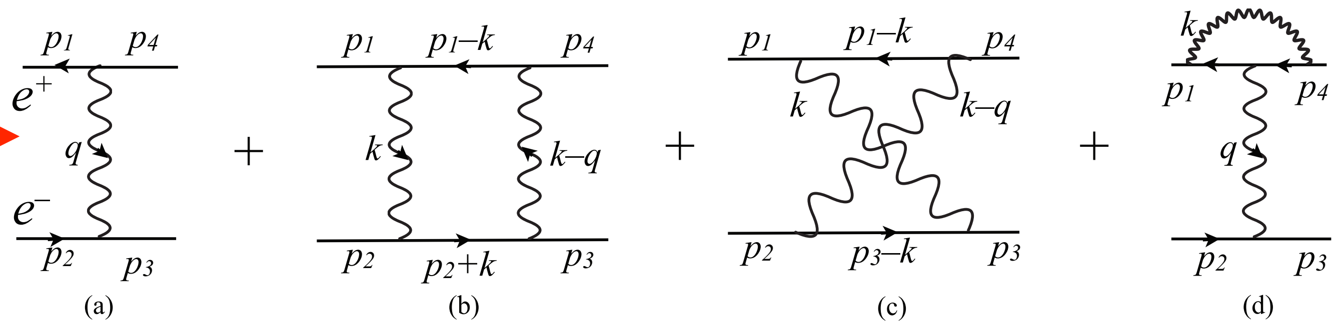

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QED: Sum of "ladder diagrams" generates the classical field $V(r) = -\dfrac{\alpha}{r}$

QCD: $V(r) = V' r$ Free quarks & gluons are an unlikely start for confinement

$\Rightarrow$ **Need to derive the Schrödinger equation with proper boundary conditions**

Instantaneous (Δt = 0) interactions

Equal-time $q\bar{q}$ Fock states must be bound by instantaneous interactions.

Cf. the $V(r) = -\alpha/r$ potential of the NR Schrödinger equation.

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Instantaneous gauge interactions for

$\nabla\cdot\boldsymbol{A}(t,\boldsymbol{x}) = 0$ (Coulomb gauge)

$A^0(t,\boldsymbol{x}) = 0$ (Temporal gauge)

Canonical quantization

Conjugate field π_α

$$\pi_\alpha(t,\boldsymbol{x}) = \frac{\partial \mathcal{L}(\varphi, \partial\varphi)}{\partial[\partial_0 \varphi_\alpha(t,\boldsymbol{x})]}$$

Commutation relations

$$[\varphi_\alpha(t,\boldsymbol{x}), \pi_\beta(t,\boldsymbol{y})]_\pm = i\delta_{\alpha\beta}\delta^3(\boldsymbol{x}-\boldsymbol{y})$$

A^0 has no conjugate field, due to the absence of $\partial_0 A^0$ in $\mathscr{L}_{QED}$.

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This determines $\nabla\cdot\boldsymbol{E}_L$ in terms of the charge distribution in the state.

The classical, instantaneous field E_L

$$\frac{\delta \mathcal{S}_{QED}}{\delta A^0(x)}\,|phys\rangle = 0$$ is not an operator relation, it is a constraint on $|phys\rangle$

$$\frac{\delta \mathcal{S}_{QED}}{\delta A^0(x)}\,|0\rangle = 0$$ implies $\boldsymbol{E}_L = 0$ in the vacuum.

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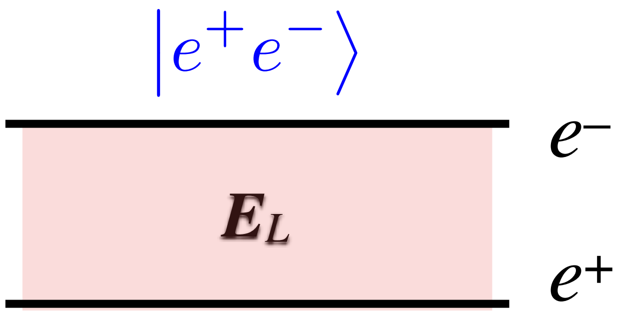
Temporal gauge allows to understand the weak-strong paradox of hadrons

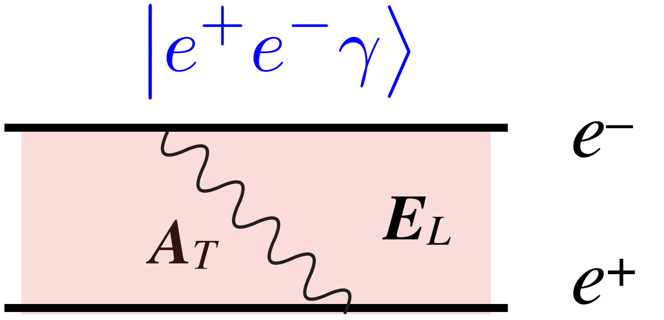
Bound state calculations generally use Coulomb gauge with constraints

Fock state expansion for Positronium in $A^0=0$ gauge

The perturbative expansion in α starts from the $|e^+e^-\rangle$ Fock state, bound by its classical field $\boldsymbol{E}_L$:



Higher order corrections include states with transverse photons and e^+e^- pairs, as determined by $H_{QED}\,|e^+e^-\rangle$



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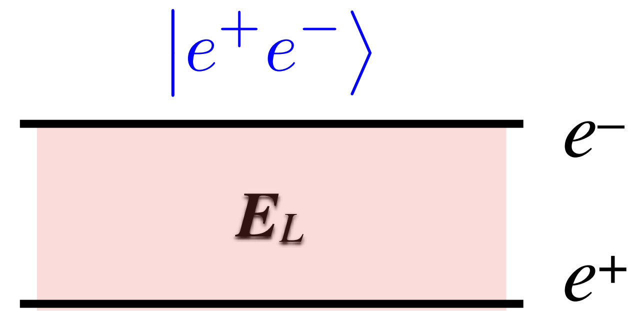

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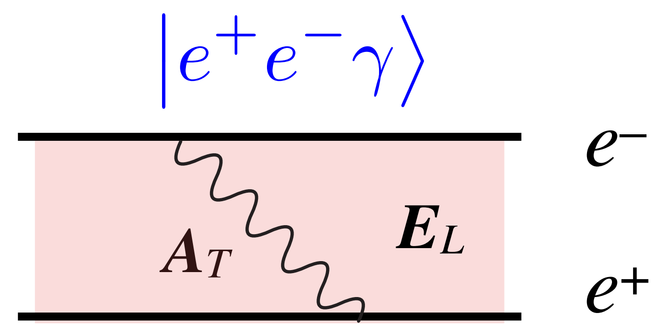

Each Fock component of the bound state includes its particular instantaneous $\boldsymbol{E}_L$ field.

This Fock expansion is valid in any frame, and is formally exact at $O(\alpha^\infty)$.

Positronium in motion: Contraction

The binding energy in the rest frame ($P = 0$) is $E_b = -\alpha^2 m_e/4 + O(\alpha^4)$

At large momenta P the binding is $\propto 1/P$:

$$\Delta E(P) \equiv \sqrt{P^2 + (2m_e + E_b)^2} - \sqrt{P^2 + 4m_e^2} = \frac{2m_e E_b}{P} + \mathcal{O}\left(\alpha^4\right)$$

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Hence the Coulomb potential
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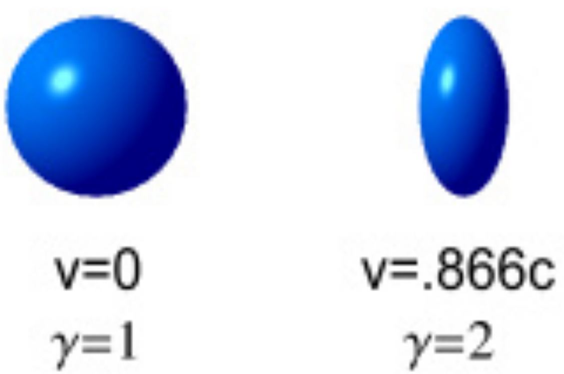

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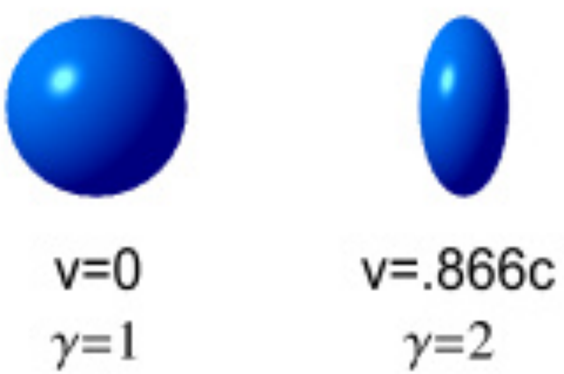

Hence the Coulomb potential
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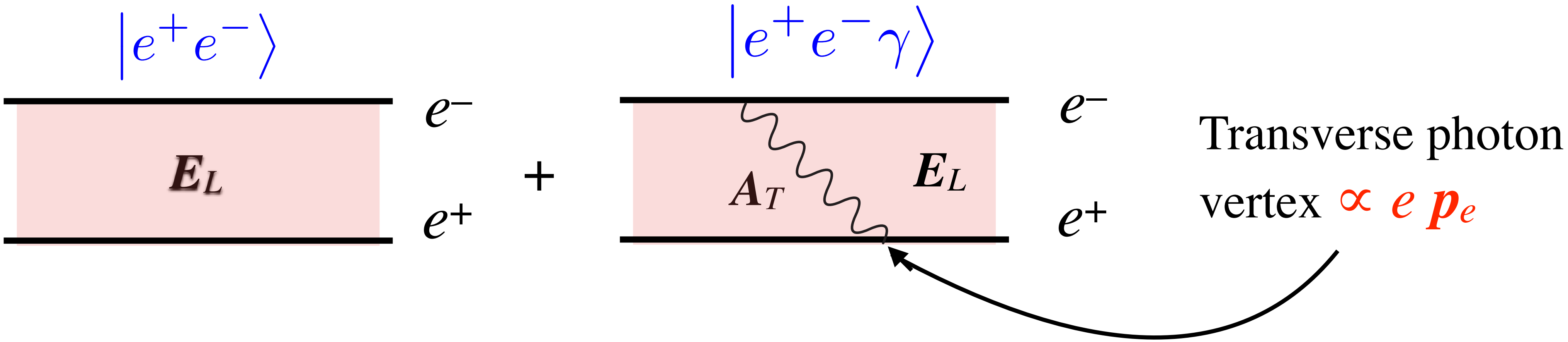
There must be more than contraction going on!

Positronium in motion: Fock expansion



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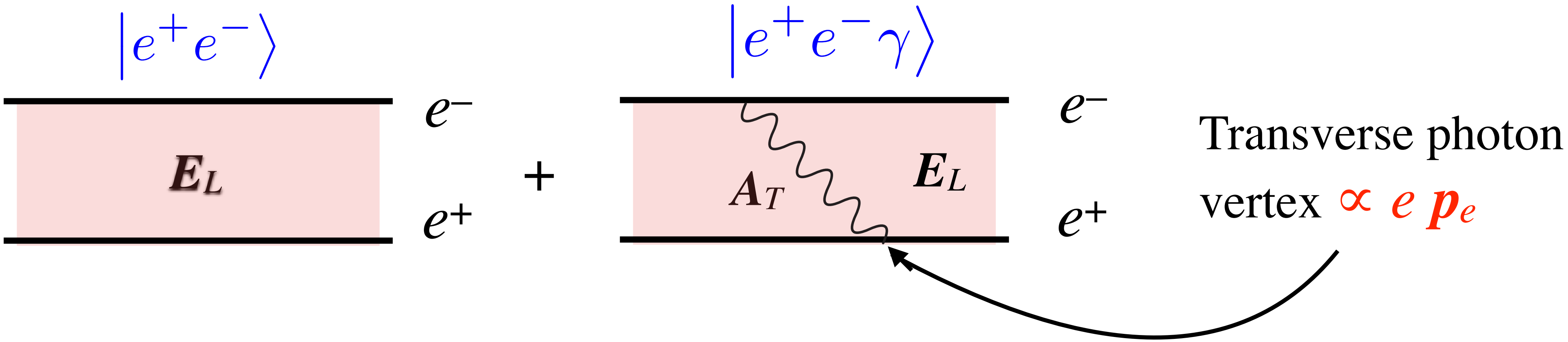

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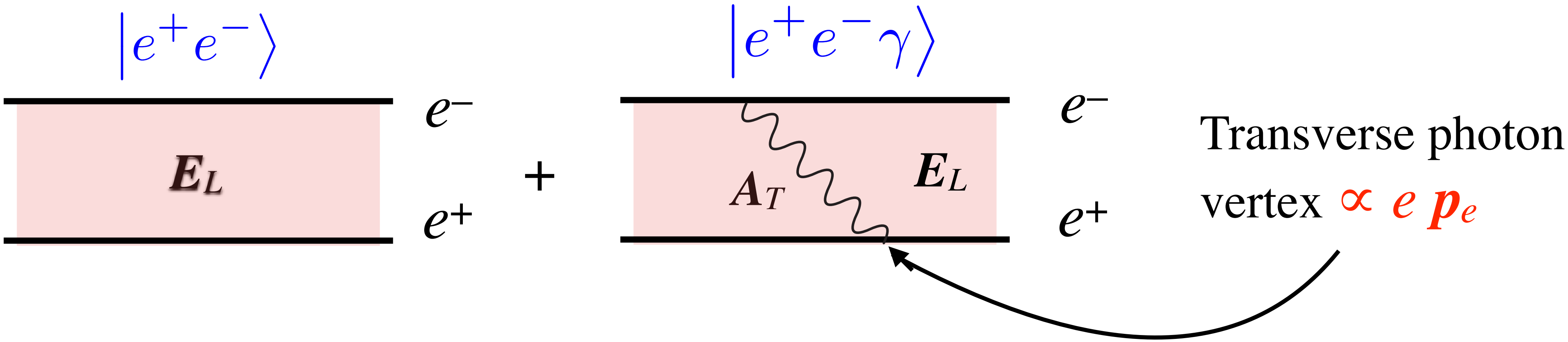

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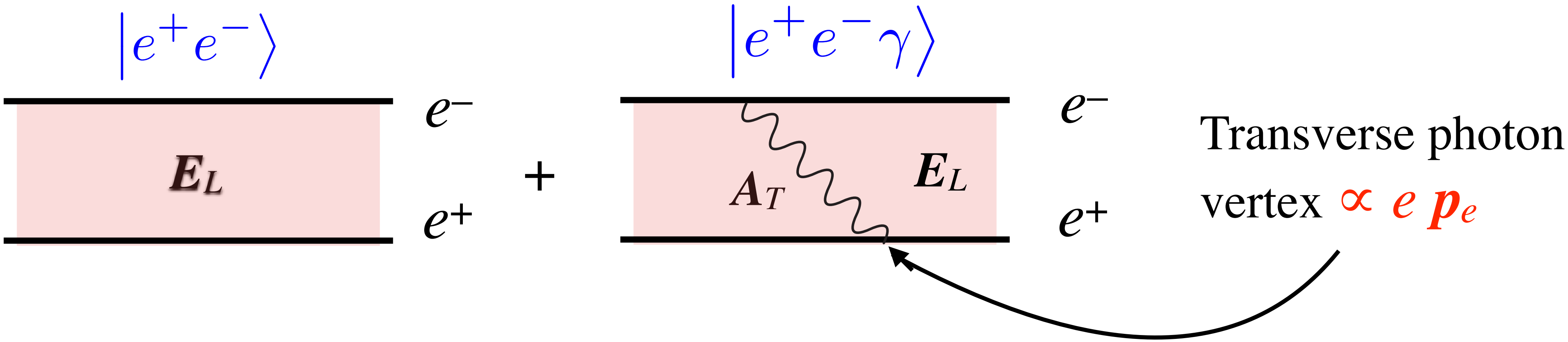

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QFT gets things right when it is treated correctly

III. Application to QCD

Temporal gauge in QCD: $A_a^0 = 0$

The temporal gauge constraint determines $\nabla\cdot \boldsymbol{E}_{L,a}$ for each state:

$$\partial_i E^i_{L,a}(\boldsymbol{x})\,|phys\rangle = g\big[-f_{abc}A^i_b E^i_c + \psi^\dagger T^a \psi(\boldsymbol{x})\big]\,|phys\rangle$$

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The homogeneous solution $\propto \kappa$ is the only one that is compatible with invariance under space translations and rotations

Including the κ ≠ 0 homogeneous solution for $E^i_{L,a}$

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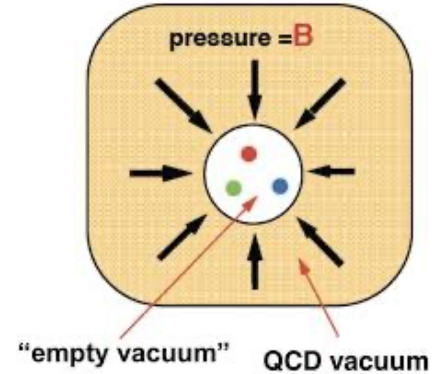

"Bag model without a bag"

Meson Fock state potential

$$|q(\boldsymbol{x}_1)\bar{q}(\boldsymbol{x}_2)\rangle \equiv \sum_A \bar{\psi}^A(\boldsymbol{x}_1)\,\psi^A(\boldsymbol{x}_2)\,|0\rangle$$ globally color singlet

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$$V_{q\bar{q}}(\boldsymbol{x}_1,\boldsymbol{x}_2) = \Lambda^2|\boldsymbol{x}_1-\boldsymbol{x}_2| - C_F\frac{\alpha_s}{|\boldsymbol{x}_1-\boldsymbol{x}_2|}$$ Cornell potential

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This potential is valid also for relativistic $q\bar{q}$ Fock states,
in any frame

Baryon Fock state potential

Baryon: $$|q(\boldsymbol{x}_1)q(\boldsymbol{x}_2)q(\boldsymbol{x}_3)\rangle \equiv \sum_{A,B,C} \epsilon_{ABC}\psi^\dagger_A(\boldsymbol{x}_1)\,\psi^\dagger_B(\boldsymbol{x}_2)\,\psi^\dagger_C(\boldsymbol{x}_3)\,|0\rangle$$

$$V_{qqq}(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3) = \Lambda^2 d_{qqq}(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3) - \frac{2}{3}\,\alpha_s\Big(\frac{1}{|\boldsymbol{x}_1-\boldsymbol{x}_2|} + \frac{1}{|\boldsymbol{x}_2-\boldsymbol{x}_3|} + \frac{1}{|\boldsymbol{x}_3-\boldsymbol{x}_1|}\Big)$$

$$d_{qqq}(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3) \equiv \frac{1}{\sqrt{2}}\sqrt{(\boldsymbol{x}_1-\boldsymbol{x}_2)^2+(\boldsymbol{x}_2-\boldsymbol{x}_3)^2+(\boldsymbol{x}_3-\boldsymbol{x}_1)^2}$$

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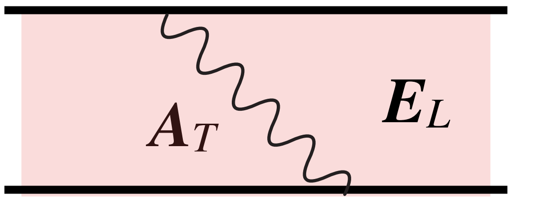
When two of the quarks coincide the potential reduces to the $q\bar{q}$ potential:

$$V_{qqq}(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_2) = \Lambda^2|\boldsymbol{x}_1-\boldsymbol{x}_2| - \frac{4}{3}\,\frac{\alpha_s}{|\boldsymbol{x}_1-\boldsymbol{x}_2|} = V_{q\bar{q}}(\boldsymbol{x}_1,\boldsymbol{x}_2)$$

Analogous potentials are obtained for any quark and gluon Fock state, such as $q\bar{q}g$ and gg.

The qgq̄ potential

A qq̄ state, with the emission of a transverse gluon:



$$|q(\boldsymbol{x}_1)g(\boldsymbol{x}_g)\bar{q}(\boldsymbol{x}_2)\rangle \equiv \sum_{A,B,b} \bar{\psi}_A(\boldsymbol{x}_1)\, A_b^j(\boldsymbol{x}_g)T^b_{AB}\psi_B(\boldsymbol{x}_2)\,|0\rangle$$

$$V^{(0)}_{qgq}(\boldsymbol{x}_1,\boldsymbol{x}_g,\boldsymbol{x}_2) = \frac{\Lambda^2}{\sqrt{C_F}}\, d_{qgq}(\boldsymbol{x}_1,\boldsymbol{x}_g,\boldsymbol{x}_2)$$ (universal Λ)

$$d_{qgq}(\boldsymbol{x}_1,\boldsymbol{x}_g,\boldsymbol{x}_2) \equiv \sqrt{\tfrac{1}{4}(N-2/N)(\boldsymbol{x}_1-\boldsymbol{x}_2)^2 + N(\boldsymbol{x}_g - \tfrac{1}{2}\boldsymbol{x}_1 - \tfrac{1}{2}\boldsymbol{x}_2)^2}$$

$$V^{(1)}_{qgq}(\boldsymbol{x}_1,\boldsymbol{x}_g,\boldsymbol{x}_2) = \tfrac{1}{2}\,\alpha_s\Big[\frac{1}{N}\,\frac{1}{|\boldsymbol{x}_1-\boldsymbol{x}_2|} - N\Big(\frac{1}{|\boldsymbol{x}_1-\boldsymbol{x}_g|} + \frac{1}{|\boldsymbol{x}_2-\boldsymbol{x}_g|}\Big)\Big]$$

When q and g coincide:

$$V^{(0)}_{qgq}(\boldsymbol{x}_1=\boldsymbol{x}_g,\boldsymbol{x}_2) = \Lambda^2|\boldsymbol{x}_1-\boldsymbol{x}_2| = V^{(0)}_{q\bar{q}}$$

$$V^{(1)}_{qgq}(\boldsymbol{x}_1=\boldsymbol{x}_g,\boldsymbol{x}_2) = V^{(1)}_{q\bar{q}}$$

The gg potential

A "glueball" component: $$|g(\boldsymbol{x}_1)g(\boldsymbol{x}_2)\rangle \equiv \sum_a A_a^i(\boldsymbol{x}_1)\, A_a^j(\boldsymbol{x}_2)\, |0\rangle$$

has the potential $$V_{gg} = \sqrt{\frac{N}{C_F}}\,\Lambda^2\,|\boldsymbol{x}_1 - \boldsymbol{x}_2| - N\,\frac{\alpha_s}{|\boldsymbol{x}_1 - \boldsymbol{x}_2|}$$

This agrees with the q$g\bar{\text{q}}$ potential where the quarks coincide:

$$V_{gg}(\boldsymbol{x}, \boldsymbol{x}_g) = V_{qg\bar{q}}(\boldsymbol{x}, \boldsymbol{x}_g, \boldsymbol{x})$$

It is straightforward to work out the instantaneous potential for any Fock state.

$\mathcal{O}\left(\alpha_s^0\right)$ q$\bar{\text{q}}$ bound states

An $\mathcal{O}\left(\alpha_s^0\right)$ meson state with $\boldsymbol{P} = 0$ and wave function Φ:

$$|M\rangle = \sum_{A,B;\alpha,\beta} \int d\boldsymbol{x}_1 d\boldsymbol{x}_2 \, \bar{\psi}_\alpha^A(t=0,\boldsymbol{x}_1)\delta^{AB}\Phi_{\alpha\beta}(\boldsymbol{x}_1-\boldsymbol{x}_2)\psi_\beta^B(t=0,\boldsymbol{x}_2)\,|0\rangle$$

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The (rest frame) bound state condition $H\,|M\rangle = M\,|M\rangle$ gives, at $\mathcal{O}\left(\alpha_s^0\right)$

$$\left[i\gamma^0\boldsymbol{\gamma}\cdot\overrightarrow{\boldsymbol{\nabla}} + m\gamma^0\right]\Phi(\boldsymbol{x}) + \Phi(\boldsymbol{x})\left[i\gamma^0\boldsymbol{\gamma}\cdot\overleftarrow{\boldsymbol{\nabla}} - m\gamma^0\right] = \left[M - V(|\boldsymbol{x}|)\right]\Phi(\boldsymbol{x})$$

where $\boldsymbol{x} \equiv \boldsymbol{x}_1 - \boldsymbol{x}_2$ and $V(|\,\boldsymbol{x}\,|) = V'|\,\boldsymbol{x}\,| = \Lambda^2|\,\boldsymbol{x}\,|$.

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In the non-relativistic limit ($m \gg \Lambda$) this reduces to the Schrödinger equation.
Including the instantaneous gluon exchange potential:

$\Rightarrow$ The quarkonium phenomenology with the Cornell potential.

Separation of radial and angular variables

$$i\boldsymbol{\nabla}\cdot\left\{\gamma^0\boldsymbol{\gamma},\Phi(\boldsymbol{x})\right\}+m\left[\gamma^0,\Phi(\boldsymbol{x})\right]=\left[M-V(\boldsymbol{x})\right]\Phi(\boldsymbol{x})$$

Expanding the 4×4 wave function in a basis of 16 Dirac structures $\Gamma_i(\boldsymbol{x})$

$$\Phi(\boldsymbol{x})=\sum_i\Gamma_i(\boldsymbol{x})F_i(r)Y_{j\lambda}(\hat{\boldsymbol{x}})$$

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We may use rotational, parity and charge conjugation invariance to determine which $\Gamma_i(\boldsymbol{x})$ may occur for a state of given j^{PC}:

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0^{--} trajectory $[s=1,\ \ell=j]:$ $\eta_P=\eta_C=-(-1)^j$ $\gamma^0\gamma_5\,\boldsymbol{\alpha}\cdot\boldsymbol{x},\ \gamma^0\gamma_5\,\boldsymbol{\alpha}\cdot\boldsymbol{x}\times\boldsymbol{L},\ \boldsymbol{\alpha}\cdot\boldsymbol{L},\ \gamma^0\,\boldsymbol{\alpha}\cdot\boldsymbol{L}$

0^{++} trajectory $[s=1,\ \ell=j\pm 1]:$ $\eta_P=\eta_C=+(-1)^j$ $1,\ \boldsymbol{\alpha}\cdot\boldsymbol{x},\ \gamma^0\boldsymbol{\alpha}\cdot\boldsymbol{x},\ \boldsymbol{\alpha}\cdot\boldsymbol{x}\times\boldsymbol{L},\ \gamma^0\boldsymbol{\alpha}\cdot\boldsymbol{x}\times\boldsymbol{L},\ \gamma^0\gamma_5\,\boldsymbol{\alpha}\cdot\boldsymbol{L}$

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$\Rightarrow$ There are no solutions for quantum numbers that would be exotic in the NR quark model (despite the relativistic dynamics)

The BSE gives the radial equations for the $F_i(r)$

(There are two coupled radial equations for the 0^{++} trajectory)

Example: 0^{-+} trajectory wf's

$$\Phi_{-+}(\boldsymbol{x}) = \Big[\frac{2}{M-V}(i\boldsymbol{\alpha}\cdot\overset{\rightarrow}{\boldsymbol{\nabla}} + m\gamma^0) + 1\Big]\gamma_5\, F_1(r)Y_{j\lambda}(\hat{\boldsymbol{x}})$$

$\eta_P = (-1)^{j+1}$

$\eta_C = (-1)^j$

Radial equation: $F_1'' + \Big(\frac{2}{r} + \frac{V'}{M-V}\Big)F_1' + \Big[\frac{1}{4}(M-V)^2 - m^2 - \frac{j(j+1)}{r^2}\Big]F_1 = 0$

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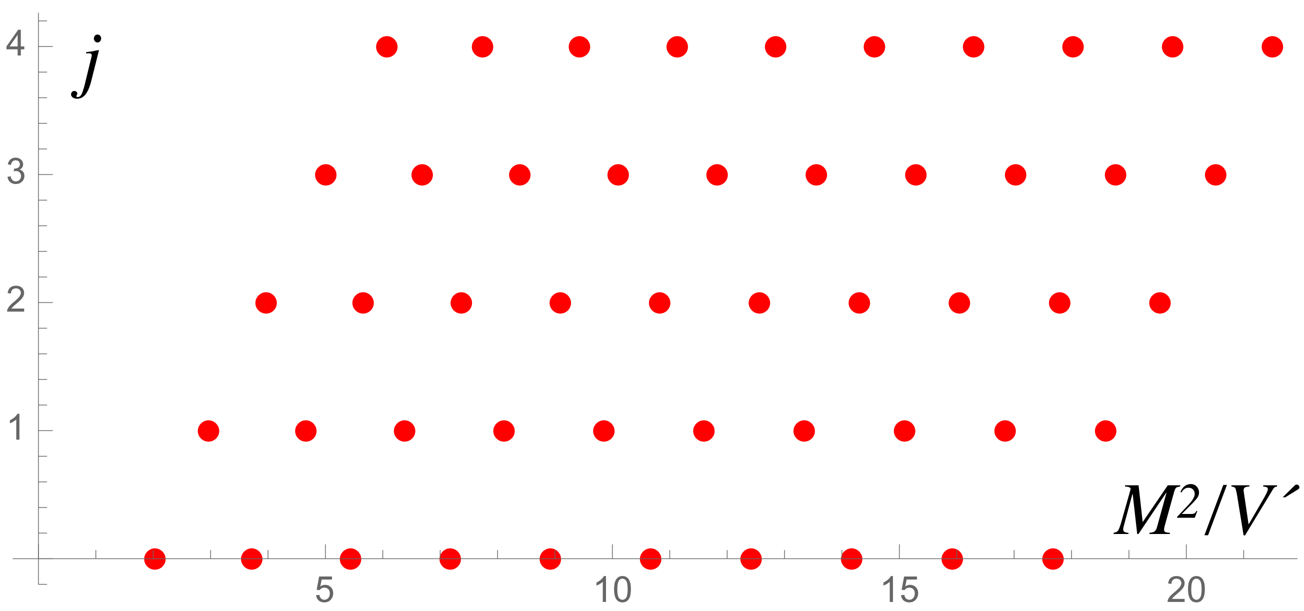
$m = 0$

Mass spectrum:

Linear Regge trajectories with daughters

Spectrum similar to dual models



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Many features of hadrons thus obtained look promising & intriguing

PH 2109.06257

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