§5.2 Effective Surface Mass Density and Limber's Equation

- The metric perturbations ϕ are due to density perturbations $\delta \equiv \frac{\delta \rho_m}{\bar{\rho}_m} \simeq \frac{\delta \rho}{\bar{\rho}_m}$

(we assume that only matter density perturbations are significant)

In local coordinates $\nabla^2 \phi = 4\pi G \delta\rho$ $\Rightarrow$ In comoving coordinates

$$\nabla^2 \phi = 4\pi G a^2 \delta\rho = 4\pi G a^2 \bar{\rho}_m \delta = 4\pi G \bar{\rho}_{m0} a^{-1} \delta = \frac{3}{2} H_0^2 \Omega_{m0} a^{-1} \delta$$

$$(9) \Rightarrow \nabla^2 \psi = 2 \int_0^w dw' \frac{f_k(w-w') f_k(w')}{f_k(w)} \nabla_\perp^2 \phi(f_k(w')\vec{\theta}, w')$$

We want to replace this with $\nabla^2 \phi = \nabla_\perp^2 \phi + \frac{\partial^2}{\partial x_3^2} \phi$

We did this in homework 10.3, where we integrated over a single lens, $\int \frac{\partial^2 \phi}{\partial x_3^2} dx_3 = \Delta \left(\frac{\partial \phi}{\partial x_3} \right)$, the difference in $\frac{\partial \phi}{\partial x_3}$ between in front and behind the lens, assumed negligible sufficiently far from the lens. Now the situation is trickier, since we integrate over the entire light path, and do not assume that ϕ is negligible anywhere. Also the time evolution of ϕ plays a role, since the integration is along light cone,* not at constant time. However, since ϕ is a perturbation, which takes both positive and negative values and should average to zero over the entire universe, we may expect some averaging out of the $\frac{\partial^2}{\partial x_3^2} \phi$ part.

- Bartelmann & Schneider Phys. Rep. 340, 291 (2001) p.392: "We can augment it $[\nabla_\perp^2 \phi]$ by $\frac{\partial^2 \phi}{\partial x_3^2}$, which involves only derivatives along the light path, because these average to zero in the limit to which we are working; the validity of this approximation has been verified with numerical simulations by White and Hu (1999)."

$\therefore$ Let's believe this and replace $\nabla_\perp^2 \phi$ with $\nabla^2 \phi = \frac{3}{2} H_0^2 \Omega_{m0} a^{-1} \delta$

$$\Rightarrow \nabla^2 \psi = 3 H_0^2 \Omega_{m0} \int_0^w dw' \frac{f_k(w') f_k(w-w')}{f_k(w)} \frac{\delta(f_k(w')\vec{\theta}, w')}{a(w')} \equiv 2\kappa(\vec{\theta}, w) \qquad (10)$$

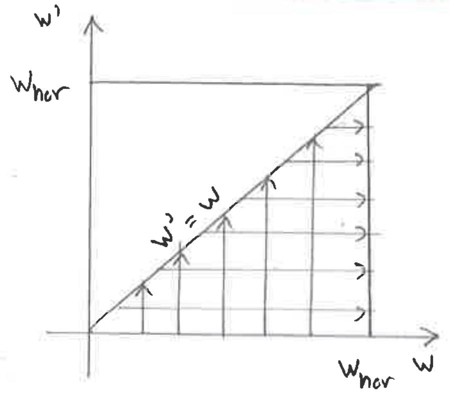
*) $x_3 = w'$, except that $\int dw'$ was assumed to be along the light cone, so that $\phi(\vec{x}, w')$ meant $\phi(\tau_0 - w, \vec{x}, w')$

- The surface mass density $\kappa(\vec{\vartheta}, w)$ in direction $\vec{\vartheta}$ depends on the distance w to the source.

Consider a distribution $p_w(w)$ of sources over different distances w.

The effective surface mass density $\kappa(\vec{\vartheta})$ is defined as the average over this distribution:

$$\kappa(\vec{\vartheta}) \equiv \int dw\, p_w(w)\, \kappa(\vec{\vartheta}, w) = \frac{3}{2} H_0^2 \Omega_{m0} \int_0^{w_{hor}} dw \int_0^{w} dw' \frac{p_w(w) f_K(w') f_K(w-w')}{f_K(w)} \frac{\delta(f_K(w')\vec{\vartheta}, w')}{a(w')}$$



$$= \frac{3}{2} H_0^2 \Omega_{m0} \int_0^{w_{hor}} dw' \int_{w'}^{w_{hor}} dw \frac{p_w(w) f_K(w') f_K(w-w')}{f_K(w)} \frac{\delta(f_K(w')\vec{\vartheta}, w')}{a(w')}$$

call $w \Leftrightarrow w'$

$$= \frac{3}{2} H_0^2 \Omega_{m0} \int_0^{w_{hor}} dw \frac{f_K(w)\, \delta(f_K(w)\vec{\vartheta}, w)}{a(w)} \underbrace{\int_w^{w_{hor}} dw'\, p_w(w') \frac{f_K(w'-w)}{f_K(w')}}_{\equiv\, g(w)} \qquad (11)$$

Compared to (10), the factor $\frac{f_K(w-w')}{f_K(w)}$ was replaced by the source-redshift weighted lens efficiency $\left(\frac{D_{ds}}{D_s}\right)$ factor $g(w)$. (w_{hor} indicates maximum distance we can see, i.e. $p_w(w) = 0$ for $w > w_{hor}$)

- Eq. (11) expresses $\kappa(\vec{\vartheta})$ as a weighted projection of δ. This resembles the relation between the angular (galaxy number) density perturbation $\frac{\delta n}{\langle n \rangle}$ and the 3D density perturbation $\delta = \frac{\delta\rho}{\langle\rho\rangle}$ from GSC part 1, §4.1:

$$\frac{\delta n}{\langle n \rangle}(\vec{\vartheta}) = \frac{1}{V_s} \int_0^\infty \delta(\vec{r})\, S(r)\, r^2\, dr \equiv \int_0^\infty dr\, q(r)\, \delta(\vec{r}), \qquad (12)$$

where $S(r)$ was the selection function, $V_s = \frac{\langle n \rangle}{\langle \rho \rangle}$, and $q(r) \equiv \frac{1}{V_s} S(r) r^2$ is a weight function.

For the relation between the power spectra of $\delta n/\langle n\rangle$ and δ we obtained in the small-angle / flat sky limit (in §4.2.1) Limber's equation

$$\underbrace{\mathcal{P}_g(\ell)}_{\frac{\ell^2}{2\pi} P_g(\ell)} = \frac{\pi}{\ell} \cdot \frac{1}{V_s^2} \int_0^\infty dr\, r^5 S^2(r)\, \mathcal{P}_\delta\!\left(\frac{\ell}{r}, t(r)\right) = \frac{\pi}{\ell} \int_0^\infty dr\, r\, q^2(r)\, \underbrace{\mathcal{P}_\delta\!\left(\frac{\ell}{r}, t(r)\right)}_{\frac{1}{r^3}\frac{\ell^3}{2\pi^2} P_\delta\left(\frac{\ell}{r}, t(r)\right)}$$

$$\Rightarrow P_g(\ell) = \int_0^\infty dr\, r^{-2} q^2(r)\, P_\delta\!\left(\frac{\ell}{r}, t(r)\right) \qquad (13)$$

- Limber's equation (13) was derived for $K=0$ in GSC part 1. The generalization for $K \neq 0$ is

$$P_\vartheta(L) = \int_0^\infty dw \, \frac{g(w)^2}{f_K(w)^2} \, P_\delta\left(\frac{L}{f_K(w)}, w\right) \qquad (14)$$

(where w again indicates location on the light cone, i.e., implies also time)

- Since (11) is like (12), except now $g(w) = \frac{3}{2} H_0^2 \Omega_{m0} \frac{f_K(w) g(w)}{a(w)}$

we can apply (14) to relate the power spectra of $\varkappa(\vec{\vartheta})$ and $\delta(\vec{r})$:

$$P_\varkappa(L) = \frac{9}{4} H_0^4 \Omega_{m0}^2 \int_0^{w_{hor}} dw \, \frac{g^2(w)}{a^2(w)} \, P_\delta\left(\frac{L}{f_K(w)}, w\right) \qquad (15)$$

Thus the convergence power spectrum $P_\varkappa$ is a probe of the density power spectrum P_δ, of all matter, including dark matter, not just galaxies. A cosmological model and parameters, which predict P_δ, predict thus also $P_\varkappa$.

- The remaining issue is that we cannot observe $\varkappa$ directly. Observations of galaxy ellipticities measure the reduced shear $g = \frac{\gamma}{1-\varkappa}$. For cosmic shear, mostly $\varkappa \ll 1$, so that we can ignore the difference between g and γ.

We shall now look at how to determine $P_\varkappa(L)$ from observations of shear γ.

§5.3 Shear and Convergence

- In §1.2 we had the integral relation between ψ and $\varkappa$:

$$\psi(\vec{\vartheta}) = \frac{1}{\pi}\int d^2\vartheta' \, \varkappa(\vec{\vartheta}') \ln|\vec{\vartheta}-\vec{\vartheta}'| \qquad (16)$$

Derivating this twice for $\gamma_1 = \frac{1}{2}(\psi_{,11}-\psi_{,22})$ and $\gamma_2 = \psi_{,12}$ we find (exercise)

$$\gamma \equiv \gamma_1 + i\gamma_2 = \frac{1}{\pi}\int d^2\vartheta' \, \varkappa(\vec{\vartheta}') D(\vec{\vartheta}-\vec{\vartheta}') \qquad (17)$$

where $$D(\vec{\vartheta}) = -\frac{\vartheta_1^2-\vartheta_2^2+2i\vartheta_1\vartheta_2}{|\vec{\vartheta}|^4} = \frac{-1}{(\vartheta_1-i\vartheta_2)^2} \qquad (18)$$

- Fourier transform

$$\psi(\vec{l}) = \int d^2\vartheta \, e^{-i\vec{l}\cdot\vec{\vartheta}}\psi(\vec{\vartheta}) \qquad \gamma_1(\vec{l}) = \int d^2\vartheta \, e^{-i\vec{l}\cdot\vec{\vartheta}}\gamma_1(\vec{\vartheta})$$

$$\varkappa(\vec{l}) = \int d^2\vartheta \, e^{-i\vec{l}\cdot\vec{\vartheta}}\varkappa(\vec{\vartheta}) \qquad \gamma_2(\vec{l}) = \int d^2\vartheta \, e^{-i\vec{l}\cdot\vec{\vartheta}}\gamma_2(\vec{\vartheta})$$

$$\varkappa(\vec{\vartheta}) = \frac{1}{2}\nabla^2\psi(\vec{\vartheta}) \Rightarrow \varkappa(\vec{l}) = -\frac{1}{2}l^2\psi(\vec{l}) = -\frac{1}{2}(l_1^2+l_2^2)\psi(\vec{l}) \qquad (19a)$$

$$\gamma_1(\vec{\vartheta}) = \frac{1}{2}(\psi_{,11}-\psi_{,22}) \Rightarrow \gamma_1(\vec{l}) = -\frac{1}{2}(l_1^2-l_2^2)\psi(\vec{l}) \qquad (19b)$$

$$\gamma_2(\vec{\vartheta}) = \psi_{,12}(\vec{\vartheta}) \Rightarrow \gamma_2(\vec{l}) = -l_1 l_2\psi(\vec{l}) \qquad (19c)$$

Eq. (17) is a convolution in $\vec{\vartheta}$ space $\Rightarrow$ it becomes a product in Fourier space

$$\gamma(\vec{l}) \equiv \gamma_1(\vec{l}) + i\gamma_2(\vec{l}) = \frac{D(\vec{l})\varkappa(\vec{l})}{\pi} \quad \text{where} \quad D(\vec{l}) = \pi\frac{l_1^2-l_2^2+2il_1l_2}{l^2} \qquad (20)$$

(Exercise; easiest to show using (19)).

We have $D(\vec{l})D(\vec{l})^* = \pi^2$ (exercise) $\Rightarrow$ $$\varkappa(\vec{l}) = \frac{D^*(\vec{l})\gamma(\vec{l})}{\pi} \qquad (21)$$

These relations hold only for $\vec{l} \neq 0$, $D(0)$ is undefined.