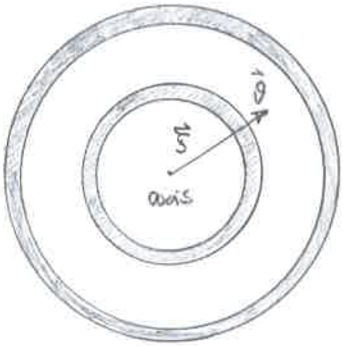
§1.5 Axially Symmetric Lenses, part I

Mass distribution symmetric wrt rotation around optical axis $\Rightarrow$ $\kappa(\vec{\vartheta})$ circularly symmetric, divided into homogeneous circular shells in the lens plane



In 3D gravity, for spherically symmetric mass distributions:

1) Gravity of outer shells (observer inside) = 0
2) Gravity of inner shells (observer outside) = as if mass concentrated at the center.

Similar result holds for "2D gravity" of lensing, for axisymmetric mass distributions. (The integrals to prove this seem more difficult than for 3D, so let's just believe this.)

Thus

$$\boxed{\begin{aligned} \hat{\vec{\alpha}}(\vec{\xi}) &= 4G\frac{\vec{\xi}}{\xi^2}M(\xi) = 4G\frac{\vec{\xi}}{\xi^2}\int_0^{\xi} 2\pi\xi' d\xi' \,\Sigma(\xi') \\ \vec{\alpha}(\vec{\vartheta}) &= \frac{2\vec{\vartheta}}{\vartheta^2}\int_0^{\vartheta}\vartheta'\kappa(\vartheta')\,d\vartheta' \equiv m(\vartheta)\frac{\vec{\vartheta}}{\vartheta^2} \equiv \bar{\kappa}(\vartheta)\vec{\vartheta} \end{aligned}}$$

where $m(\vartheta) \equiv 2\int_0^{\vartheta}\vartheta'\kappa(\vartheta')\,d\vartheta' = \frac{D_{ds}}{D_d D_s}4GM(\xi)$ dimensionless mass inside circle ϑ

$\bar{\kappa}(\vartheta) \equiv \frac{2}{\vartheta^2}\int_0^{\vartheta}d\vartheta'\vartheta'\kappa(\vartheta') = \frac{m(\vartheta)}{\vartheta^2}$ mean convergence within ϑ

The deflection potential is

$$\psi(\vartheta) = m(\vartheta)\ln\vartheta + 2\int_{\vartheta}^{\infty}\vartheta'\kappa(\vartheta')\ln\vartheta'\,d\vartheta'$$

Exercise: Show that $\nabla\psi = \bar{\kappa}(\vartheta)\vec{\vartheta}$

- Constant $\kappa = \bar{\kappa}$ over a disk, with axisymmetric edges:

$$\Rightarrow \quad \vec{\alpha}(\vec{\vartheta}) = \bar{\kappa}\vec{\vartheta}$$

$$\psi(\vec{\vartheta}) = \frac{1}{2}\bar{\kappa}|\vec{\vartheta}|^2 + \text{const} \qquad \text{(exercise)}$$

§1.6 Mass-Sheet Degeneracy

- From observations = images on lens plane, determine lense structure $\kappa(\vec{\vartheta})$ and sources:

  $\kappa(\vec{\vartheta})$ lens model that fits observations $\Rightarrow$ sources at $\vec{\beta} = \vec{\vartheta} - \vec{\alpha}(\vec{\vartheta})$

  obtained from $\kappa(\vec{\vartheta})$

Consider another model $\kappa_\lambda(\vec{\vartheta}) = \underbrace{(1-\lambda)} + \lambda\kappa(\vec{\vartheta})$ $\quad [(1-\kappa_\lambda) = \lambda(1-\kappa)]$

a constant over the relevant region, with axisymmetric edge

$$\Rightarrow \quad \vec{\alpha}_\lambda = (1-\lambda)\vec{\vartheta} + \lambda\vec{\alpha}(\vec{\vartheta})$$

$$\Rightarrow \quad \vec{\beta}_\lambda = \vec{\vartheta} - (1-\lambda)\vec{\vartheta} - \lambda\vec{\alpha}(\vec{\vartheta}) = \lambda[\vec{\vartheta} - \vec{\alpha}(\vec{\vartheta})] = \lambda\vec{\beta}$$

$\therefore$ The other model produces the same observations from a source scaled by λ

If no independent information about source size or lens mass, cannot distinguish those models.

- Since $\vec{\beta}$ was rescaled by λ, $A = \frac{\partial\vec{\beta}}{\partial\vec{\vartheta}}$ is rescaled by λ $\Rightarrow$ $1-\kappa$, γ_i rescaled by λ

  $\Rightarrow$ det A rescaled by λ^2, $\mu_\lambda = \frac{1}{\lambda^2}\mu$; reduced shear $g_i = \frac{\gamma_i}{1-\kappa}$ unaffected

  (relation between source and image shapes unaffected)

  Critical curves (det $A = 0$) and curves with $\kappa = 1$ unaffected

- Fermat potential $\tau(\vec{\vartheta};\vec{\beta}) = \frac{1}{2}(\vec{\vartheta}-\vec{\beta})^2 - \psi(\vec{\vartheta})$ is transformed

$$\vec{\beta}_\lambda = \lambda\vec{\beta}, \qquad \psi_\lambda(\vec{\vartheta}) = \frac{1}{2}(1-\lambda)|\vec{\vartheta}|^2 + \lambda\psi(\vec{\vartheta})$$

$$\Rightarrow \quad \tau_\lambda(\vec{\vartheta};\vec{\beta}_\lambda) = \frac{1}{2}(\vec{\vartheta}-\lambda\vec{\beta})^2 - \frac{1}{2}(1-\lambda)|\vec{\vartheta}|^2 - \lambda\psi(\vec{\vartheta})$$

$$= -\lambda\vec{\vartheta}\cdot\vec{\beta} + \frac{1}{2}\lambda^2|\vec{\beta}|^2 + \frac{1}{2}\lambda|\vec{\vartheta}|^2 - \lambda\psi(\vec{\vartheta})$$

$$= \lambda\cdot\left[\frac{1}{2}(\vec{\vartheta}-\vec{\beta})^2 - \psi(\vec{\vartheta})\right] + \underbrace{\frac{1}{2}(\lambda^2-\lambda)|\vec{\beta}|^2}$$

depends only on source $\vec{\beta}$, same for all of its images

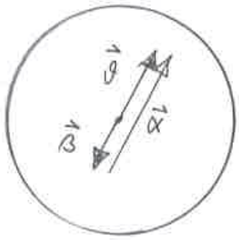
$\Rightarrow$ time delay between images scaled by λ; measuring H breaks the mass-sheet degeneracy

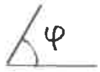
(but is degenerate with H_0 determination)

- Can be broken with sources at different distances, since for a given Σ, κ depends on distances, which scale κ, not $1-\kappa$.

2. SIMPLE LENS MODELS [SKW Part I, Sec. 3]

§2.1 Axially Symmetric Lenses, part II





Axial symmetry $\Rightarrow$ $\vec\alpha \parallel \vec\vartheta \parallel \vec\beta$

The lens equation $\vec\beta = \vec\vartheta - \alpha(\vec\vartheta)$ can be written

$$\underline{\beta = \vartheta - \alpha(\vartheta)}$$

α, ϑ, β may here be positive or negative

symmetry $\Rightarrow$ $\alpha(-\vartheta) = -\alpha(\vartheta)$

From §1.5: $\quad \alpha(\vartheta) = \dfrac{m(\vartheta)}{\vartheta} = \bar\kappa(\vartheta)\vartheta$

$\vartheta_1 = \vartheta\cos\varphi$

$\vartheta_2 = \vartheta\sin\varphi$

$$\Rightarrow \quad \vec\beta = \left[1-\bar\kappa(\vartheta)\right]\vec\vartheta \qquad \text{or} \qquad \beta_j = \vartheta_j - \bar\kappa(\vartheta)\vartheta_j$$

Jacobian
$$A_{ij} = \frac{\partial\beta_j}{\partial\vartheta_i} = \delta_{ij} - \bar\kappa'\overbrace{\partial_i\vartheta}^{\vartheta_i/\vartheta}\,\vartheta_j - \bar\kappa\delta_{ij} = (1-\bar\kappa)\delta_{ij} - \bar\kappa'\frac{\vartheta_i\vartheta_j}{\vartheta}$$

$$\bar\kappa(\vartheta) = \frac{2}{\vartheta^2}\int_0^\vartheta d\vartheta'\vartheta'\kappa(\vartheta') \quad \Rightarrow \quad \bar\kappa' = -\frac{2}{\vartheta}\bar\kappa + \frac{2}{\vartheta}\kappa = \frac{2}{\vartheta}(\kappa-\bar\kappa)$$

$$\Rightarrow \quad A_{ij} \equiv \begin{bmatrix} 1-\kappa-\gamma_1 & -\gamma_2 \\ -\gamma_2 & 1-\kappa+\gamma_1 \end{bmatrix} = \begin{bmatrix} 1-\bar\kappa & \\ & 1-\bar\kappa \end{bmatrix} - \frac{2}{\vartheta^2}(\kappa-\bar\kappa)\begin{bmatrix} \vartheta_1\vartheta_1 & \vartheta_1\vartheta_2 \\ \vartheta_1\vartheta_2 & \vartheta_2\vartheta_2 \end{bmatrix}$$

$$\Rightarrow \quad \gamma_1 = \tfrac{1}{2}(A_{22}-A_{11}) = -(\kappa-\bar\kappa)(\sin^2\varphi - \cos^2\varphi) = (\kappa-\bar\kappa)\cos 2\varphi$$

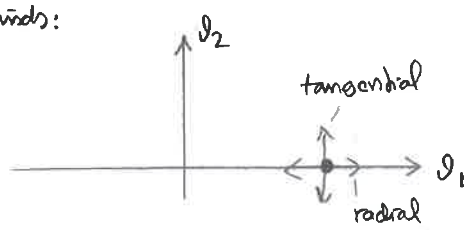
$$\gamma_2 = -A_{12} = 2(\kappa-\bar\kappa)\cos\varphi\sin\varphi = (\kappa-\bar\kappa)\sin 2\varphi \qquad \Rightarrow \quad \gamma_1^2+\gamma_2^2 = (\bar\kappa-\kappa)^2$$

$$\Rightarrow \quad \det A = (1-\kappa)^2 - (\gamma_1^2+\gamma_2^2) = (1-\kappa)^2 - (\kappa-\bar\kappa)^2 = \ldots = (1-\bar\kappa)(1+\bar\kappa-2\kappa)$$

factorizes

- Critical curves, $\det A = 0$, are now circles. There are two kinds:

$1-\bar\kappa = 0$	:	tangential critical curves
$1+\bar\kappa-2\kappa = 0$	:	radial critical curves



Consider image on ϑ_1 axis: $\vartheta_1 = \vartheta$, $\vartheta_2 = 0$ $\Rightarrow$ A diagonal

$A_{11} = 1+\bar\kappa-2\kappa$ inverse of magnification in ϑ_1 direction (radial)

$A_{22} = 1-\bar\kappa$ inverse of magnification in ϑ_2 direction (tangential)

$$\partial_1\vartheta = \frac{\partial\sqrt{\vartheta_1^2+\vartheta_2^2}}{\partial\vartheta_1} = \frac{2\vartheta_1}{2\sqrt{\vartheta_1^2+\vartheta_2^2}} = \frac{\vartheta_1}{\vartheta}$$

$$\beta = (1-\bar\kappa)\vartheta$$

$$\Rightarrow \quad \boxed{\frac{d\beta}{d\vartheta} = 1-\bar\kappa-\bar\kappa'\vartheta = 1+\bar\kappa-2\kappa} = 1-\frac{d\alpha}{d\vartheta}$$

$$\frac{d\alpha}{d\vartheta} = \bar\kappa + \bar\kappa'\vartheta = 2\kappa - \bar\kappa$$

Tangential critical curves: $\bar{\varkappa} = 1$

$$\Rightarrow \quad m(\vartheta) = \frac{D_{ds}}{D_d D_s} 4GM(\xi) = \bar{\varkappa}\vartheta^2 = \vartheta^2 \quad \Rightarrow \quad \vartheta = \vartheta_E \equiv \sqrt{4GM \frac{D_{ds}}{D_d D_s}} = \sqrt{m(\vartheta_E)}$$

Observing a tangential critical curve: its radius $\vartheta = \vartheta_E$ tells us the mass inside it

- Numbers: $2GM_\odot = 2.95$ km (Sun's Schwarzschild radius)

  $M = 10^{12} M_\odot$ (large galaxy) $\Rightarrow$ $4GM = 5.9 \times 10^{12}$ km $= 0.19$ pc

  small compared to size of galaxy or cosmological distances for D_d, D_s $\Rightarrow$ $\vartheta_E \ll 1$

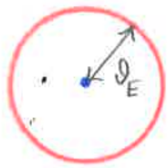
$$\therefore \quad \vartheta_E = \sqrt{\frac{M}{10^{12} M_\odot}} \sqrt{\frac{D_{ds} \cdot 1\text{Gpc}}{D_d D_s}} \cdot \underbrace{\sqrt{\frac{0.19\,\text{pc}}{1\,\text{Gpc}}}}_{\sqrt{1.9\times10^{-10}} = 1.38\times10^{-5}\,\text{rad} = 7.9\times10^{-4}\,\text{deg} = 2''.85}$$

(arcsec)

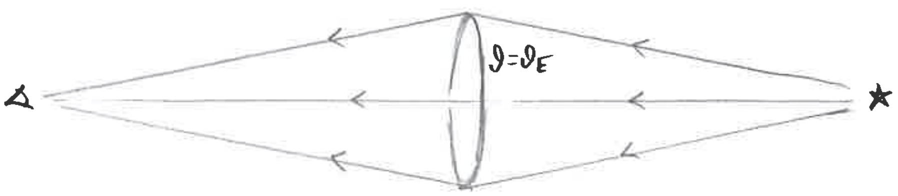
$2''.85$ corresponds to 13.8 kpc at 1 Gpc distance

- Caustic $\vec{\beta} = (1-\bar{\varkappa})\vec{\vartheta} = 0$ reduces to a point at center of symmetry

  $\therefore$ A source located at $\vec{\beta} = 0$ is seen as an Einstein ring, filling the tangential critical curve



An additional image appears at the center, since symmetry $\Rightarrow$ $\vec{\alpha}(0) = 0$

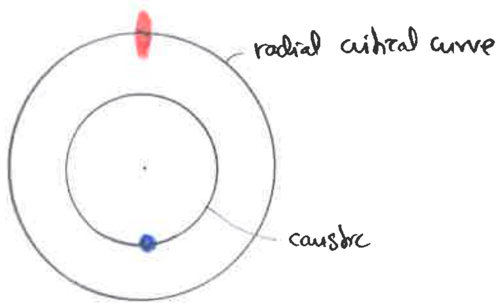


Radial critical curves: $1+\bar{\kappa}-2\kappa = 0$

$$\beta = (1-\bar{\kappa})\theta \Rightarrow \frac{d\beta}{d\theta} = 1-\bar{\kappa}-\bar{\kappa}'\theta = 1+\bar{\kappa}-2\kappa$$

$\therefore$ At radial critical curves $\frac{d\beta}{d\theta} = 0 \Leftrightarrow \frac{d\alpha}{d\theta} = 1$

An object located at the caustic (a ring in source plane w radius β) is seen (its image is) stretched in the radial direction $(d\theta \gg d\beta)$



Magnification

- The magnification is the product of radial and transverse stretching of the image:

$$\boxed{\mu = \underbrace{\frac{d\theta}{d\beta}}_{\text{radial}} \cdot \underbrace{\frac{\theta}{\beta}}_{\text{transverse}} = \frac{1}{1+\bar{\kappa}-2\kappa} \cdot \frac{1}{1-\bar{\kappa}}}$$