The (non-)perturbative nature of atoms and hadrons

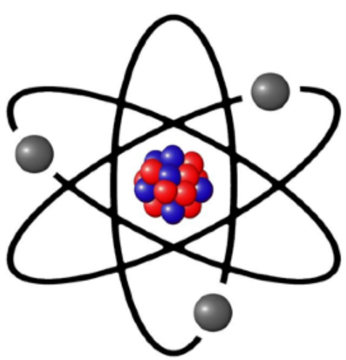
Complutense University, Madrid, 13 October 2021

Paul Hoyer
University of Helsinki

I. Features of bound states

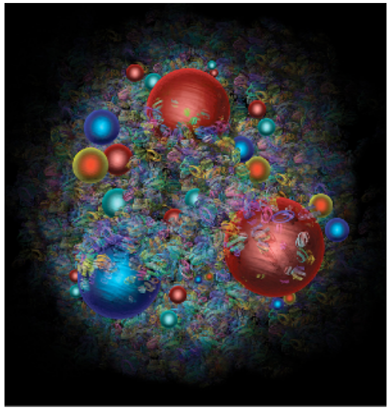
II. A method for all frames

III. Applications to hadrons



Atom

From QED to QCD



Nucleon

THE STATE IS NOT ABOLISHED, IT WITHERS AWAY: HOW QUANTUM FIELD THEORY BECAME A THEORY OF SCATTERING

Alexander S. Blum†

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12th November 2020

Learning quantum field theory (QFT) for the first time, after first learning quantum mechanics (QM), one is (or maybe, rather, I was) struck by the change of emphasis: The notion of the quantum state, which plays such an essential role in QM, from the stationary states of the Bohr atom, over the Schrödinger equation to the interpretation debates over measurement and collapse, seems to fade from view when doing QFT.

2011.0598

I. Features of bound states

Atoms from the QED action

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In QFT it should be derived from S_{QED}. *C.f.*: $\sqrt{M^2+P^2} \simeq M + P^2/2M$

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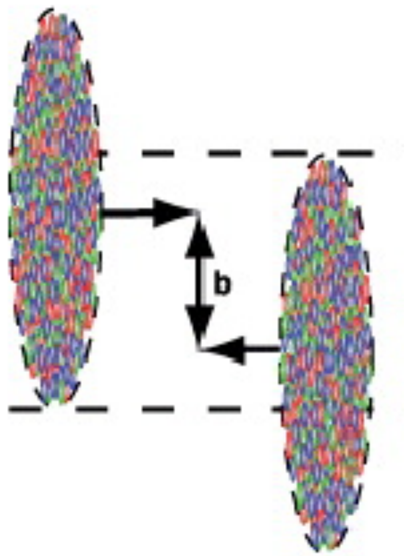
Moving bound states are often depicted as ellipses due to Lorentz contraction



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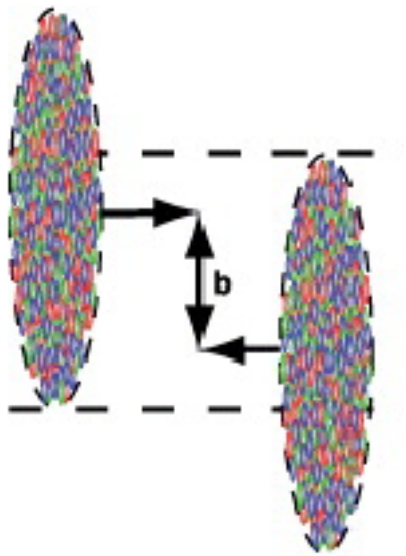

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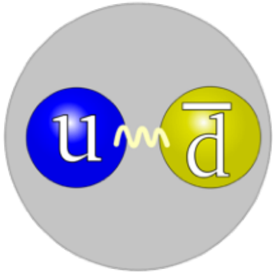
What is the wave function of Positronium in motion?

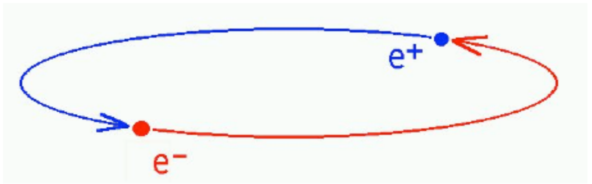
The unbearable likeness of hadrons and atoms



QCD Meson

Hadrons are strongly bound

$$M_N \gg 2m_u + m_d$$



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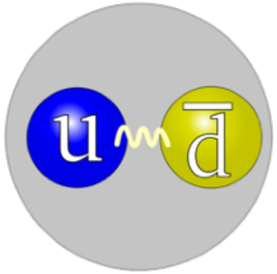

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Yet hadron quantum numbers reflect their valence quarks:

$q\bar{q}, qqq$ $\quad n^{2s+1}\ell_J$

Paradox:
Hadrons are strongly bound, but their quantum numbers indicates weak binding.

PDG

$n^{2s+1}\ell_J$	J^{PC}	$I = 1$ $u\bar{d}$, $\bar{u}d$, $\frac{1}{\sqrt{2}}(d\bar{d} - u\bar{u})$	$I = \frac{1}{2}$ $u\bar{s}$, $d\bar{s}$; $\bar{d}s$, $\bar{u}s$	$I = 0$ f'	$I = 0$ f	$\theta_{\rm quad}$ [°]	$\theta_{\rm lin}$ [°]
1^1S_0	0^{-+}	π	K	η	$\eta'(958)$	−11.3	−24.5
1^3S_1	1^{--}	$\rho(770)$	$K^*(892)$	$\phi(1020)$	$\omega(782)$	39.2	36.5
1^1P_1	1^{+-}	$b_1(1235)$	$K_{1B}{}^\dagger$	$h_1(1415)$	$h_1(1170)$		
1^3P_0	0^{++}	$a_0(1450)$	$K_0^*(1430)$	$f_0(1710)$	$f_0(1370)$		
1^3P_1	1^{++}	$a_1(1260)$	$K_{1A}{}^\dagger$	$f_1(1420)$	$f_1(1285)$		
1^3P_2	2^{++}	$a_2(1320)$	$K_2^*(1430)$	$f_2'(1525)$	$f_2(1270)$	29.6	28.0
1^1D_2	2^{-+}	$\pi_2(1670)$	$K_2(1770)^\dagger$	$\eta_2(1870)$	$\eta_2(1645)$		
1^3D_1	1^{--}	$\rho(1700)$	$K^*(1680)^\ddagger$		$\omega(1650)$		
1^3D_2	2^{--}		$K_2(1820)^\dagger$				
1^3D_3	3^{--}	$\rho_3(1690)$	$K_3^*(1780)$	$\phi_3(1850)$	$\omega_3(1670)$	31.8	30.8
1^3F_4	4^{++}	$a_4(1970)$	$K_4^*(2045)$	$f_4(2300)$	$f_4(2050)$		
1^3G_5	5^{--}	$\rho_5(2350)$	$K_5^*(2380)$				
2^1S_0	0^{-+}	$\pi(1300)$	$K(1460)$	$\eta(1475)$	$\eta(1295)$		
2^3S_1	1^{--}	$\rho(1450)$	$K^*(1410)^\ddagger$	$\phi(1680)$	$\omega(1420)$		
2^3P_1	1^{++}	$a_1(1640)$					
2^3P_2	2^{++}	$a_2(1700)$	$K_2^*(1980)$	$f_2(1950)$	$f_2(1640)$		

Strongly bound Positronium in QED_2 (D = 1+1)

In QED_2 the spectrum can be determined both for weak ($e/m << 1$) and strong ($e/m >> 1$) coupling

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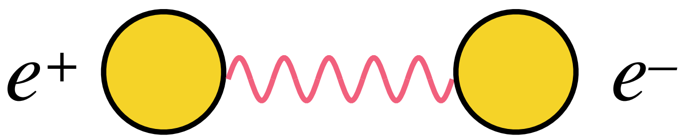

Bound states of weakly interacting fermions

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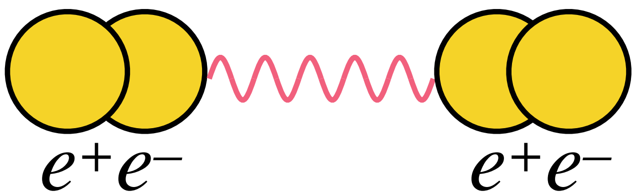

Bound states of weakly interacting bosons

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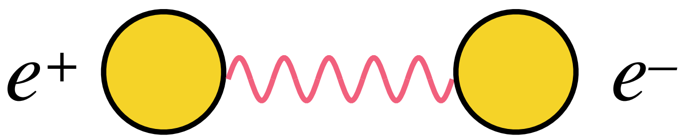

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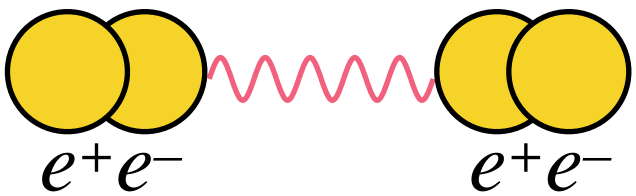

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For $e/m \to \infty$ QED_2 describes a non-interacting, pointlike boson field.

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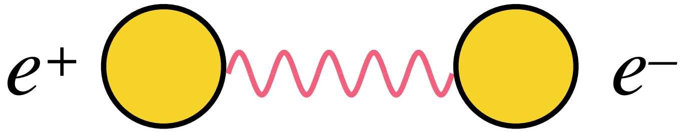

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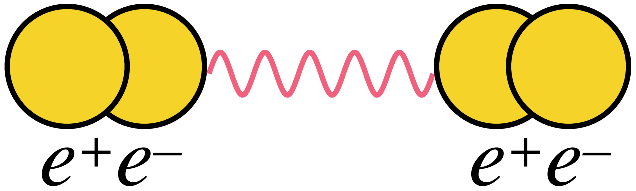

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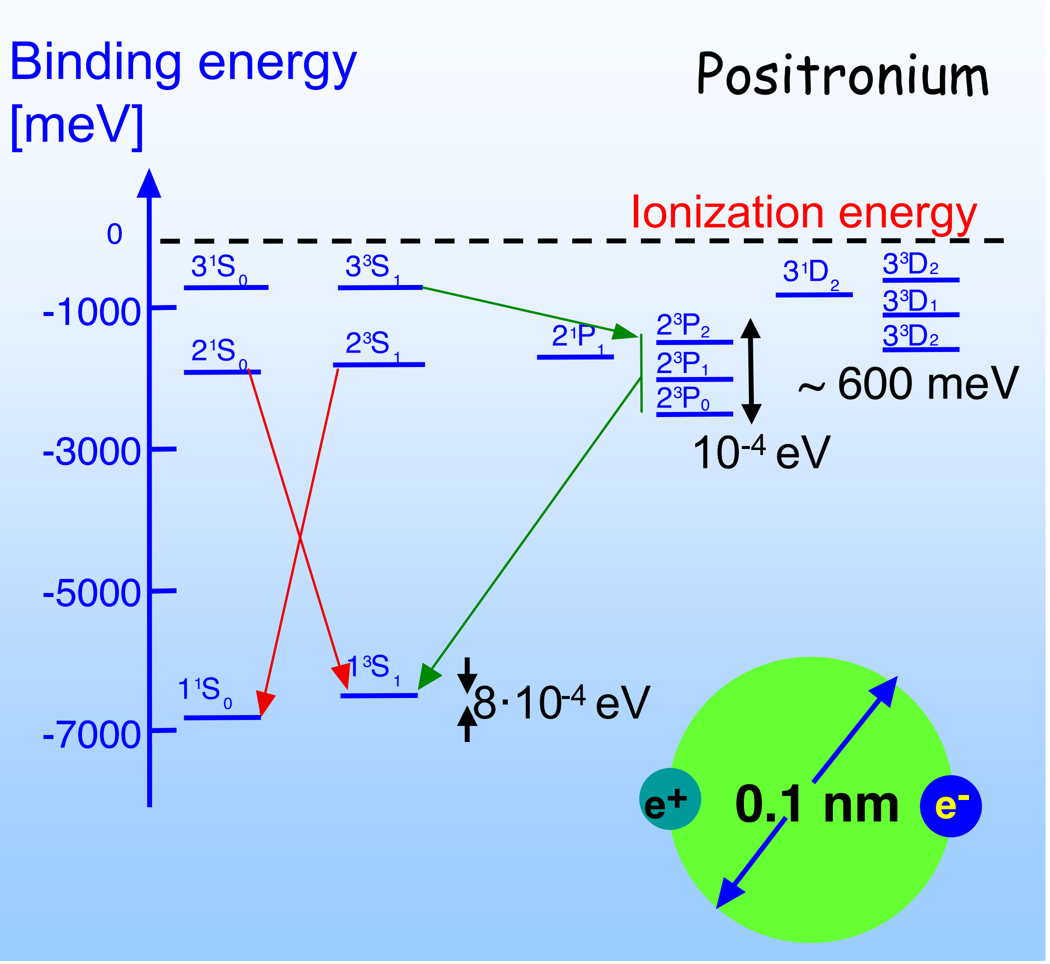
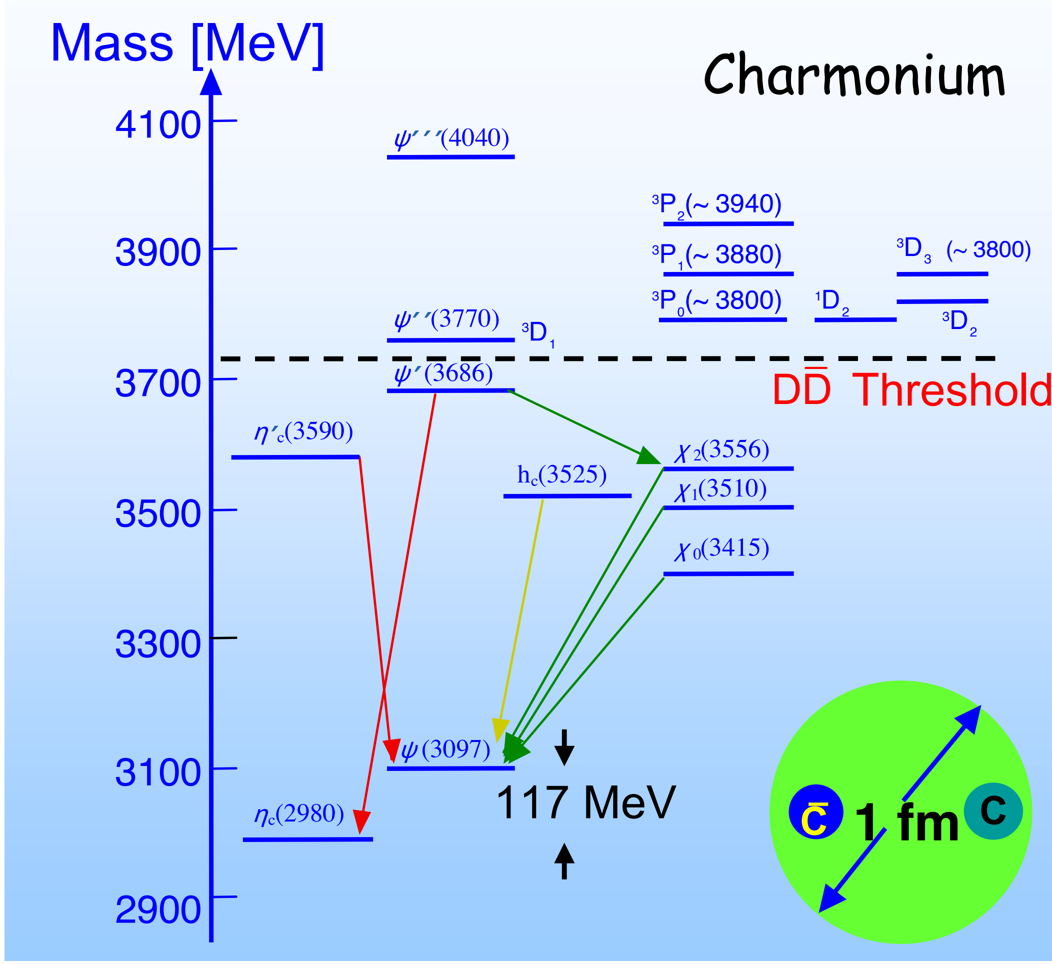
The hadron spectrum suggests weakly bound valence quarks, yet the binding energies are large, indicating strong coupling.

Quarkonia are like atoms with confinement





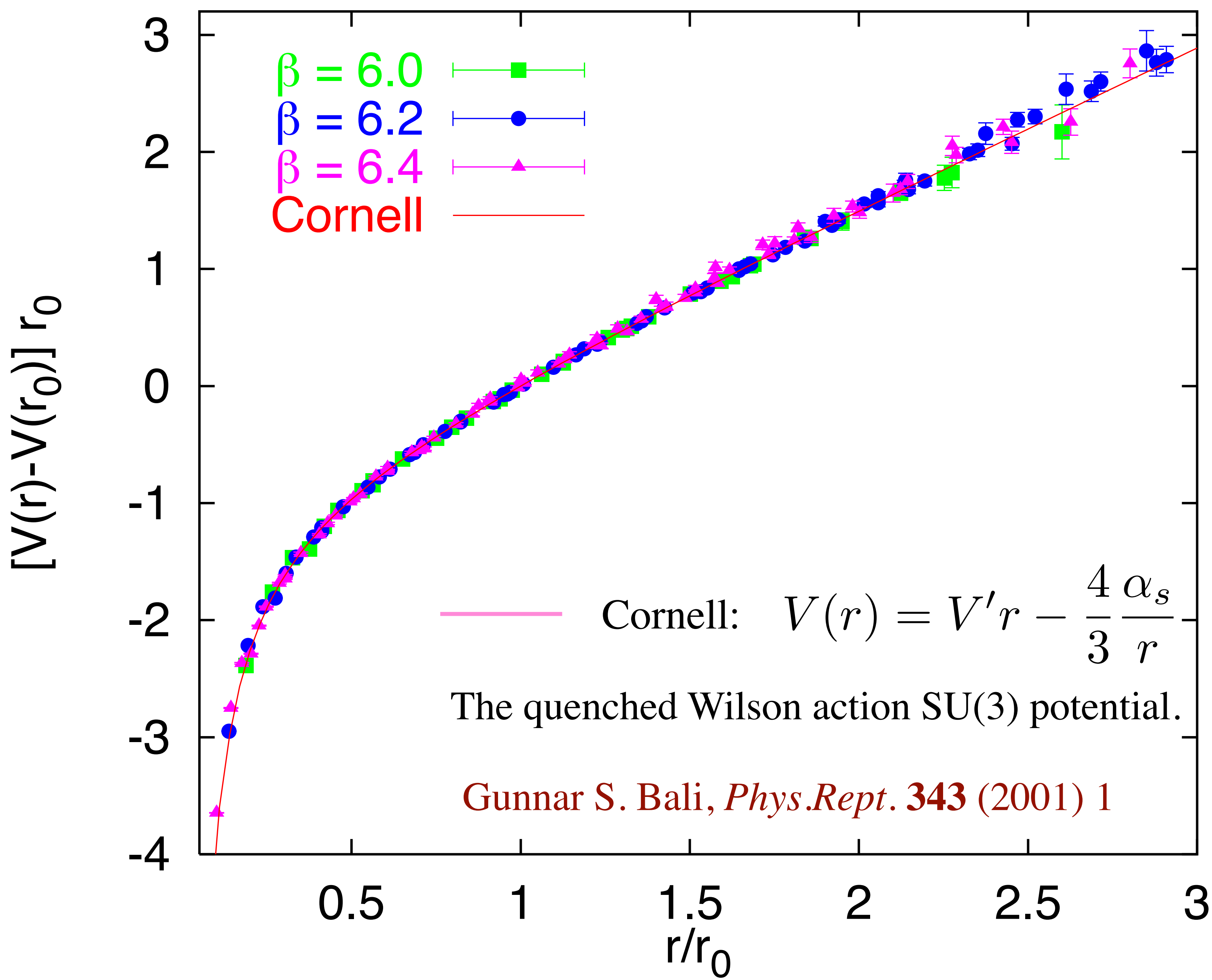
$$V(r) = -\frac{\alpha}{r}$$

$$V(r) = V'r - \frac{4}{3}\frac{\alpha_s}{r} \quad (1980)$$

E. Eichten, S. Godfrey, H. Mahlke and J. L. Rosner,
Rev. Mod. Phys. **80** (2008) 1161

"The J/ψ is the Hydrogen atom of QCD"

Lattice QCD agrees with the Cornell potential



Confinement with a classical potential

The Cornell potential with the Schrödinger equation

$$V(r) = V'r - \frac{4}{3}\frac{\alpha_s}{r} \quad \text{with} \;\; V' \simeq 0.18 \text{ GeV}^2, \quad \alpha_s \simeq 0.39$$

describes confinement with a classical gluon field.

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A classical field does not create particles.

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Consider the perturbative methods developed for QED atoms

PQED for atoms is impressive

Example: Hyperfine splitting in Positronium

G. S. Adkins,
Hyperfine Interact. **233** (2015) 59

$$\Delta\nu_{QED} = m_e\alpha^4\left\{\frac{7}{12} - \frac{\alpha}{\pi}\left(\frac{8}{9} + \frac{\ln 2}{2}\right)\right.$$
$$+\frac{\alpha^2}{\pi^2}\left[-\frac{5}{24}\pi^2\ln\alpha + \frac{1367}{648} - \frac{5197}{3456}\pi^2 + \left(\frac{221}{144}\pi^2 + \frac{1}{2}\right)\ln 2 - \frac{53}{32}\zeta(3)\right]$$
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Yet the principles of PQED remain obscure:

"Bound state theory is non-perturbative, but it is possible to develop expressions in increasing orders of α. There is an art in developing a theoretical expression in this manner"

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Only the rest frame is considered.

II. A method for all frames

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Physics guides the proper choice of initial state.

Instantaneous interactions

Positronium $|e^+e^-\rangle$ states have an instantaneous $-\alpha/r$ potential.
Retardation effects are described by transverse photons ($|e^+e^-\gamma\rangle$ states).

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Initial bound states are simplified by an instantaneous potential

$\Rightarrow$ Coulomb $(\nabla\cdot\boldsymbol{A}_L = 0)$ or temporal $(A^0 = 0)$ gauge are preferable

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- Canonical quantisation straightforward (unlike in $\nabla\cdot \boldsymbol{A} = 0$ gauge)

$$\left[E^i(t,\boldsymbol{x}), A^j(t,\boldsymbol{y})\right] = i\delta^{ij}\delta(\boldsymbol{x}-\boldsymbol{y})$$

- Time-independent gauge transformations are fixed by Gauss constraint

$$\frac{\delta S}{\delta A^0(t,\boldsymbol{x})}\,|phys\rangle = 0$$

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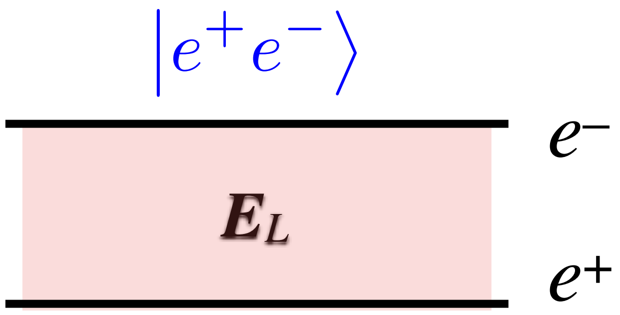
$\mathcal{H}_V \equiv \frac{1}{2}\int d\boldsymbol{x}\,\boldsymbol{E}_L^2$ gives the potential energy. For $|e^-(\boldsymbol{x}_1)\; e^+(\boldsymbol{x}_2)\rangle$,

$$\mathcal{H}_V\,\bar\psi_\alpha(\boldsymbol{x}_1)\psi_\beta(\boldsymbol{x}_2)\left|0\right\rangle = -\frac{\alpha}{|\boldsymbol{x}_1-\boldsymbol{x}_2|}\bar\psi_\alpha(\boldsymbol{x}_1)\psi_\beta(\boldsymbol{x}_2)\left|0\right\rangle$$

Fock state expansion for Positronium in $A^0=0$ gauge

The initial state is chosen to be the $|e^+e^-\rangle$ Fock state, bound by the classical field $\boldsymbol{E}_L$ of its constituents:



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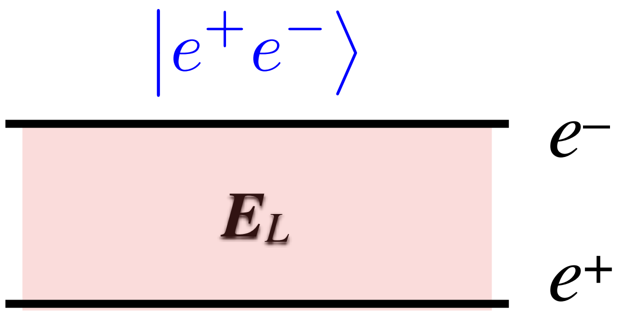
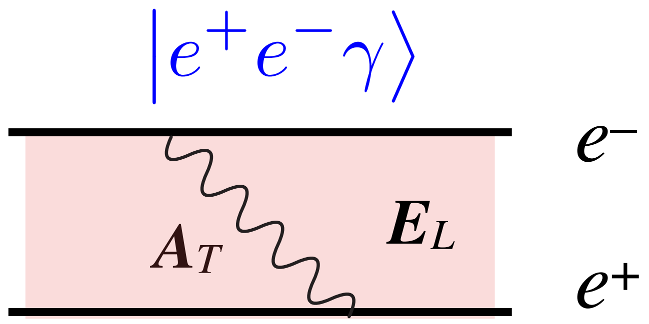


Higher order corrections given by states with transverse photons and e^+e^- pairs, as determined by $H\,|e^+e^-\rangle$, *etc*.



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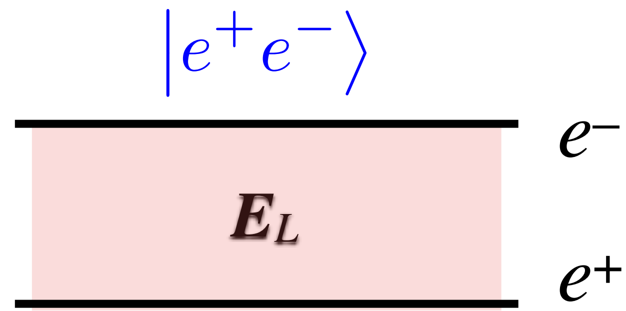

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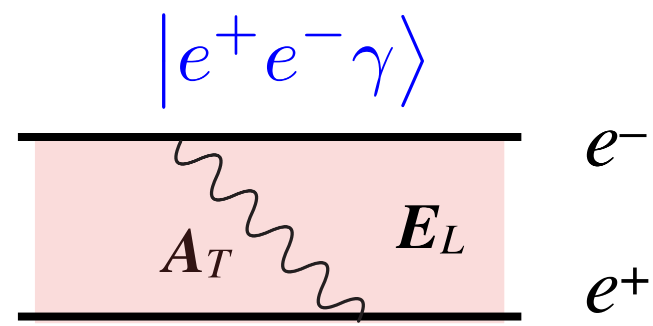

Each Fock component of the bound state includes the instantaneous $\boldsymbol{E}_L$ field in H_V.

This Fock expansion is valid in any frame.

Positronium in motion: Contraction

The binding energy in the rest frame ($P = 0$) is $E_b = -\,\alpha^2\, m_e/4 + O(\alpha^4)$

At large momenta P the binding is $\propto 1/P$:

$$\Delta E(P) \equiv \sqrt{P^2 + (2m_e + E_b)^2} - \sqrt{P^2 + 4m_e^2} = \frac{2m_e E_b}{P} + \mathcal{O}\left(\alpha^4\right)$$

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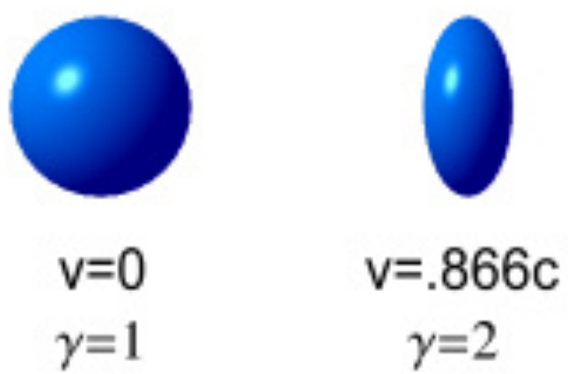

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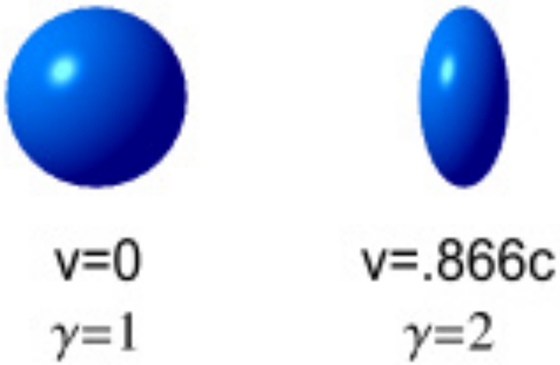
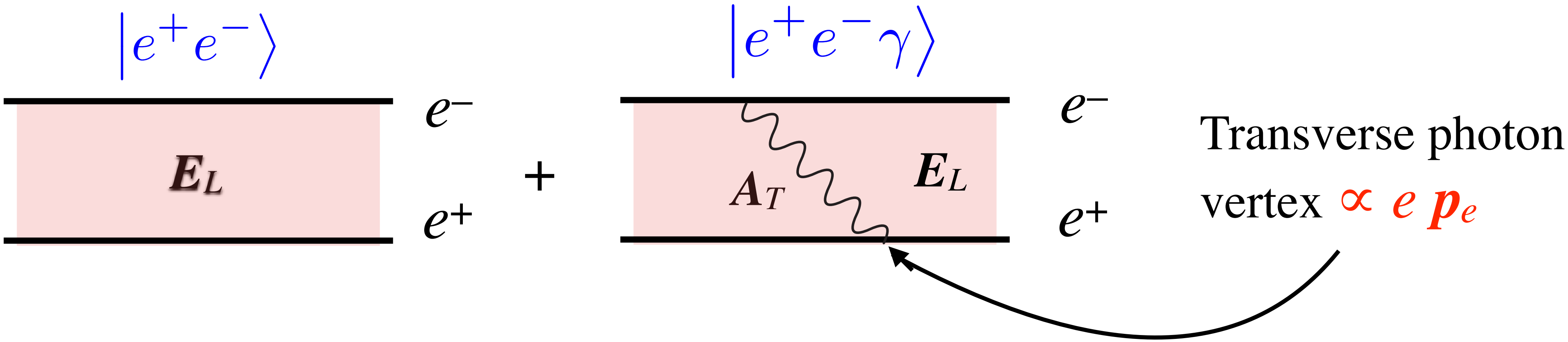


There must be more than contraction going on!

Positronium in motion: Fock expansion



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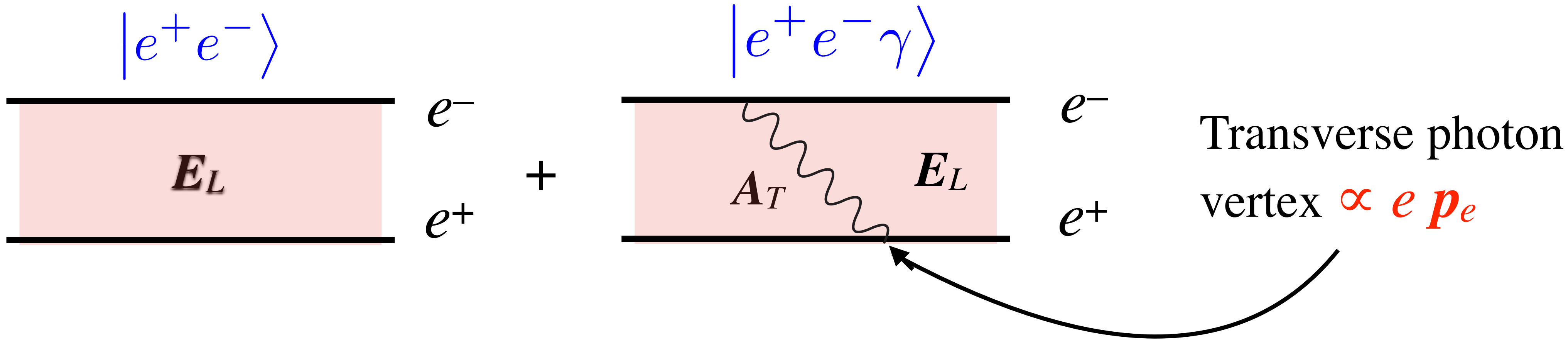

In the rest frame: $\boldsymbol{p}_e \simeq \alpha\, m_e$: transverse photon contribution is $O(\alpha^4)$

For $P > 0$: $\boldsymbol{p}_e \simeq P/2$: transverse photon contribution is leading, $O(\alpha^2)$

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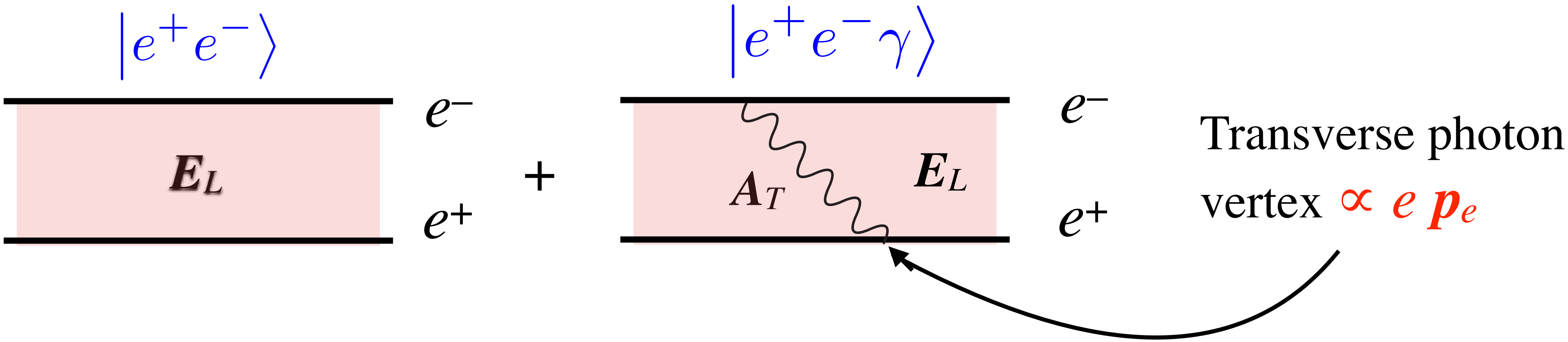

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M. Järvinen, Phys. Rev. **D71** (2005) 085006, PH 2101.06721

Higher Fock states do not contribute to the binding energy at $O(\alpha^2)$

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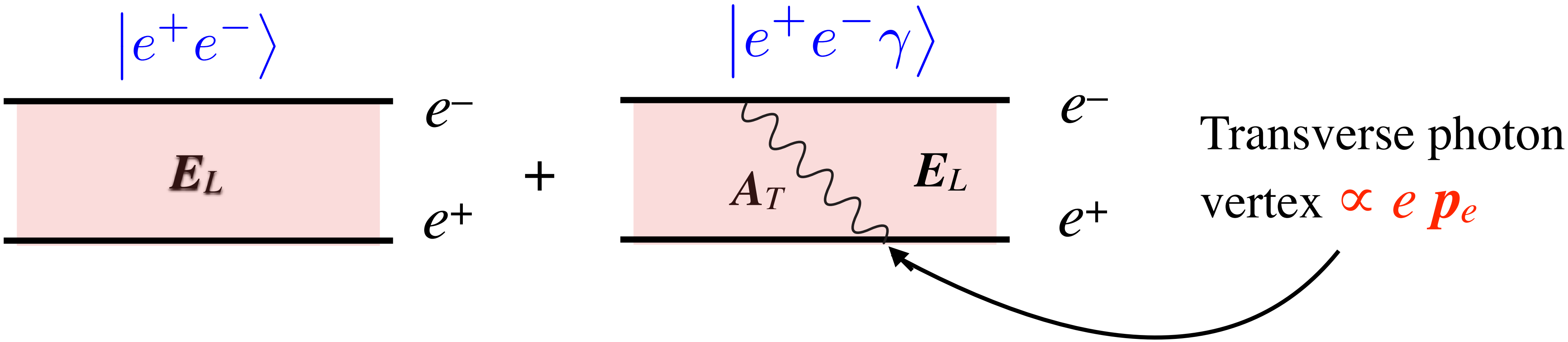

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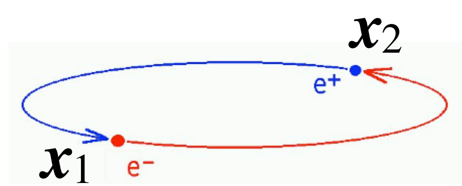
QFT gets things right when it is treated correctly

III. Applications to hadrons

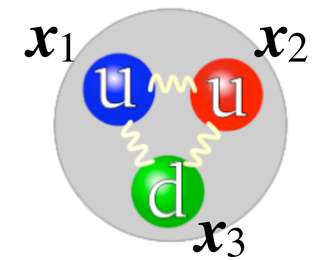
The classical fields of QED and QCD differ

Global gauge invariance allows a classical gauge field for neutral atoms, but not a color octet gluon field for color singlet hadrons.

Positronium (QED)



Proton (QCD)



$$\boldsymbol{E}_L(\boldsymbol{x}) = -\frac{e}{4\pi}\boldsymbol{\nabla}_x\left(\frac{1}{|\boldsymbol{x}-\boldsymbol{x}_1|} - \frac{1}{|\boldsymbol{x}-\boldsymbol{x}_2|}\right)$$

$$\boldsymbol{E}_L^a(\boldsymbol{x}) = 0 \quad \text{for all } \boldsymbol{x}$$

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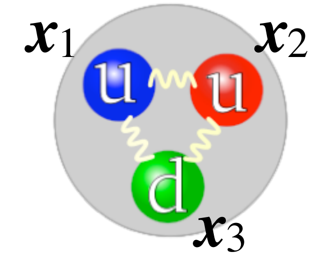

$$\boldsymbol{E}_L^a(\boldsymbol{x}) = 0$$ for all $\boldsymbol{x}$

However:
There is a classical gluon field for each color component C of the proton

$$\boldsymbol{E}_L^a(\boldsymbol{x}, C) \neq 0$$

The blue quark is bound by the $\boldsymbol{E}_L{}^a(\boldsymbol{x},C)$ field of the red and green quarks.

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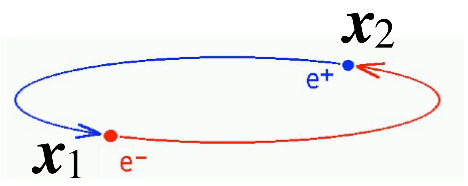

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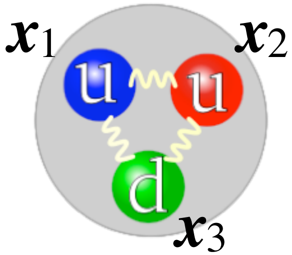

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An external observer sees no field:
The gluon field generated by a color singlet state vanishes.

$$\sum_C \boldsymbol{E}_L^a(\boldsymbol{x}, C) = 0$$

Temporal gauge in QCD: $A_a^0 = 0$

Gauss' gauge constraint determines $\boldsymbol{E}_{L,a}$ for all hadron Fock states:

$$\partial_i E^i_{L,a}(\boldsymbol{x})\,|phys\rangle = g\big[-f_{abc}A^i_b E^i_c + \psi^\dagger T^a \psi(\boldsymbol{x})\big]\,|phys\rangle$$

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$\Rightarrow$ We may consider a homogeneous solution of Gauss' constraint

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The homogeneous solution $\propto \varkappa$ of the gauge constraint is the only one that is compatible with Poincaré invariance

Including the κ ≠ 0 homogeneous solution for $E^i_{L,a}$

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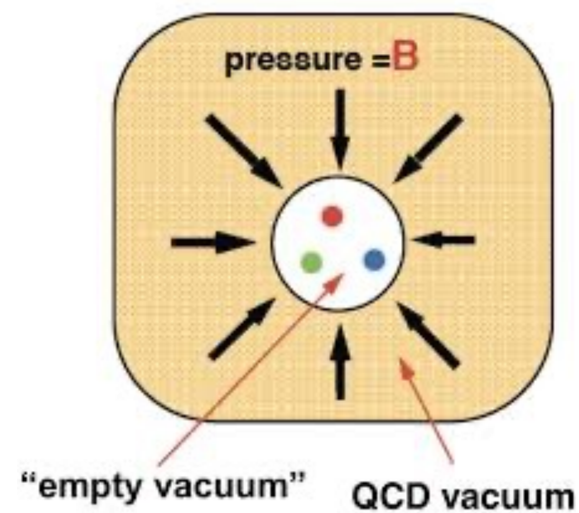

"Bag model without a bag"

The potential energy $\mathcal{H}_V = \frac{1}{2}\int d\boldsymbol{x} \sum_a \boldsymbol{E}_L^a \cdot \boldsymbol{E}_L^a$

$$H_V = \int d\boldsymbol{y} d\boldsymbol{z} \Big\{ \boldsymbol{y}\cdot\boldsymbol{z} \Big[\tfrac{1}{2}\kappa^2 \int d\boldsymbol{x} + g\kappa \Big] + \tfrac{1}{2}\frac{\alpha_s}{|\boldsymbol{y}-\boldsymbol{z}|} \Big\} \mathcal{E}_a(\boldsymbol{y})\mathcal{E}_a(\boldsymbol{z})$$

Recall: $\mathcal{E}_a(\boldsymbol{y}) = -f_{abc}A_b^i E_c^i(\boldsymbol{y}) + \psi^\dagger T^a \psi(\boldsymbol{y})$

Gives translation invariant potentials only for (globally) color singlet states

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Meson component: $|q(\boldsymbol{x}_1)\bar{q}(\boldsymbol{x}_2)\rangle \equiv \sum_A \bar{\psi}^A(\boldsymbol{x}_1)\,\psi^A(\boldsymbol{x}_2)\,|0\rangle$

$$V_{q\bar{q}}(\boldsymbol{x}_1,\boldsymbol{x}_2) = \Lambda^2|\boldsymbol{x}_1-\boldsymbol{x}_2| - C_F\frac{\alpha_s}{|\boldsymbol{x}_1-\boldsymbol{x}_2|}$$

Field energy density:

$$\langle \mathcal{H}_V \rangle = \frac{\Lambda^4}{2g^2 C_F}$$

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Field energy density:

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This potential is valid also for relativistic $q\bar{q}$ Fock states, in any frame

The linear potential is of $\mathcal{O}(\alpha^0)$

Baryon Fock state potential

Baryon: $|q(\boldsymbol{x}_1)q(\boldsymbol{x}_2)q(\boldsymbol{x}_3)\rangle \equiv \sum_{A,B,C} \epsilon_{ABC}\psi_A^\dagger(\boldsymbol{x}_1)\,\psi_B^\dagger(\boldsymbol{x}_2)\,\psi_C^\dagger(\boldsymbol{x}_3)\,|0\rangle$

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$$V_{qqq}(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3) = \Lambda^2 d_{qqq}(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3) - \frac{2}{3}\,\alpha_s\Big(\frac{1}{|\boldsymbol{x}_1-\boldsymbol{x}_2|}+\frac{1}{|\boldsymbol{x}_2-\boldsymbol{x}_3|}+\frac{1}{|\boldsymbol{x}_3-\boldsymbol{x}_1|}\Big)$$

$$d_{qqq}(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3) \equiv \frac{1}{\sqrt{2}}\sqrt{(\boldsymbol{x}_1-\boldsymbol{x}_2)^2+(\boldsymbol{x}_2-\boldsymbol{x}_3)^2+(\boldsymbol{x}_3-\boldsymbol{x}_1)^2}$$

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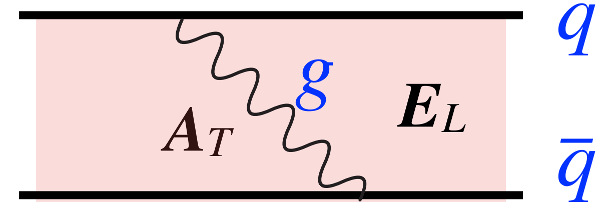
For $\boldsymbol{x}_2 = \boldsymbol{x}_3$ the baryon potential reduces to the meson one:

$$V_{qqq}(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_2) = \Lambda^2|\boldsymbol{x}_1-\boldsymbol{x}_2| - \frac{4}{3}\frac{\alpha_s}{|\boldsymbol{x}_1-\boldsymbol{x}_2|} = V_{q\bar{q}}(\boldsymbol{x}_1,\boldsymbol{x}_2)$$

The $qg\bar{q}$ potential

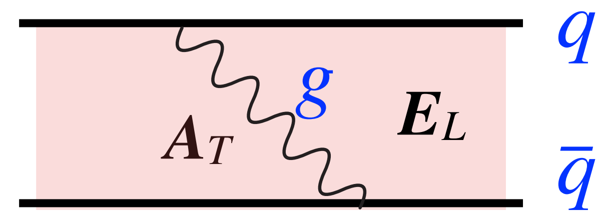
A $q\bar{q}$ state, after the emission of a transverse gluon:



$$|q(\boldsymbol{x}_1)g(\boldsymbol{x}_g)\bar{q}(\boldsymbol{x}_2)\rangle \equiv \sum_{A,B,b} \bar{\psi}_A(\boldsymbol{x}_1)\, A_b^j(\boldsymbol{x}_g)T_{AB}^b\psi_B(\boldsymbol{x}_2)\,|0\rangle$$

The qgq̄ potential

A qq̄ state, after the emission of a transverse gluon:



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$$V_{qgq}^{(0)}(\boldsymbol{x}_1,\boldsymbol{x}_g,\boldsymbol{x}_2) = \frac{\Lambda^2}{\sqrt{C_F}}\, d_{qgq}(\boldsymbol{x}_1,\boldsymbol{x}_g,\boldsymbol{x}_2) \qquad \text{(universal } \Lambda\text{)}$$

$$d_{qgq}(\boldsymbol{x}_1,\boldsymbol{x}_g,\boldsymbol{x}_2) \equiv \sqrt{\tfrac{1}{4}(N-2/N)(\boldsymbol{x}_1-\boldsymbol{x}_2)^2 + N(\boldsymbol{x}_g - \tfrac{1}{2}\boldsymbol{x}_1 - \tfrac{1}{2}\boldsymbol{x}_2)^2}$$

$$V_{qgq}^{(1)}(\boldsymbol{x}_1,\boldsymbol{x}_g,\boldsymbol{x}_2) = \tfrac{1}{2}\,\alpha_s\Big[\frac{1}{N}\,\frac{1}{|\boldsymbol{x}_1-\boldsymbol{x}_2|} - N\Big(\frac{1}{|\boldsymbol{x}_1-\boldsymbol{x}_g|} + \frac{1}{|\boldsymbol{x}_2-\boldsymbol{x}_g|}\Big)\Big]$$

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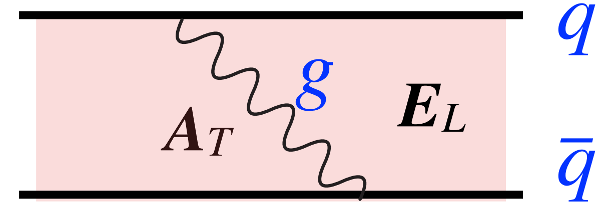
A $q\bar{q}$ state, after the emission of a transverse gluon:



$$|q(\boldsymbol{x}_1)g(\boldsymbol{x}_g)\bar{q}(\boldsymbol{x}_2)\rangle \equiv \sum_{A,B,b} \bar{\psi}_A(\boldsymbol{x}_1)\, A_b^j(\boldsymbol{x}_g) T_{AB}^b \psi_B(\boldsymbol{x}_2)\, |0\rangle$$

$$V_{qgq}^{(0)}(\boldsymbol{x}_1, \boldsymbol{x}_g, \boldsymbol{x}_2) = \frac{\Lambda^2}{\sqrt{C_F}}\, d_{qgq}(\boldsymbol{x}_1, \boldsymbol{x}_g, \boldsymbol{x}_2) \qquad \text{(universal } \Lambda\text{)}$$

$$d_{qgq}(\boldsymbol{x}_1, \boldsymbol{x}_g, \boldsymbol{x}_2) \equiv \sqrt{\tfrac{1}{4}(N - 2/N)(\boldsymbol{x}_1 - \boldsymbol{x}_2)^2 + N(\boldsymbol{x}_g - \tfrac{1}{2}\boldsymbol{x}_1 - \tfrac{1}{2}\boldsymbol{x}_2)^2}$$

$$V_{qgq}^{(1)}(\boldsymbol{x}_1, \boldsymbol{x}_g, \boldsymbol{x}_2) = \tfrac{1}{2}\,\alpha_s \Big[\frac{1}{N}\,\frac{1}{|\boldsymbol{x}_1 - \boldsymbol{x}_2|} - N\Big(\frac{1}{|\boldsymbol{x}_1 - \boldsymbol{x}_g|} + \frac{1}{|\boldsymbol{x}_2 - \boldsymbol{x}_g|}\Big)\Big]$$

When q and g coincide:

$$V_{qgq}^{(0)}(\boldsymbol{x}_1 = \boldsymbol{x}_g, \boldsymbol{x}_2) = \Lambda^2 |\boldsymbol{x}_1 - \boldsymbol{x}_2| = V_{q\bar{q}}^{(0)}$$

$$V_{qgq}^{(1)}(\boldsymbol{x}_1 = \boldsymbol{x}_g, \boldsymbol{x}_2) = V_{q\bar{q}}^{(1)}$$

The gg potential

A "glueball" component: $$|g(\boldsymbol{x}_1)g(\boldsymbol{x}_2)\rangle \equiv \sum_a A_a^i(\boldsymbol{x}_1)\, A_a^j(\boldsymbol{x}_2)\, |0\rangle$$

has the potential $$V_{gg} = \sqrt{\frac{N}{C_F}}\,\Lambda^2\,|\boldsymbol{x}_1 - \boldsymbol{x}_2| - N\,\frac{\alpha_s}{|\boldsymbol{x}_1 - \boldsymbol{x}_2|}$$

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This agrees with the $qg\bar{q}$ potential where the quarks coincide:

$$V_{gg}(\boldsymbol{x}, \boldsymbol{x}_g) = V_{qg\bar{q}}(\boldsymbol{x}, \boldsymbol{x}_g, \boldsymbol{x})$$

It is straightforward to work out the instantaneous potential for any Fock state.

$\mathcal{O}\left(\alpha_s^0\right)$ $q\bar{q}$ bound states

An $\mathcal{O}\left(\alpha_s^0\right)$ meson state with $\boldsymbol{P} = 0$ and wave function Φ:

$$|M\rangle = \sum_{A,B;\alpha,\beta} \int d\boldsymbol{x}_1 d\boldsymbol{x}_2\, \bar{\psi}_\alpha^A(t=0,\boldsymbol{x}_1)\delta^{AB}\Phi_{\alpha\beta}(\boldsymbol{x}_1-\boldsymbol{x}_2)\psi_\beta^B(t=0,\boldsymbol{x}_2)\,|0\rangle$$

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The (rest frame) bound state condition $H\,|M\rangle = M\,|M\rangle$ gives, at $\mathcal{O}\left(\alpha_s^0\right)$

$$\left[i\gamma^0\boldsymbol{\gamma}\cdot\overrightarrow{\boldsymbol{\nabla}} + m\gamma^0\right]\Phi(\boldsymbol{x}) + \Phi(\boldsymbol{x})\left[i\gamma^0\boldsymbol{\gamma}\cdot\overleftarrow{\boldsymbol{\nabla}} - m\gamma^0\right] = \left[M - V(|\boldsymbol{x}|)\right]\Phi(\boldsymbol{x})$$

where $\boldsymbol{x} \equiv \boldsymbol{x}_1 - \boldsymbol{x}_2$ and $V(|\,\boldsymbol{x}\,|) = V'|\,\boldsymbol{x}\,| = \Lambda^2|\,\boldsymbol{x}\,|$.

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In the non-relativistic limit ($m \gg \Lambda$) this reduces to the Schrödinger equation.
If we add the instantaneous gluon exchange potential:

$\Rightarrow$ The quarkonium phenomenology with the Cornell potential.

Separation of radial and angular variables

$$i\boldsymbol{\nabla}\cdot\left\{\gamma^0\boldsymbol{\gamma},\Phi(\boldsymbol{x})\right\}+m\left[\gamma^0,\Phi(\boldsymbol{x})\right]=\left[M-V(\boldsymbol{x})\right]\Phi(\boldsymbol{x})$$

Expanding the 4×4 wave function in a basis of 16 Dirac structures $\Gamma_i(\boldsymbol{x})$

$$\Phi(\boldsymbol{x})=\sum_i\Gamma_i(\boldsymbol{x})F_i(r)Y_{j\lambda}(\hat{\boldsymbol{x}})$$

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We may use rotational, parity and charge conjugation invariance to determine which $\Gamma_i(\boldsymbol{x})$ may occur for a state of given j^{PC}:

0^{-+} trajectory	$[s=0,\ \ell=j]:$	$-\eta_P=\eta_C=(-1)^j$	$\gamma_5,\ \gamma^0\gamma_5,\ \gamma_5\,\boldsymbol{\alpha}\cdot\boldsymbol{x},\ \gamma_5\,\boldsymbol{\alpha}\cdot\boldsymbol{x}\times\boldsymbol{L}$
0^{--} trajectory	$[s=1,\ \ell=j]:$	$\eta_P=\eta_C=-(-1)^j$	$\gamma^0\gamma_5\,\boldsymbol{\alpha}\cdot\boldsymbol{x},\ \gamma^0\gamma_5\,\boldsymbol{\alpha}\cdot\boldsymbol{x}\times\boldsymbol{L},\ \boldsymbol{\alpha}\cdot\boldsymbol{L},\ \gamma^0\,\boldsymbol{\alpha}\cdot\boldsymbol{L}$
0^{++} trajectory	$[s=1,\ \ell=j\pm 1]:$	$\eta_P=\eta_C=+(-1)^j$	$1,\ \boldsymbol{\alpha}\cdot\boldsymbol{x},\ \gamma^0\boldsymbol{\alpha}\cdot\boldsymbol{x},\ \boldsymbol{\alpha}\cdot\boldsymbol{x}\times\boldsymbol{L},\ \gamma^0\boldsymbol{\alpha}\cdot\boldsymbol{x}\times\boldsymbol{L},\ \gamma^0\gamma_5\,\boldsymbol{\alpha}\cdot\boldsymbol{L}$
0^{+-} trajectory	[exotic] :	$\eta_P=-\eta_C=(-1)^j$	$\gamma^0,\ \gamma_5\,\boldsymbol{\alpha}\cdot\boldsymbol{L}$

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$\Rightarrow$ There are no solutions for quantum numbers that would be exotic in the quark model (despite the relativistic dynamics)

The BSE gives the radial equations for the $F_i(r)$

(There are two coupled radial equations for the 0^{++} trajectory)

Example: 0⁻⁺ trajectory wf's

$$\Phi_{-+}(\boldsymbol{x}) = \Big[\frac{2}{M-V}(i\boldsymbol{\alpha}\cdot\overrightarrow{\boldsymbol{\nabla}} + m\gamma^0) + 1\Big]\gamma_5\, F_1(r) Y_{j\lambda}(\hat{\boldsymbol{x}})$$

$\eta_P = (-1)^{j+1}$

$\eta_C = (-1)^j$

Radial equation: $F_1'' + \Big(\frac{2}{r} + \frac{V'}{M-V}\Big)F_1' + \Big[\frac{1}{4}(M-V)^2 - m^2 - \frac{j(j+1)}{r^2}\Big]F_1 = 0$

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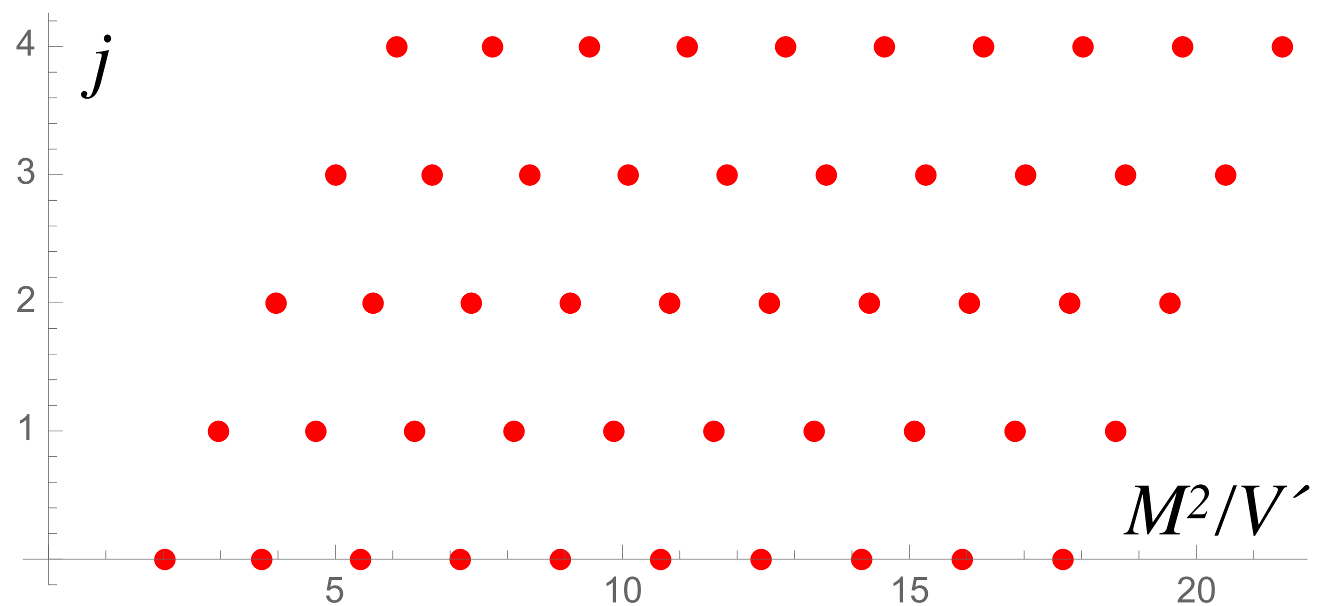
$m = 0$

Mass spectrum:

Linear Regge trajectories with daughters

Spectrum similar to dual models



Bound states in motion

An $\mathcal{O}\left(\alpha_s^0\right)$ $q\bar{q}$ bound state with CM momentum $\boldsymbol{P}$ may be expressed as

$$|M,\boldsymbol{P}\rangle = \int dx_1\, dx_2\, \bar{\psi}(t=0,x_1)\, e^{i\boldsymbol{P}\cdot(\boldsymbol{x}_1+\boldsymbol{x}_2)/2}\, \Phi^{(\boldsymbol{P})}(x_1-x_2)\, \psi(t=0,x_2)\, |0\rangle$$

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D=1+1: The P-dependence reduces to Lorentz contraction only at weak coupling.

D=3+1: No contribution from transverse gluons at $O(\alpha_s{}^0)$

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Brave new QCD world!

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