

Cosmological Perturbation Theory II

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1 Scalar Fields in Minkowski Space

In theoretical physics it has proved fruitful to use the *Principle of Least Action* to find laws of nature. In its first applications it was phrased so that nature minimizes (“Least”) some quantity S (“Action”). However, in some applications one could just as well choose the opposite sign in definition of S , so that one is “extremizing” (minimizing or maximizing) S . In practice the law of nature is derived by requiring that a small variation of the physical configuration from the one nature has chosen (or will choose) will not change S to 1st order of the variation, i.e., $\delta S = 0$. This does not guarantee that S is at its minimum or maximum, we could be at an inflection point or a saddle point. These could be distinguished by studying the second order variation of S ; but we will not consider the possible role of it here; and we take the meaning of Least Action just to be that $\delta S = 0$ to 1st order. Thus it would be more accurately called the Principle of Stationary Action.

For a field theory in Minkowski space the action takes the form

$$S = \int_{t_1}^{t_2} L dt = \int \mathcal{L} d^4x, \quad \text{where } L = \int \mathcal{L} d^3x. \quad (1.1)$$

where L is the *Lagrange function* and \mathcal{L} , the *Lagrangian* (density), is a scalar function of the fields and their spacetime derivatives, and we use Cartesian spacetime coordinates. The Principle of Least Action states that: For fixed initial ($t = t_1$) and final ($t = t_2$) field configurations (see Fig. 1), the fields evolve so as to extremize (or at least to choose a stationary state of) the action S . Thus the principle is related to a boundary value problem: situation at initial and final times is given and we want to find out how Nature chooses to evolve from the initial to the final state.

We consider a theory of N scalar fields φ_I , $I = 1, \dots, N$. The Lagrangian of the theory is thus

$$\mathcal{L} = \mathcal{L}(\varphi_1, \dots, \varphi_N, \partial_\mu \varphi_1, \dots, \partial_\mu \varphi_N). \quad (1.2)$$

We can divide the Lagrangian into two parts, a part that depends only on the fields, and a part that depends also on the derivatives. The first part is called the potential term or the *field potential* $V(\varphi_1, \dots, \varphi_N)$, and the second part is called the *kinetic term*. The simplest case for the latter is the *canonical kinetic term*

$$\frac{1}{2} \partial_\mu \varphi_I \partial^\mu \varphi_I. \quad (1.3)$$

We restrict our study to the case with a canonical kinetic term, so that

$$\mathcal{L}(\varphi_1, \dots, \varphi_N, \partial_\mu \varphi_1, \dots, \partial_\mu \varphi_N) = -\frac{1}{2} \partial_\mu \varphi_I \partial^\mu \varphi_I - V(\varphi_1, \dots, \varphi_N) \quad (1.4)$$

(implied summation over both μ and I in $\partial_\mu \varphi_I \partial^\mu \varphi_I$; and $\partial_\mu \varphi_1$ etc. on the left really means $\partial_0 \varphi_1, \dots, \partial_3 \varphi_1$ etc. The negative sign is a convention—it does not affect physics.

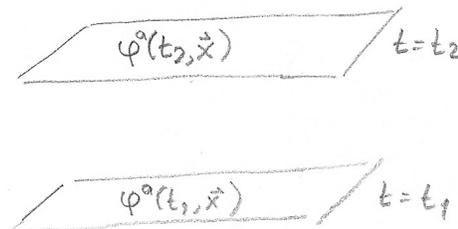


Figure 1: Initial and final field configurations.

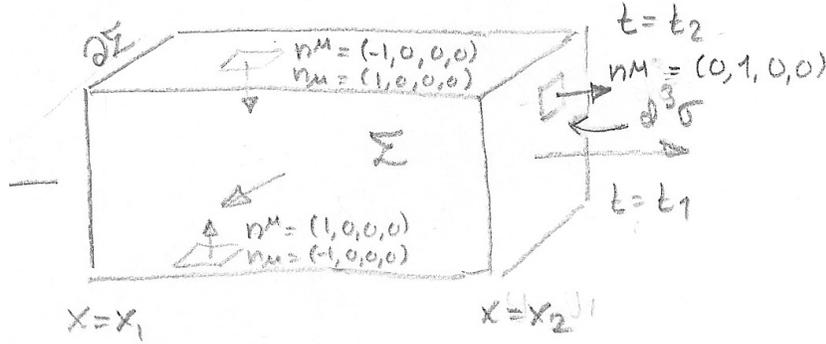


Figure 2: The 3-boundary $\partial\Sigma$ to the integration 4-volume Σ (one space dimension suppressed). Due to the Minkowski geometry, the normal vector to timelike surfaces points inward. (For a lightlike surface the normal vector would be parallel to the surface, since a lightlike vector is normal to itself, being a null vector.)

Let us now vary the fields

$$\varphi_I \rightarrow \varphi_I + \delta\varphi_I \quad \Rightarrow \quad \partial_\mu\varphi_I \rightarrow \partial_\mu\varphi_I + \partial_\mu(\delta\varphi_I) \quad (1.5)$$

around the configuration nature must have chosen (stationary point of the action in the field configuration space), keeping $\varphi_I(t_1, \vec{x})$ and $\varphi_I(t_2, \vec{x})$ fixed. The variation of S (to first order in $\delta\varphi_I$) is then

$$\delta S = \int d^4x \left[\frac{\partial\mathcal{L}}{\partial\varphi_I} \delta\varphi_I + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_I)} \partial_\mu(\delta\varphi_I) \right] = 0 \quad (1.6)$$

for all variations $\delta\varphi_I$. Integrating the second term by parts, this becomes

$$\delta S = \int d^4x \partial_\mu \left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_I)} \delta\varphi_I \right] + \int d^4x \left\{ \frac{\partial\mathcal{L}}{\partial\varphi_I} - \partial_\mu \left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_I)} \right] \right\} \delta\varphi_I = 0, \quad (1.7)$$

The first integral is a 4-volume integral of a divergence, and it can be converted by Gauss theorem to a 3-surface integral over the boundary of the volume (see Fig. 2)

$$\int_\Sigma d^4x \partial_\mu \left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_I)} \delta\varphi_I \right] = \int_{\partial\Sigma} d^3\sigma n_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_I)} \delta\varphi_I, \quad (1.8)$$

where n_μ is the unit normal to the boundary. The boundary consists of the initial and final time slices, where $\delta\varphi_I$ vanishes by assumption (“for fixed initial and final field configurations”), but also of spatial boundaries of the volume we are considering. We have to assume that the variations φ_I vanish also at these spatial boundaries. We can either say that we move these boundaries so far away that they are outside our region of interest, or to just include this condition in the initial formulation of the Principle: “for fixed boundary conditions”. When we move to curved spacetime (Sec. 3) a distinction between temporal and spatial boundaries becomes artificial anyway.

Since the variations $\delta\varphi_I$ vanish at $\partial\Sigma$, (1.8) gives 0 for the first integral in (1.7) and the condition $\delta S = 0$ for all variations $\delta\varphi_I$ gives the Euler–Lagrange equations

$$\frac{\partial\mathcal{L}}{\partial\varphi_I} - \partial_\mu \left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_I)} \right] = 0, \quad (1.9)$$

where

$$\frac{\partial\mathcal{L}}{\partial\varphi_I} = -V_I \quad \text{and} \quad \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_I)} = -\partial^\mu\varphi_I \quad (1.10)$$

and we use the short-hand notation

$$V_I \equiv \frac{\partial V}{\partial \varphi_I}. \quad (1.11)$$

This gives us the field equations

$$\partial_\mu \partial^\mu \varphi_I - V_I = 0. \quad (1.12)$$

The Lagrangian (1.4) depends on the spacetime location (or, on coordinates) only through its dependence on φ_I and $\partial_\mu \varphi_I$, not directly. This means that the action S is invariant under a homogenous translation of the field configuration in space and time. According to Noether's theorem, such invariances lead to conserved currents.

Let us derive the resulting conservation law directly. Shift the whole field configuration by an infinitesimal translation $a^\nu = \text{const}$. This means that at spacetime point P the new values of the fields φ'_I and their gradients $\partial_\mu \varphi'_I$ are what they used to be at another point P' , such that

$$x^\nu(P') = x^\nu(P) - a^\nu. \quad (1.13)$$

Thus

$$\begin{aligned} \varphi'_I(P) &= \varphi_I(P') = \varphi_I(P) - a^\nu \partial_\nu \varphi_I(P) \equiv \varphi_I(P) + \delta \varphi_I(P) \\ \partial_\mu \varphi'_I(P) &= \partial_\mu \varphi_I(P') = \partial_\mu \varphi_I(P) - a^\nu \partial_\nu \partial_\mu \varphi_I(P) \equiv \partial_\mu \varphi_I(P) + \delta(\partial_\mu \varphi_I(P)) \\ \mathcal{L}'(P) &= \mathcal{L}(P') = \mathcal{L}(P) - a^\nu \partial_\nu \mathcal{L}(P) \equiv \mathcal{L}(P) + \delta \mathcal{L}(P). \end{aligned} \quad (1.14)$$

Since the Lagrangian \mathcal{L} depends only on φ_I and $\partial_\mu \varphi_I$, we can write

$$\begin{aligned} \delta \mathcal{L} &= \frac{\partial \mathcal{L}}{\partial \varphi_I} \delta \varphi_I + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_I)} \delta(\partial_\mu \varphi_I) \\ &= -a^\nu \left[\frac{\partial \mathcal{L}}{\partial \varphi_I} \partial_\nu \varphi_I + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_I)} \partial_\nu \partial_\mu \varphi_I \right] \\ &= -a^\nu \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_I)} \partial_\nu \varphi_I \right], \end{aligned} \quad (1.15)$$

where we used Eq. (1.9). Thus we have that

$$\delta \mathcal{L} = -a^\nu \partial_\nu \mathcal{L} \equiv -a^\nu \partial_\mu (\delta^\mu_\nu \mathcal{L}) = -a^\nu \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_I)} \partial_\nu \varphi_I \right]. \quad (1.16)$$

Since this holds for all (infinitesimal) a^ν , we have the conservation law

$$\partial_\mu T^\mu_\nu = 0, \quad (1.17)$$

where

$$T^\mu_\nu \equiv \delta^\mu_\nu \mathcal{L} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_I)} \partial_\nu \varphi_I, \quad (1.18)$$

is called the energy tensor. It's contravariant components are

$$\begin{aligned} T^{\mu\nu} &\equiv \eta^{\mu\nu} \mathcal{L} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_I)} \partial^\nu \varphi_I \\ &= \eta^{\mu\nu} \left[-\frac{1}{2} \partial_\rho \varphi_I \partial^\rho \varphi_I - V(\varphi_I) \right] + \partial^\mu \varphi_I \partial^\nu \varphi_I. \end{aligned} \quad (1.19)$$

In particular (**exercise**),

$$\begin{aligned} \rho &\equiv T^{00} = \frac{1}{2} \sum_I \dot{\varphi}_I^2 + \frac{1}{2} \sum_I (\nabla \varphi_I)^2 + V(\varphi_I) \\ p &\equiv \frac{1}{3} T^i_i = \frac{1}{2} \sum_I \dot{\varphi}_I^2 - \frac{1}{6} \sum_I (\nabla \varphi_I)^2 - V(\varphi_I). \end{aligned} \quad (1.20)$$

2 Curved Spacetime

When the spacetime is curved, the metric becomes a dynamical variable, and we need a law of nature for it also. Consider first the case of empty spacetime, i.e., there are no fields or other forms of matter or energy. Then the metric is the only variable and the action principle should be based on some scalar quantity obtained from the metric and its derivatives taking the role of the Lagrangian density.

2.1 Hilbert Action

The simplest candidate for the spacetime curvature action is the Hilbert action

$$S_H = \int R\sqrt{-g}d^4x, \quad (2.1)$$

where $R \equiv g^{\mu\nu}R_{\mu\nu}$ is the scalar curvature, $R_{\mu\nu}$ is the Ricci curvature tensor and $g^{\mu\nu}$ is the inverse metric. Since we are now using an arbitrary coordinate system in a curved spacetime, the correct normalization of the volume element in the integral requires the $\sqrt{-g}$ factor, where $g \equiv \det[g_{\mu\nu}]$ is the determinant of the metric. Since the time coordinate can be chosen arbitrarily, the action principle is formulated by specifying some spacetime region Σ and fixing quantities on its boundary $\partial\Sigma$, instead of initial and final times. The action principle then states that inside this spacetime region nature chooses the metric so that it corresponds to a stationary point of the action. When we then vary the metric around this stationary configuration we should find that $\delta S_H = 0$.

For notational convenience we choose to vary the inverse metric $g^{\mu\nu}$, instead of $g_{\mu\nu}$. Varying $g^{\mu\nu} \rightarrow g^{\mu\nu} + \delta g^{\mu\nu}$, we have

$$\delta S_H = \int d^4x \sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu} + \int d^4x \sqrt{-g} R_{\mu\nu} \delta g^{\mu\nu} + \int d^4x R \delta \sqrt{-g}. \quad (2.2)$$

Consider first $\delta R_{\mu\nu}$. The Riemann and Ricci tensors are given by

$$\begin{aligned} R^\rho_{\mu\lambda\nu} &= \partial_\lambda \Gamma^\rho_{\nu\mu} + \Gamma^\rho_{\lambda\sigma} \Gamma^\sigma_{\nu\mu} - (\lambda \leftrightarrow \nu) \\ R_{\mu\nu} &= R^\lambda_{\mu\lambda\nu}. \end{aligned} \quad (2.3)$$

In GR the connection is the Christoffel connection, so $\delta\Gamma^\rho_{\nu\mu}$ can be expressed in terms of $\delta g_{\mu\nu}$, but consider first an arbitrary torsion-free ($\Gamma^\rho_{\nu\mu} = \Gamma^\rho_{\mu\nu}$) connection and variation

$$\Gamma^\rho_{\nu\mu} \rightarrow \Gamma^\rho_{\nu\mu} + \delta\Gamma^\rho_{\nu\mu} \quad \Rightarrow \quad \delta R^\rho_{\mu\lambda\nu} = \partial_\lambda (\delta\Gamma^\rho_{\nu\mu}) + \delta\Gamma^\rho_{\lambda\sigma} \Gamma^\sigma_{\nu\mu} + \Gamma^\rho_{\lambda\sigma} \delta\Gamma^\sigma_{\nu\mu} - (\lambda \leftrightarrow \nu). \quad (2.4)$$

Since $\delta\Gamma^\rho_{\nu\mu}$ is a difference between two connections, it is a tensor field, and we can define its covariant derivative (in terms of the unvaried connection $\Gamma^\rho_{\nu\mu}$)

$$\nabla_\lambda (\delta\Gamma^\rho_{\nu\mu}) = \partial_\lambda (\delta\Gamma^\rho_{\nu\mu}) + \Gamma^\rho_{\lambda\sigma} \delta\Gamma^\sigma_{\nu\mu} - \Gamma^\sigma_{\lambda\nu} \delta\Gamma^\rho_{\sigma\mu} - \Gamma^\sigma_{\lambda\mu} \delta\Gamma^\rho_{\nu\sigma} \quad (2.5)$$

and we see that

$$\delta R^\rho_{\mu\lambda\nu} = \nabla_\lambda (\delta\Gamma^\rho_{\nu\mu}) - (\lambda \leftrightarrow \nu). \quad (2.6)$$

This gives the *Palatini identity*

$$\delta R_{\mu\nu} = \delta R^\lambda_{\mu\lambda\nu} = \nabla_\lambda (\delta\Gamma^\lambda_{\nu\mu}) - \nabla_\nu (\delta\Gamma^\lambda_{\lambda\mu}) \quad (2.7)$$

and the first term in (2.2) is

$$\delta S_1 \equiv \int d^4x \sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu} = \int d^4x \sqrt{-g} g^{\mu\nu} \left[\nabla_\lambda (\delta\Gamma^\lambda_{\nu\mu}) - \nabla_\nu (\delta\Gamma^\lambda_{\lambda\mu}) \right]. \quad (2.8)$$

We now assume that the connection is the Christoffel (Levi-Civita) connection

$$\boxed{\Gamma_{\mu\nu}^{\sigma} \equiv \frac{1}{2}g^{\sigma\rho}(\partial_{\mu}g_{\nu\rho} + \partial_{\nu}g_{\mu\rho} - \partial_{\rho}g_{\mu\nu})}, \quad (2.9)$$

for which we have two important results, *metric compatibility* (of the connection)

$$\boxed{\nabla_{\lambda}g_{\mu\nu} = 0}, \quad (2.10)$$

and the general relativistic version of the *Stokes theorem*,

$$\boxed{\int_{\Sigma} \nabla_{\mu}v^{\mu}\sqrt{-g}d^4x = \int_{\partial\Sigma} n_{\mu}v^{\mu}\sqrt{|\gamma|}d^3x}, \quad (2.11)$$

where $\partial\Sigma$ is the boundary of the spacetime region Σ and γ is the determinant of the induced metric of the boundary.

Using metric compatibility, we can convert the integrand in (2.8) to a total derivative, so that we can use the Stokes theorem to convert

$$\delta S_1 = \int d^4x \sqrt{-g} \nabla_{\sigma} \left(g^{\mu\nu} \delta \Gamma_{\nu\mu}^{\sigma} - g^{\mu\sigma} \delta \Gamma_{\lambda\mu}^{\lambda} \right) = \int_{\partial\Sigma} n_{\sigma} \left(g^{\mu\nu} \delta \Gamma_{\nu\mu}^{\sigma} - g^{\mu\sigma} \delta \Gamma_{\lambda\mu}^{\lambda} \right) \sqrt{|\gamma|} d^3x = 0 \quad (2.12)$$

into a boundary integral, which vanishes, assuming the variation $\delta \Gamma_{\nu\mu}^{\sigma}$ vanishes at the boundary.¹

For the last term in (2.2) we derived in the GR course that

$$\boxed{\delta \sqrt{-g} = -\frac{1}{2} \sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu}}. \quad (2.13)$$

Altogether the requirement that the variation gives $\delta S_H = 0$ for arbitrary $\delta g^{\mu\nu}$ leads to the condition

$$\frac{1}{\sqrt{-g}} \frac{\delta S_H}{\delta g^{\mu\nu}} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \equiv G_{\mu\nu} = 0, \quad (2.14)$$

which is the vacuum Einstein equation.

In case the spacetime contains some forms of matter or energy, we need to add them to the action:

$$S = S_H + S_{\text{matter}}. \quad (2.15)$$

The requirement that $\delta S = 0$ becomes $\delta S_H = -\delta S_{\text{matter}}$ and (2.14) is replaced by

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \equiv G_{\mu\nu} = -\frac{1}{\sqrt{-g}} \frac{\delta S_{\text{matter}}}{\delta g^{\mu\nu}} \equiv 8\pi G T_{\mu\nu}. \quad (2.16)$$

Derivation of Friedmann equations. We review here the derivation of the (flat universe) Friedmann equations as practice for the modified Friedmann equations in $f(R)$ gravity (Sec. 2.2). From Part 1 of these lecture notes we have for the flat RW metric $g_{\mu\nu} = a^2 \eta_{\mu\nu}$:

$$\begin{aligned} \Gamma_{00}^0 &= \mathcal{H} & \Gamma_{0k}^0 &= 0 & \Gamma_{ij}^0 &= \mathcal{H} \delta_{ij} \\ \Gamma_{00}^i &= 0 & \Gamma_{0j}^i &= \mathcal{H} \delta_j^i & \Gamma_{kl}^i &= 0 \\ R_{00} &= -3\mathcal{H}' & R_{0i} &= 0 & R_{ij} &= (\mathcal{H}' + 2\mathcal{H}^2) \delta_{ij} \\ R_0^0 &= 3a^{-2}\mathcal{H}' & R_i^0 &= R_0^i = 0 & R_j^i &= a^{-2}(\mathcal{H}' + 2\mathcal{H}^2) \delta_j^i \\ R &= 6a^{-2}(\mathcal{H}' + \mathcal{H}^2), \end{aligned} \quad (2.17)$$

¹This variation is (exercise) $\delta \Gamma_{\nu\mu}^{\lambda} = -\frac{1}{2} [g_{\rho\mu} \nabla_{\nu} (\delta g^{\rho\lambda}) + g_{\rho\nu} \nabla_{\mu} (\delta g^{\rho\lambda}) - g_{\mu\alpha} g_{\nu\beta} \nabla^{\lambda} (\delta g^{\alpha\beta})]$. Thus we need to assume a version of the action principle, where the boundary conditions fix both the metric and its derivatives.

and for the energy tensor

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu} \quad \text{where } u_\mu = a(-1, \vec{0}). \quad (2.18)$$

This gives the Einstein equations

$$\begin{aligned} R_{00} - \frac{1}{2}g_{00}R &= 3\mathcal{H}^2 = 8\pi GT_{00} = 8\pi G\rho a^2 \\ R_{11} - \frac{1}{2}g_{11}R &= -2\mathcal{H}' - \mathcal{H}^2 = 8\pi GT_{11} = 8\pi Gpa^2 \end{aligned} \quad (2.19)$$

(the $_{22}$ and $_{33}$ components of (3.9) give the same as the above $_{11}$ component and the off-diagonal components give $0 = 0$). We shall later refer to the first one and the sum of the two:

$$\begin{aligned} 3\mathcal{H}^2 &= 8\pi G\rho a^2 \\ 2\mathcal{H}^2 - 2\mathcal{H}' &= 8\pi G(\rho + p)a^2 \end{aligned} \quad (2.20)$$

2.2 f(R) Gravity

Note: I put this section here, since it is a generalization of the previous section. Otherwise it does not logically belong here, since the next section continues with scalar fields in standard GR – as does the remaining part of these lecture notes until Sec. 15. We return to modified gravity in Sec. 16.

A popular class of modified gravity theories is $f(R)$ gravity. In $f(R)$ gravity one replaces the scalar curvature R in the Hilbert action with some scalar function $f(R)$ of it:

$$S = \int f(R)\sqrt{-g} d^4x. \quad (2.21)$$

Varying $g^{\mu\nu} \rightarrow g^{\mu\nu} + \delta g^{\mu\nu}$,

$$\delta S = \int d^4x \sqrt{-g} F(R) g^{\mu\nu} \delta R_{\mu\nu} + \int d^4x \sqrt{-g} F(R) R_{\mu\nu} \delta g^{\mu\nu} - \frac{1}{2} \int d^4x \sqrt{-g} f(R) g_{\mu\nu} \delta g^{\mu\nu}, \quad (2.22)$$

where

$$F(R) \equiv \frac{df(R)}{dR}. \quad (2.23)$$

In the Palatini identity (2.7),

$$\delta R_{\mu\nu} = \nabla_\lambda (\delta \Gamma_{\nu\mu}^\lambda) - \nabla_\nu (\delta \Gamma_{\lambda\mu}^\lambda), \quad (2.24)$$

we have

$$\delta \Gamma_{\nu\mu}^\lambda = -\frac{1}{2} \left[g_{\rho\mu} \nabla_\nu (\delta g^{\rho\lambda}) + g_{\rho\nu} \nabla_\mu (\delta g^{\rho\lambda}) - g_{\mu\alpha} g_{\nu\beta} \nabla^\lambda (\delta g^{\alpha\beta}) \right], \quad (2.25)$$

for the Christoffel connection. Using metric compatibility, $\nabla_\lambda g_{\mu\nu} = 0$, and the Stokes theorem, with boundary terms vanishing, to move $g_{\mu\nu}$ and ∇_λ around, we get (**exercise**)

$$\delta S = \int d^4x \sqrt{-g} \delta g^{\mu\nu} \left[-\nabla_\mu \nabla_\nu F(R) + g_{\mu\nu} \nabla_\lambda \nabla^\lambda F(R) + F(R) R_{\mu\nu} - g_{\mu\nu} \frac{1}{2} f(R) \right]. \quad (2.26)$$

Requiring $\delta S = 0$ for all variations $\delta g^{\mu\nu}$ of the (inverse) metric, we obtain the modified Einstein equation for vacuum,

$$F(R) R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} f(R) = (\nabla_\mu \nabla_\nu - g_{\mu\nu} \square) F(R). \quad (2.27)$$

In standard GR, $f(R) = R$ and $F(R) \equiv 1$, so this becomes $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0$, the (unmodified) Einstein equation for vacuum. The energy tensor is obtained as before from the matter/field part of the action, so the full field equation of $f(R)$ gravity is

$$\boxed{F(R)R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}f(R) = (\nabla_\mu\nabla_\nu - g_{\mu\nu}\square)F(R) + 8\pi GT_{\mu\nu}}. \quad (2.28)$$

Since R contains second derivatives of the metric, (2.28) will contain fourth derivatives of the metric.

The trace of (2.28) is

$$F(R)R - 2f(R) = -3\square F(R) + 8\pi GT, \quad (2.29)$$

where the trace of the energy-momentum tensor is $T \equiv g^{\mu\nu}T_{\mu\nu} = -\rho + 3p$.

In GR, the Ricci tensor vanishes in vacuum (defined here as $T_{\mu\nu} = 0$). If $f(R) = \alpha R^2$, then $F(R)R - 2f(R) = 0$ and there exists a vacuum solution $R_{\mu\nu} = \Lambda g_{\mu\nu}$, where $\Lambda = \text{const}$. Now also $R = 4\Lambda$ and $F(R)$ are constants, so that $\nabla_\mu\nabla_\nu F = 0$ and $\square F = 0$, and (2.28) is satisfied. We know from GR that this is the de Sitter space. This theory is very different from GR, so it is not a viable modified gravity theory as such. However, if instead $f(R) = R + \alpha R^2$, this theory becomes GR in the limit of small curvature $\alpha R^2 \ll R$, but in the limit of large curvature, which we expect to encounter in the very early universe, the αR^2 part will dominate, and the theory has a solution that approaches de Sitter in the early universe (R^2 inflation).

(This is $f(R)$ gravity of the metric formalism. There is another variant of $f(R)$ gravity, the Palatini formalism, see Sec. 18.)

Tools. Basic tools for operating on, e.g., the action variation integral δS are metric compatibility

$$\nabla_\lambda g_{\mu\nu} = 0, \quad (2.30)$$

which means that the metric commutes with covariant derivative, so that one can move $g_{\mu\nu}$ and $g^{\mu\nu}$ in and out of ∇_λ , e.g.,

$$g_{\mu\nu}\nabla_\lambda(g^{\nu\sigma}A_{\sigma\rho}) = g_{\mu\nu}g^{\nu\sigma}\nabla_\lambda A_{\sigma\rho} = \nabla_\lambda A_{\mu\rho}, \quad (2.31)$$

and the Stokes theorem (2.11) together with the Leibniz rule for the covariant derivative, $\nabla_\mu(A^{\alpha\beta}B_{\gamma\delta\epsilon}) = \nabla_\mu(A^{\alpha\beta})B_{\gamma\delta\epsilon} + A^{\alpha\beta}\nabla_\mu(B_{\gamma\delta\epsilon})$, which can be used for partial integration, e.g.,

$$\int_\Sigma A^\mu_{\alpha\beta}\nabla_\mu B^{\alpha\beta}\sqrt{-g}d^4x = \int_{\partial\Sigma} n_\mu A^\mu_{\alpha\beta}B^{\alpha\beta}\sqrt{|\gamma|}d^3x - \int_\Sigma B^{\alpha\beta}\nabla_\mu A^\mu_{\alpha\beta}\sqrt{-g}d^4x, \quad (2.32)$$

where the boundary integral will vanish if it contains a variation that we assume to vanish at the boundary.

Flat FRW Universe in f(R) gravity. Following the example in Sec. 3 we now want to derive the modified Friedmann equations we get when (2.28) is applied to $g_{\mu\nu} = a^2\eta_{\mu\nu}$. We can still use (2.17) and (2.18). To keep the notation compact, we write just f and F for $f(R)$ and $F(R)$. Since f and F are scalars, $\nabla_\mu F = \partial_\mu F$, so we have $\nabla_0 F = F'$ and $\nabla_i F = 0$ (FRW is homogeneous). For the second covariant derivatives we get (**exercise**)

$$\begin{aligned} \nabla_0\nabla_0 F &= \partial_0\partial_0 F - \Gamma_{00}^0\partial_0 F = F'' - \mathcal{H}F' \\ \nabla_i\nabla_j F &= \partial_i\partial_j F - \Gamma_{ij}^0\partial_0 F = -\mathcal{H}\delta_{ij}F' \\ \square F &= g^{00}\nabla_0\nabla_0 F + g^{ij}\nabla_i\nabla_j F = -a^{-2}(F'' + 2\mathcal{H}F') \end{aligned} \quad (2.33)$$

Thus the $_{00}$ and $_{11}$ components of (2.28) become (**exercise**)

$$\begin{aligned} -3\mathcal{H}'F + \frac{1}{2}a^2 f + 3\mathcal{H}F' &= 8\pi G\rho a^2 \\ (\mathcal{H}' + 2\mathcal{H}^2)F - \frac{1}{2}a^2 f - \mathcal{H}F' - F'' &= 8\pi Gp a^2. \end{aligned} \quad (2.34)$$

It is easier to compare these to (2.20) when we use $\frac{1}{2}a^2 R = 3\mathcal{H}' + 3\mathcal{H}^2$ to replace $-3\mathcal{H}'F = -\frac{1}{2}a^2 FR + 3\mathcal{H}^2 F$, so the first equation and the sum of the two become

$$\begin{aligned} 3F\mathcal{H}^2 &= 8\pi G\rho a^2 + \frac{1}{2}a^2(FR - f) - 3\mathcal{H}F' \\ (2\mathcal{H}^2 - 2\mathcal{H}')F &= 8\pi G(\rho + p)a^2 - 2\mathcal{H}F' + F'' . \end{aligned} \quad (2.35)$$

These are the *modified Friedmann equations* for $f(R)$ gravity. It is easy to see that for $f = R \Rightarrow F = 1$ these become (2.20).

In terms of cosmic time t (instead of conformal time η),

$$R = 6\dot{H} + 12H^2 \quad (2.36)$$

and (2.35a) becomes (**exercise**)

$$3FH^2 = 8\pi G\rho + \frac{1}{2}(FR - f) - 3HF' . \quad (2.37)$$

The energy continuity equation is the same as in GR:

$$\rho' = -3\mathcal{H}(\rho + p) \quad \text{or} \quad \dot{\rho} = -3H(\rho + p) . \quad (2.38)$$

Cosmological constant. A trivial example of $f(R)$ gravity is just adding a constant:

$$f(R) = R - 2\Lambda \quad \Rightarrow \quad F(R) = 1 . \quad (2.39)$$

The field equation (2.28) becomes

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 8\pi GT_{\mu\nu} \quad (2.40)$$

and the modified Friedmann equation (2.37) becomes

$$3H^2 = 8\pi G\rho + \Lambda . \quad (2.41)$$

Starobinsky gravity. The simplest nontrivial $f(R)$ gravity is the one with [1]

$$f(R) = R + \alpha R^2 \quad \Rightarrow \quad F(R) = 1 + 2\alpha R , \quad (2.42)$$

where

$$\alpha = \frac{1}{6M^2} \quad (2.43)$$

is a constant (and M is another constant, with dimension of mass). This can be motivated by quantum corrections to gravity (with M presumably of similar order of magnitude as the Planck mass). Equation (2.37) becomes (**exercise**)

$$H^2 + \frac{1}{M^2} \left(2H\ddot{H} - \dot{H}^2 + 6H^2\dot{H} \right) = \frac{8\pi G}{3}\rho . \quad (2.44)$$

Written for the scale factor $a(t)$ this becomes

$$\left(\frac{\dot{a}}{a} \right)^2 + \frac{1}{M^2} \left[2\frac{\dot{a}\ddot{a}}{a^2} - \left(\frac{\ddot{a}}{a} \right)^2 + 2\left(\frac{\dot{a}}{a} \right)^2 \frac{\ddot{a}}{a} - 3\left(\frac{\dot{a}}{a} \right)^4 \right] = \frac{8\pi G}{3}\rho \quad (2.45)$$

(with $\rho = 0$ this becomes Eq. (4) in [1], when one sets $K = k_2 = 0$ in it).

Starobinsky gravity accepts inflationary solutions (Starobinsky inflation, or R^2 inflation), where H is changing (decreasing) slowly and the dynamics is dominated by spacetime curvature (ρ is negligible compared to the LHS of (2.44)). Approximating $\rho = 0$, (2.44) becomes

$$H^2 = -\frac{H^2}{M^2} \left[6\dot{H} + 2\frac{\ddot{H}}{H} - \left(\frac{\dot{H}}{H} \right)^2 \right] . \quad (2.46)$$

Try for a solution ([2], p. 257)

$$H(t) = \frac{1}{6}M^2(t_1 - t), \quad (2.47)$$

where $t \ll t_1$ (both positive). Substituting in (2.46) gives

$$M^2(t_1 - t)^2 = 1 + M^2(t_1 - t)^2, \quad (2.48)$$

so (2.47) is not an exact solution, but is an approximate solution if

$$t_1 - t \gg 1/M, \quad (2.49)$$

or $t_1 \gg 1/M$ (as $t \ll t_1$). The scale factor corresponding to (2.47) is

$$a(t) \propto \exp \left[\frac{M^2}{12}(2t_1 - t)t \right]. \quad (2.50)$$

As t grows to become no longer $\ll t_1$, the approximation (2.47) becomes worse and inflation ends.

We shall return to Starobinsky inflation in Sec. 19.1.

A large R means high curvature, so forms of $f(R)$ that deviate from R significantly for large values of R , like (2.42), modify gravity at high-curvature conditions; in cosmology this corresponds to the early universe. (Note that R does not contain all aspects of spacetime curvature, since in GR $R = 0$ in vacuum, like we may have near a black hole.) If we want to explain the acceleration of the late-time universe with $f(R)$ gravity instead of dark energy, we need a form of $f(R)$ that deviates from R at small values of R . The first thing that comes to mind is then

$$f(R) = R - \frac{\alpha}{R^n} \quad (n > 0). \quad (2.51)$$

However, these theories violate local constraints on gravity that we have from experiments in the solar system. The Euclid Theory Working Group (Amendola et al. [3]) lists four types of viable $f(R)$ models² (i.e., models that can at the same time satisfy current experimental limits and provide late-time acceleration without dark energy):

$$f(R) = R - \mu R_c \left(\frac{R}{R_c} \right)^p \quad (0 < p < 1, \mu, R_c > 0) \quad (\text{Amendola et al.}[4]) \quad (2.52)$$

$$f(R) = R - \mu R_c \frac{(R/R_c)^{2n}}{(R/R_c)^{2n} + 1} \quad (n, \mu, R_c > 0) \quad (\text{Hu and Sawicki}[5]) \quad (2.53)$$

$$f(R) = R - \mu R_c \left[1 - \left(1 + \frac{R^2}{R_c^2} \right)^{-n} \right] \quad (n, \mu, R_c > 0) \quad (\text{Starobinsky}[6]) \quad (2.54)$$

$$f(R) = R - \mu R_c \tanh(R/R_c) \quad (\mu, R_c > 0) \quad (\text{Tsujiikawa}[7]) \quad (2.55)$$

As one sees from the above example the field equations in $f(R)$ gravity become high (fourth order differential equations. (Equation (2.45) is just third order, but the second modified Friedmann equation is fourth order.) These are quite cumbersome to work with.

To make the equations easier to work with, one can introduce a conformal transformation

$$\tilde{g}_{\mu\nu} = \omega^2 g_{\mu\nu}, \quad \text{where } \omega^2 = F(R). \quad (2.56)$$

The conformal metric $\tilde{g}_{\mu\nu}$ will then satisfy the usual Einstein equation but with the scalar field ω appearing in the energy tensor. I will return to this in Sections 15 and 16. For now, this is an additional motivation for the study of cosmological perturbation theory with scalar fields that will follow.

²These may not look like plausible models to you. Some reviewer (I wish I could locate the reference) said about dark energy vs. modified gravity that “the more you study one the more convinced you become that the other is the way to go”. Anyway, the way to proceed in cosmology is to make more accurate observations to improve the constraints on modified gravity, and these models serve as tools for this work.

3 Scalar Fields in Curved Spacetime

In curved spacetime the Lagrangian (1.4) is replaced by

$$\boxed{\mathcal{L}_\varphi = -\frac{1}{2}g^{\mu\nu}\nabla_\mu\varphi_I\nabla_\nu\varphi_I - V(\varphi_1, \dots, \varphi_N) = -\frac{1}{2}\partial_\rho\varphi_I\partial^\rho\varphi_I - V(\varphi_1, \dots, \varphi_N)}. \quad (3.1)$$

and the action (1.1) by

$$S_\varphi = \int \mathcal{L}_\varphi \sqrt{-g} d^4x, \quad (3.2)$$

where

$$g \equiv \det [g_{\mu\nu}]. \quad (3.3)$$

In Eq. (3.1), $\nabla_\mu\varphi_I = \partial_\mu\varphi_I$, since φ_I are scalar fields, but for the derivation of the Euler-Lagrange equation, where one needs to use the Stokes theorem (2.11), it is more clear to write it as the covariant derivative.

We get the general relativistic theory with scalar field sources minimally coupled to gravity by varying the action

$$S = \frac{1}{16\pi G} S_H + S_\varphi = \frac{1}{16\pi G} \int R \sqrt{-g} d^4x + \int \mathcal{L}_\varphi \sqrt{-g} d^4x, \quad (3.4)$$

where \mathcal{L}_φ is the Lagrangian (3.1). By varying with respect to the scalar fields, we get the Euler-Lagrange equation

$$\boxed{\frac{\partial \mathcal{L}_\varphi}{\partial \varphi_I} - \nabla_\mu \left[\frac{\partial \mathcal{L}_\varphi}{\partial (\nabla_\mu \varphi_I)} \right] = 0}, \quad (3.5)$$

where

$$\frac{\partial \mathcal{L}_\varphi}{\partial \varphi_I} = -V_I \quad \text{and} \quad \frac{\partial \mathcal{L}_\varphi}{\partial (\nabla_\mu \varphi_I)} = -\nabla^\mu \varphi_I. \quad (3.6)$$

This gives us the field equations

$$\boxed{\square \varphi_I - V_I = 0}, \quad (3.7)$$

where

$$\boxed{\square \varphi_I \equiv \nabla_\mu \nabla^\mu \varphi_I = g^{\mu\nu} \nabla_\mu \nabla_\nu \varphi_I = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} \partial^\mu \varphi_I)}. \quad (3.8)$$

By varying with respect to the inverse metric we get the Einstein equations

$$\boxed{R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}}, \quad (3.9)$$

where

$$T_{\mu\nu} \equiv -2 \frac{1}{\sqrt{-g}} \frac{\delta S_\varphi}{\delta g^{\mu\nu}}. \quad (3.10)$$

From (3.1)

$$\frac{\delta \mathcal{L}_\varphi}{\delta g^{\mu\nu}} = -\frac{1}{2} \nabla_\mu \varphi_I \nabla_\nu \varphi_I = -\frac{1}{2} \partial_\mu \varphi_I \partial_\nu \varphi_I \quad (3.11)$$

so (using also Eq. 2.13)

$$\delta S_\varphi = \int \delta \mathcal{L}_\varphi \sqrt{-g} d^4x + \int \mathcal{L}_\varphi \delta \sqrt{-g} d^4x = \int d^4x \sqrt{-g} \left(-\frac{1}{2} \partial_\mu \varphi_I \partial_\nu \varphi_I - \frac{1}{2} \mathcal{L}_\varphi g_{\mu\nu} \right) \delta g^{\mu\nu}, \quad (3.12)$$

so that

$$\begin{aligned} T_{\mu\nu} &\equiv -2 \frac{1}{\sqrt{-g}} \frac{\delta S_\varphi}{\delta g^{\mu\nu}} = \partial_\mu \varphi_I \partial_\nu \varphi_I + g_{\mu\nu} \mathcal{L}_\varphi \\ &= \partial_\mu \varphi_I \partial_\nu \varphi_I - \frac{1}{2} g_{\mu\nu} \partial_\rho \varphi_I \partial^\rho \varphi_I - g_{\mu\nu} V(\varphi_I), \end{aligned} \quad (3.13)$$

or

$$\boxed{T_\nu^\mu = \partial^\mu \varphi_I \partial_\nu \varphi_I - \frac{1}{2} \delta_\nu^\mu \partial_\rho \varphi_I \partial^\rho \varphi_I - \delta_\nu^\mu V(\varphi_I)}, \quad (3.14)$$

which is the general relativistic version of (1.19), i.e., $\eta_{\mu\nu}$ replaced by $g_{\mu\nu}$.

4 Background Universe

Let us now apply the equations from Sec. 3 to our flat FRW background universe, with the metric

$$ds^2 = a^2(\eta)(-d\eta^2 + dx^2 + dy^2 + dz^2) \Rightarrow \sqrt{-g} = a^4. \quad (4.1)$$

In the background universe the scalar fields are homogeneous,

$$\bar{\varphi}_I = \bar{\varphi}_I(\eta). \quad (4.2)$$

The background field equations are **(exercise)**

$$\boxed{\bar{\varphi}_I'' + 2\mathcal{H}\bar{\varphi}_I' = -a^2 \frac{\partial V}{\partial \varphi_I}}, \quad (4.3)$$

and the background energy tensor is **(exercise)**

$$\begin{aligned} \bar{T}_0^0 &= -\frac{1}{2}a^{-2} \sum_I (\bar{\varphi}_I')^2 - V(\bar{\varphi}_1, \dots, \bar{\varphi}_N) = -\bar{\rho} \\ \bar{T}_i^0 &= \bar{T}_0^i = 0 \\ \bar{T}_j^i &= \delta_j^i \left[\frac{1}{2}a^{-2} \sum_I (\bar{\varphi}_I')^2 - V(\bar{\varphi}_1, \dots, \bar{\varphi}_N) \right] = \delta_j^i \bar{p}. \end{aligned} \quad (4.4)$$

From this we have that

$$\bar{\rho} + \bar{p} = a^{-2} \sum_I (\bar{\varphi}_I')^2 \quad (4.5)$$

$$\bar{\rho} - \bar{p} = 2V \quad (4.6)$$

and the Friedmann equations are

$$\begin{aligned} \mathcal{H}^2 &= \frac{8\pi G}{3} \bar{\rho} a^2 = \frac{8\pi G}{3} \left[\frac{1}{2} \sum_I (\bar{\varphi}_I')^2 + a^2 V \right] \\ \mathcal{H}' &= -\frac{4\pi G}{3} (\bar{\rho} + 3\bar{p}) = -\frac{8\pi G}{3} \left[\sum_I (\bar{\varphi}_I')^2 - a^2 V \right]. \end{aligned} \quad (4.7)$$

Different combinations of the Friedmann equations give

$$-2\mathcal{H}' - \mathcal{H}^2 = 8\pi G \bar{p} a^2 = 8\pi G \left[\frac{1}{2} \sum_I (\bar{\varphi}_I')^2 - a^2 V \right] \quad (4.8)$$

$$-\mathcal{H}' + \mathcal{H}^2 = 4\pi G (\bar{\rho} + \bar{p}) a^2 = 4\pi G \sum_I (\bar{\varphi}_I')^2 \quad (4.9)$$

$$\mathcal{H}' + 2\mathcal{H}^2 = 4\pi G (\bar{\rho} - \bar{p}) a^2 = 8\pi G a^2 V. \quad (4.10)$$

In terms of ordinary cosmic time t , (4.3) and (4.7) become

$$\begin{aligned} \ddot{\bar{\varphi}}_I + 3H\dot{\bar{\varphi}}_I &= -V_I \\ H^2 &= \frac{8\pi G}{3} \bar{\rho} = \frac{8\pi G}{3} \left[\frac{1}{2} \sum_I (\dot{\bar{\varphi}}_I)^2 + V \right] \\ \dot{H} &= -4\pi G (\bar{\rho} + \bar{p}) = -4\pi G \sum_I \dot{\bar{\varphi}}_I^2. \end{aligned} \quad (4.11)$$

5 Perturbed Universe

The metric of the perturbed universe is

$$\begin{aligned}
g_{\mu\nu} &= \bar{g}_{\mu\nu} + \delta g_{\mu\nu} = a^2(\eta_{\mu\nu} + h_{\mu\nu}) \\
&= a^2 \begin{bmatrix} -1 - 2A & -B_i \\ -B_i & (1 - 2D)\delta_{ij} + 2E_{ij} \end{bmatrix} \\
g^{\mu\nu} &= a^{-2} \begin{bmatrix} -1 + 2A & -B_i \\ -B_i & (1 + 2D)\delta_{ij} - 2E_{ij} \end{bmatrix}.
\end{aligned} \tag{5.1}$$

From this, the metric determinant is (all products except the diagonal one are at least second order small)

$$\begin{aligned}
g &\approx -a^8(1 + 2A)(1 - 2D + 2E_{11})(1 - 2D + 2E_{22})(1 - 2D + 2E_{33}) \\
&\approx -a^8 [1 + 2A - 6D + 2(E_{11} + E_{22} + E_{33})] = -a^8(1 + 2A - 6D),
\end{aligned} \tag{5.2}$$

since E_{ij} is traceless. From this,

$$\sqrt{-g} = a^4(1 + A - 3D). \tag{5.3}$$

In this perturbed metric, the field equations (3.7) become **(exercise)**³

$$\varphi_I'' + 2\mathcal{H}\varphi_I' - \nabla^2\varphi_I - (A' + 3D' - B_{i,i})\varphi_I' = -a^2(1 + 2A)\frac{\partial V}{\partial\varphi_I}. \tag{5.4}$$

Divide now the fields into a background and a perturbation part,

$$\varphi_I = \bar{\varphi}_I(\eta) + \delta\varphi_I(\eta, \vec{x}). \tag{5.5}$$

The potential becomes

$$V(\varphi_1, \dots, \varphi_N) = V(\bar{\varphi}_1 + \delta\varphi_1, \dots, \bar{\varphi}_N + \delta\varphi_N) = V(\bar{\varphi}_1, \dots, \bar{\varphi}_N) + \frac{\partial V}{\partial\varphi_I}\delta\varphi_I, \tag{5.6}$$

and its derivatives become

$$\begin{aligned}
V_I &\equiv \frac{\partial V}{\partial\varphi_I} = \frac{\partial V}{\partial\varphi_I}(\bar{\varphi}_1 + \delta\varphi_1, \dots, \bar{\varphi}_N + \delta\varphi_N) \\
&= \frac{\partial V}{\partial\varphi_I}(\bar{\varphi}_1, \dots, \bar{\varphi}_N) + \sum_J \frac{\partial^2 V}{\partial\varphi_I\partial\varphi_J}\delta\varphi_J \\
&\equiv \bar{V}_I + \sum_J \bar{V}_{IJ}\delta\varphi_J.
\end{aligned} \tag{5.7}$$

Dividing Eq. (5.4) into a background and a perturbation part **(exercise)**, we get Eq. (4.3) and the field perturbation equation

$$\delta\varphi_I'' + 2\mathcal{H}\delta\varphi_I' - \nabla^2\delta\varphi_I + a^2V_{IJ}\delta\varphi_J = -2a^2V_I A + \varphi_I'(A' + 3D' - B_{i,i}), \tag{5.8}$$

where we have now *quit putting overbars on the background quantities* (here V_{IJ} , V_I , and φ'). Here $-B_{i,i} = -\nabla \cdot \vec{B} = \nabla^2 B$ (note the signs!), so the vector part of \vec{B} does not contribute. For

³In Sec. 3 we had the covariant derivative ∇_μ and other curved spacetime machinery. Here we unravel all that into ordinary functions of coordinates and ordinary partial derivatives, so that, e.g., $\nabla^2 = \sum_i \partial_i \partial_i$. We usually don't write the \sum since summation over repeated indices, e.g., i or I , is implied.

symmetry reasons, the scalar fields couple only to scalar metric perturbations, and therefore we shall just discuss those from here on.

Inserting (5.5) into (3.14) we get the background part of the energy tensor (4.4) and its perturbation (**exercise**)

$$\begin{aligned}
\delta T_0^0 &= -\sum_I \{a^{-2} [\varphi'_I \delta \varphi'_I - (\varphi'_I)^2 A] + V_I \delta \varphi_I\} = -\delta \rho \\
\delta T_i^0 &= -a^{-2} \sum_I \varphi'_I \partial_i (\delta \varphi_I) \\
\delta T_0^i &= +a^{-2} \sum_I [\varphi'_I \partial_i (\delta \varphi_I) - (\varphi'_I)^2 B_i] \\
\delta T_j^i &= \delta_j^i \sum_I \{a^{-2} [\varphi'_I \delta \varphi'_I - (\varphi'_I)^2 A] - V_I \delta \varphi_I\} = \delta_j^i \delta p.
\end{aligned} \tag{5.9}$$

We see that scalar fields have *no anisotropic stress*, and therefore the two Bardeen potentials are equal, $\Phi = \Psi$. (Note also that the shift vector B_i appears in δT_0^i , whereas in the fluid description it appeared in δT_i^0 . This is because we now base the T_ν^μ perturbation on the gradient $\partial_\nu \varphi_I$, which is a covariant vector; whereas in the fluid description we used the contravariant vector u^μ .)

We can now write the Einstein equations. (**Exercise:** Take δG_ν^μ from Appendix A of part 1 of these notes, assuming scalar perturbations.)

$$\begin{aligned}
\frac{1}{2} a^2 \delta G_0^0 &= 3\mathcal{H}(\mathcal{H}A + D') - \nabla^2(\psi - \mathcal{H}B) \\
&= -4\pi G \sum_I \{[\varphi'_I \delta \varphi'_I - (\varphi'_I)^2 A] + a^2 V_I \delta \varphi_I\}
\end{aligned} \tag{5.10}$$

$$-\frac{1}{2} a^2 \delta G_i^0 = (\psi' + \mathcal{H}A)_{,i} = 4\pi G \sum_I \varphi'_I \partial_i (\delta \varphi_I) \tag{5.11}$$

$$+\frac{1}{2} a^2 \delta G_0^i = [\psi' + \mathcal{H}A + (-\mathcal{H}' + \mathcal{H}^2)B]_{,i} = 4\pi G \sum_I [\varphi'_I \partial_i (\delta \varphi_I) + (\varphi'_I)^2 B_{,i}] \tag{5.12}$$

$$\begin{aligned}
\frac{1}{2} a^2 \delta G_j^i &= [(2\mathcal{H}' + \mathcal{H}^2)A + \mathcal{H}A' + \psi'' + 2\mathcal{H}\psi' + \frac{1}{2}\nabla^2 \mathcal{D}] \delta_j^i - \frac{1}{2} \mathcal{D}_{,ij} \\
&= 4\pi G \delta_j^i \sum_I \{[\varphi'_I \delta \varphi'_I - (\varphi'_I)^2 A] - a^2 V_I \delta \varphi_I\},
\end{aligned} \tag{5.13}$$

where $\psi \equiv D + \frac{1}{3}\nabla^2 E$ and

$$\mathcal{D} \equiv A - \psi + 2\mathcal{H}(B - E') + (B - E')' = 0 \tag{5.14}$$

since the rhs of Eq. (5.13) has no off-diagonal part.⁴ Both sides of Eqs. (5.11,5.12) are gradients, and from Eqs. (5.11,5.12,5.13) we get

$$\psi' + \mathcal{H}A = 4\pi G \sum_I \varphi'_I \delta \varphi_I \tag{5.15}$$

$$\psi' + \mathcal{H}A + (-\mathcal{H}' + \mathcal{H}^2)B = 4\pi G \sum_I [\varphi'_I \delta \varphi_I + (\varphi'_I)^2 B] \tag{5.16}$$

$$(2\mathcal{H}' + \mathcal{H}^2)A + \mathcal{H}A' + \psi'' + 2\mathcal{H}\psi' = 4\pi G \sum_I \{[\varphi'_I \delta \varphi'_I - (\varphi'_I)^2 A] - a^2 V_I \delta \varphi_I\}. \tag{5.17}$$

⁴We use similar arguments as when we derived the Einstein equations for perturbations with fluid sources in Part 1. The nondiagonal part of $\mathcal{D}_{,ij}$ does not vanish, unless $\mathcal{D} = 0$. In Fourier space $\mathcal{D}_{,ij}$ becomes $-k_i k_j \mathcal{D}_{\vec{k}}$ and for each Fourier mode we can orient the coordinate system so that at least two of the k_i are nonzero.

The difference between Eqs. (5.16) and (5.15) is just the background equation (4.9), so they are not independent perturbation equations.

Since the φ_I are scalar fields, they gauge transform as

$$\widetilde{\delta\varphi_I} = \delta\varphi_I - \bar{\varphi}'_I \xi^0. \quad (5.18)$$

6 Spatially Flat Gauge

There are different strategies for solving the scalar field perturbation equations. One strategy is to use the Einstein equations to eliminate the metric perturbations from Eq. (5.8), to get a differential equation just for $\delta\varphi_I$.

To accomplish this, we go to the *spatially flat gauge*, denoted by the sub/superscript Q and defined by

$$\psi_Q = 0. \quad (6.1)$$

Thus $D_Q = -\frac{1}{3}\nabla^2 E_Q$. Since the metric perturbation ψ transforms as

$$\tilde{\psi} = \psi + \mathcal{H}\xi^0. \quad (6.2)$$

we get to the spatially flat gauge by the gauge transformation

$$\xi^0 = -\mathcal{H}^{-1}\psi. \quad (6.3)$$

The field perturbation in the spatially flat gauge is usually denoted by Q and called the Sasaki or Mukhanov variable:

$$Q_I \equiv \delta\varphi_I^Q = \delta\varphi_I + \frac{\bar{\varphi}'_I}{\mathcal{H}}\psi. \quad (6.4)$$

(We could also say that the Sasaki–Mukhanov variable is the gauge-invariant quantity defined by the rhs of (6.4).) In the spatially flat gauge the field equations read

$$\begin{aligned} Q_I'' + 2\mathcal{H}Q_I' - \nabla^2 Q_I + a^2 V_{IJ}Q_J &= -2a^2 V_I A_Q + \varphi'_I (A'_Q + 3D'_Q + \nabla^2 B_Q) \\ &= -2a^2 V_I A_Q + \varphi'_I A'_Q - \varphi'_I \nabla^2 (E'_Q - B_Q). \end{aligned} \quad (6.5)$$

The second Einstein equation (5.15) is now

$$\mathcal{H}A_Q = 4\pi G \sum_I \varphi'_I Q_I \quad \Rightarrow \quad A_Q = 4\pi G \mathcal{H}^{-1} \sum_I \varphi'_I Q_I, \quad (6.6)$$

which allows us to eliminate A_Q from Eq. (6.5) in favor of field perturbations Q_I .

The remaining metric perturbations are the combination $\nabla^2(E'_Q - B_Q)$. We first note that the Bardeen potentials are now

$$\Phi = \Psi = \psi - \mathcal{H}(B - E') = \mathcal{H}(E'_Q - B_Q), \quad (6.7)$$

so that the last term in Eq. (6.5) is

$$-\varphi'_I \nabla^2 (E'_Q - B_Q) = -\mathcal{H}^{-1} \varphi'_I \nabla^2 \Phi, \quad (6.8)$$

where

$$\nabla^2 \Phi = 4\pi G a^2 \delta\rho^C \quad (6.9)$$

from the Einstein constraint equation. The remaining metric perturbations have now been replaced by an energy density perturbation, but in a different gauge, the comoving gauge!

Thus we now need express the comoving gauge density perturbation $\delta\rho^C$ in terms of spatially flat gauge quantities.

The comoving gauge was defined earlier by the requirement $v^C = B^C = 0$. The concept of “velocity” is perhaps not appropriate for scalar fields, although we could formally define it as

$$v_i = -\frac{\delta T_i^0}{\rho + p}, \quad (6.10)$$

so we define comoving gauge by

$$\delta T_i^0 = 0 \quad \text{and} \quad B = 0. \quad (6.11)$$

From this follows that

$$\sum_I \varphi'_I \partial_i (\delta\varphi_I^C) = 0 \quad (6.12)$$

and for the single-field case

$$\delta\varphi^C = 0 \quad (6.13)$$

(at least when $\varphi' \neq 0$). Since δT_i^0 and B transform as

$$\begin{aligned} \widetilde{\delta T}_i^0 &= \delta T_i^0 - (T_0^0 - \frac{1}{3}T_k^k)\xi^0_{,i} = \delta T_i^0 + (\rho + p)\xi^0_{,i} \\ \widetilde{B} &= B + \xi' + \xi^0, \end{aligned} \quad (6.14)$$

we get to comoving gauge by

$$\begin{aligned} \xi^0_{,i} &= -\frac{\delta T_i^0}{\rho + p} = \frac{\partial_i(\varphi'_I \delta\varphi_I)}{\sum(\varphi'_I)^2} \\ \Rightarrow \xi^0 &= \frac{\varphi'_I \delta\varphi_I}{\sum(\varphi'_I)^2} \end{aligned} \quad (6.15)$$

and

$$\xi' = -B - \xi^0. \quad (6.16)$$

For the comoving gauge density perturbation we get (**exercise**)

$$\begin{aligned} \delta\rho^C &= \delta\rho - \rho'\xi^0 \\ &= a^{-2} \sum_I [\varphi'_I(\delta\varphi'_I - \varphi'_I A) - (\varphi''_I - \mathcal{H}\varphi'_I)\delta\varphi_I] \\ &= a^{-2} \sum_I \left[\varphi'_I(\delta\varphi'_I - \varphi'_I A) + \left(3\mathcal{H}\varphi'_I + a^2 \frac{\partial V}{\partial\varphi_I} \right) \delta\varphi_I \right] \\ \Rightarrow \nabla^2\Phi &= 4\pi G \sum_I [\varphi'_I(\delta\varphi'_I - \varphi'_I A) - (\varphi''_I - \mathcal{H}\varphi'_I)\delta\varphi_I], \end{aligned} \quad (6.17)$$

where $\delta\varphi_I$ and A are in arbitrary gauge. In particular, in the spatially flat gauge

$$\nabla^2\Phi = 4\pi G \sum_I [\varphi'_I(Q'_I - \varphi'_I A_Q) - (\varphi''_I - \mathcal{H}\varphi'_I)Q_I]. \quad (6.18)$$

We can now use Eq. (6.6) (and the background Eq. 4.9) to eliminate A_Q from Eq. (6.18) (**exercise**),

$$\nabla^2\Phi = 4\pi G a^2 \delta\rho^C = 4\pi G \sum_I \varphi'_I \left(Q'_I + \frac{\mathcal{H}'}{\mathcal{H}} Q_I - \frac{\varphi''_I}{\varphi'_I} Q_I \right) \quad (6.19)$$

and to write Eq. (6.5) as a differential equation involving just the field perturbations Q_I (**exercise**):

$$\boxed{Q_I'' + 2\mathcal{H}Q_I' - \nabla^2 Q_I + \sum_J \left[a^2 V_{IJ} - \frac{8\pi G}{a^2} \left(\frac{a^2}{\mathcal{H}} \phi_I' \phi_J' \right)' \right] Q_J = 0.} \quad (6.20)$$

We see that the evolution of the different scalar field perturbations Q_I are coupled by the second derivatives of the potential V_{IJ} and the background evolution $(a^2 \mathcal{H}^{-1} \phi_I' \phi_J')'$.⁵ In terms of ordinary cosmic time t , Eq. (6.20) becomes (**exercise**)

$$\ddot{Q}_I + 3H\dot{Q}_I - \frac{1}{a^2} \nabla^2 Q_I + \sum_J \left[V_{IJ} - \frac{8\pi G}{a^3} \frac{d}{dt} \left(\frac{a^3}{H} \dot{\phi}_I \dot{\phi}_J \right) \right] Q_J = 0. \quad (6.21)$$

7 Single Field

7.1 Background

For the case of a single field, Eq. (4.4) becomes

$$\rho = \frac{1}{2} a^{-2} (\phi')^2 + V \quad \text{and} \quad p = \frac{1}{2} a^{-2} (\phi')^2 - V, \quad (7.1)$$

so that

$$\rho + p = a^{-2} (\phi')^2 \quad \rho - p = 2V \quad w \equiv \frac{p}{\rho} = \frac{(\phi')^2 - 2a^2 V}{(\phi')^2 + 2a^2 V} \quad (7.2)$$

and the Friedmann equations are

$$\mathcal{H}^2 = \frac{8\pi G}{3} \rho a^2 = \frac{4\pi G}{3} [(\phi')^2 + 2a^2 V] \quad (7.3)$$

$$\mathcal{H}' = -\frac{4\pi G}{3} (\rho + 3p) a^2 = -\frac{8\pi G}{3} [(\phi')^2 - a^2 V]. \quad (7.4)$$

The background field equation is

$$\phi'' + 2\mathcal{H}\phi' + a^2 V_\phi = 0, \quad (7.5)$$

where $V_\phi \equiv dV/d\phi$. Derivating (7.1) and using (7.5), we have

$$\rho' = -3\mathcal{H}a^{-2} (\phi')^2 \quad \text{and} \quad p' = -3\mathcal{H}a^{-2} (\phi')^2 - 2\phi' V_\phi \quad (7.6)$$

and

$$c_s^2 \equiv \frac{p'}{\rho'} = \frac{3\mathcal{H}\phi' + 2a^2 V_\phi}{3\mathcal{H}\phi'}. \quad (7.7)$$

⁵The background evolution part (the second term inside the square brackets) came from the metric perturbations. In Cosmology II we ignored the metric perturbations in this context, saying that in a suitable gauge (e.g., the spatially flat gauge) their effect is negligible during inflation. We will see later (Secs. 7.5 and 9.3) that this contribution is of 1st order in slow-roll parameters and (in Sec. 7.7) that, for single-field inflation, to calculate the primordial power spectrum to 1st order in slow-roll parameters it is enough to use the field perturbation equation to 0th order in slow-roll parameters.

7.2 Field perturbation equation

For a single scalar field, we have just one degree of freedom (for each Fourier mode \vec{k}). For that degree of freedom, we can take the metric perturbation Φ , or the field perturbation in the spatially flat gauge Q . For both approaches we get a second order differential equation, in the latter case

$$Q''_{\vec{k}} + 2\mathcal{H}Q'_{\vec{k}} + k^2Q_{\vec{k}} + a^2V_{\varphi\varphi}Q_{\vec{k}} = \frac{8\pi G}{a^2} \left[\frac{a^2}{\mathcal{H}}(\varphi')^2 \right]' Q_{\vec{k}}. \quad (7.8)$$

If we know the background solution, we can solve the evolution of the field perturbations starting from initial values $Q_{\vec{k}}$ and $Q'_{\vec{k}}$ specified at some initial time $\eta = \eta_{\text{init}}$. The field perturbations then determine all other perturbation quantities.

However, just like in the fluid case, it is useful to introduce the comoving curvature perturbation \mathcal{R} . It has a simpler behavior at superhorizon scales, which helps us understand what is going on.

7.3 Comoving curvature perturbation

The comoving curvature perturbation is defined

$$\mathcal{R} \equiv -\psi^C = -\psi - \mathcal{H}\xi^0, \quad (7.9)$$

where now, from Eq. (6.15)

$$\xi^0 = \frac{\delta\varphi}{\varphi'}, \quad (7.10)$$

so that we have

$$\mathcal{R} = -\psi - \frac{\mathcal{H}}{\varphi'}\delta\varphi = -\frac{\mathcal{H}}{\varphi'}Q. \quad (7.11)$$

Derivating this gives

$$\mathcal{R}' = -\frac{\mathcal{H}}{\varphi'} \left(Q' + \frac{\mathcal{H}'}{\mathcal{H}}Q - \frac{\varphi''}{\varphi'}Q \right). \quad (7.12)$$

Comparing to Eq. (6.19) we can write this as

$$\mathcal{R}' = -\frac{\mathcal{H}a^2}{(\varphi')^2}\delta\rho^C = -\frac{\mathcal{H}}{\rho+p}\delta\rho^C \quad (7.13)$$

$$= -\frac{\mathcal{H}}{4\pi G(\varphi')^2}\nabla^2\Phi = -\frac{2}{3\mathcal{H}}\frac{\rho}{\rho+p}\nabla^2\Phi. \quad (7.14)$$

In Fourier space

$$\mathcal{H}^{-1}\mathcal{R}'_{\vec{k}} = \frac{2}{3}\frac{\rho}{\rho+p} \left(\frac{k}{\mathcal{H}} \right)^2 \Phi, \quad (7.15)$$

showing that $\mathcal{R}_{\vec{k}} = \text{const.}$ at superhorizon ($k \ll \mathcal{H}$) scales. During inflation $\rho + p \ll \rho$; in Sec. 7.4 we show that $1 + w = (\rho + p)/\rho$ is first order in slow-roll parameters. However, for cosmologically relevant scales k/\mathcal{H} will typically shrink to something like e^{-50} , which is much smaller.

We can compare Eq. (7.13) to the equation from the first part of the course,

$$\mathcal{H}^{-1}\mathcal{R}' = -\frac{\delta p^C}{\rho+p} - \frac{2}{3}\frac{w}{1+w}\nabla^2\Pi = -\frac{\delta p^C}{\rho+p} \quad (7.16)$$

(since now $\Pi = 0$).

In the single-field case $\delta\rho^C = \delta p^C$, since $\delta\rho - \delta p = 2V_{\varphi}\delta\varphi$ and $\delta\varphi^C = 0$. *Comoving time slices are the $\varphi = \text{const}$ hypersurfaces.*

Total entropy perturbation (we do not use this for anything in these lectures). We have defined the total entropy perturbation

$$\mathcal{S} \equiv \mathcal{H} \left(\frac{\delta p}{p'} - \frac{\delta \rho}{\rho'} \right) \quad (7.17)$$

$$\Rightarrow \delta p = c_s^2 [\delta \rho - 3(\rho + p)\mathcal{S}], \quad (7.18)$$

where $c_s^2 \equiv p'/\rho'$. In the single-field case \mathcal{S} is proportional to $\delta\rho^C$, and therefore it is negligible at superhorizon scales (since $\delta\rho^C$ is), compared to quantities like \mathcal{R} and Φ .

In the single-field case

$$\mathcal{S} = \frac{c_s^2 - 1}{3c_s^2} \frac{\delta\rho^C}{\rho + p} = \frac{2a^2 V_\varphi}{3(\varphi')^2 (3\mathcal{H}\varphi' + 2a^2 V_\varphi)} a^2 \delta\rho^C = \frac{a^2 V_\varphi}{6\pi G (\varphi')^2 (3\mathcal{H}\varphi' + 2a^2 V_\varphi)} \nabla^2 \Phi, \quad (7.19)$$

so that

$$\mathcal{S}_{\vec{k}} = -\frac{2}{9} \frac{\rho}{\rho + p} \frac{2a^2 V_\varphi}{3\mathcal{H}\varphi' + 2a^2 V_\varphi} \left(\frac{k}{\mathcal{H}} \right)^2 \Phi_{\vec{k}}. \quad (7.20)$$

7.4 Slow-Roll Inflation

This subsection is completely about the background solution. We use ordinary cosmic time t , instead of the conformal time η , and write

$$V' \equiv \frac{dV}{d\varphi}. \quad (7.21)$$

The exact background equations are

$$\ddot{\varphi} + 3H\dot{\varphi} + V' = 0 \quad \text{and} \quad H^2 = \frac{8\pi G}{3} \left(\frac{1}{2}\dot{\varphi}^2 + V \right) = \frac{1}{3M^2} \left(\frac{1}{2}\dot{\varphi}^2 + V \right), \quad (7.22)$$

where

$$M \equiv \frac{1}{\sqrt{8\pi G}} \quad (7.23)$$

is the reduced Planck mass.

In the *slow-roll approximation* we assume that

$$\dot{\varphi}^2 \ll V \quad \text{and} \quad |\ddot{\varphi}| \ll |3H\dot{\varphi}|, \quad (7.24)$$

and replace Eq. (7.22) by the *slow-roll equations*

$$\begin{aligned} 3H\dot{\varphi} + V' = 0 &\Rightarrow V' = -3H\dot{\varphi} \Rightarrow \dot{\varphi} = -\frac{V'}{3H} \\ H^2 = \frac{1}{3M^2}V &\Rightarrow V = 3M^2 H^2 \Rightarrow 3H^2 = \frac{V}{M^2}, \end{aligned} \quad (7.25)$$

and from these

$$H^{-1}\dot{\varphi} = -M^2 \frac{V'}{V}. \quad (7.26)$$

To be consistent, we must now completely forget Eqs. (7.22), and use just Eqs. (7.25) in their place.

The important thing about the slow-roll approximation is that everything now depends on φ , through $V(\varphi)$ and its derivatives; we can not specify φ and $\dot{\varphi}$ separately. Derivating the slow-roll equations, we get (**exercise**)

$$\dot{H} = -\frac{(V')^2}{6V} \quad \text{and} \quad \ddot{\varphi} = \frac{M^2 V'' V'}{3V} - \frac{M^2 (V')^3}{6V^2}. \quad (7.27)$$

We define the slow-roll parameters

$$\boxed{\varepsilon \equiv \varepsilon_V \equiv \frac{M^2}{2} \left(\frac{V'}{V} \right)^2 \quad \eta \equiv \eta_V \equiv M^2 \frac{V''}{V} \quad \xi \equiv \xi_V \equiv M^4 \frac{V'''V'}{V^2}.} \quad (7.28)$$

(We do not usually write the subscript V .⁶) Using the slow-roll equations we obtain the following results (**exercise**)

$$\begin{aligned} H^{-2} \dot{H} &= -\varepsilon \\ (H^{-1} \dot{\varphi})^2 &= 2M^2 \varepsilon \\ H^{-2} \ddot{\varphi} &= H^{-1} \dot{\varphi} (\varepsilon - \eta) \\ H^{-1} \dot{\varepsilon} &= -2\varepsilon\eta + 4\varepsilon^2 \\ H^{-1} \dot{\eta} &= 2\varepsilon\eta - \xi \end{aligned} \quad (7.31)$$

and

$$\begin{aligned} \rho &= (1 + \frac{1}{3}\varepsilon)V \\ p &= (-1 + \frac{1}{3}\varepsilon)V \\ w &\equiv \frac{p}{\rho} \approx -1 + \frac{2}{3}\varepsilon \quad \Rightarrow \quad 1 + w \approx \frac{2}{3}\varepsilon \\ c_s^2 &\equiv \frac{\dot{p}}{\dot{\rho}} \approx -1 - \frac{2}{3}\varepsilon + \frac{2}{3}\eta. \end{aligned} \quad (7.32)$$

During inflation, the slow-roll parameters ε and η are typically small, and ξ even smaller: typically ξ is "second-order small" compared to ε and η . When we say that we calculate to a given order in slow-roll parameters, we refer to ε and η as being first order, and ξ as second order. We see that the time variation of ε and η is also second-order small in slow-roll parameters.

7.5 Evolution through Horizon Exit

7.5.1 Preliminaries

We want to calculate how field perturbations evolve in slow-roll inflation, starting from when the perturbations are well inside the horizon, and ending when they are well outside the horizon.

The field perturbation equation is

$$H^{-2} \ddot{Q}_{\vec{k}} + 3H^{-1} \dot{Q}_{\vec{k}} + \left(\frac{k}{aH} \right)^2 Q_{\vec{k}} = \left[\frac{8\pi G}{a^3 H^2} \frac{d}{dt} \left(\frac{a^3}{H} \dot{\varphi}^2 \right) - H^{-2} V'' \right] Q_{\vec{k}}. \quad (7.33)$$

In the slow-roll approximation the right hand side is (**exercise**)

$$[6\varepsilon - 3\eta + 6\varepsilon^2 - 4\varepsilon\eta] Q_{\vec{k}}. \quad (7.34)$$

⁶The subscript V is to emphasize that these are the *potential* slow-roll parameters, defined through $V(\varphi)$, to distinguish them from the *Hamilton-Jacobi slow-roll parameters*, defined as

$$\varepsilon_H \equiv 2M^2 \left[\frac{H'(\varphi)}{H(\varphi)} \right]^2 \quad \text{and} \quad \eta_H \equiv 2M^2 \frac{H''(\varphi)}{H(\varphi)}, \quad (7.29)$$

where $'$ denotes derivation with respect to φ . One can show that

$$\varepsilon_H = -\frac{d \ln H}{d \ln a} = -\frac{\dot{H}}{H^2} \quad \text{and} \quad \eta_H = -\frac{d \ln H'}{d \ln a} \quad (7.30)$$

and we will actually use ε_H in Sec. 7.7.2. These two kinds of slow-roll parameters are equal to first order in each other but differ in second order. The subscript V also serves to distinguish η_V from the conformal time η , so we use it when there may be a danger of confusion. If there is no subscript we mean ε_V , η_V , and ξ_V .

Here the -3η came from the V'' term, and the $6\varepsilon + 6\varepsilon^2 - 4\varepsilon\eta$ from the first term, which came from the rhs of Eq. (6.5), i.e., the metric perturbation part. Thus we see that in the spatially flat gauge, metric perturbations affect the field perturbation equation at the first-order level in slow-roll parameters. If we calculated just to 0th order in slow-roll parameters, we could ignore the metric perturbations in this gauge.

We calculate to first order in slow-roll parameters, so that the field perturbation equation is

$$H^{-2}\ddot{Q}_{\vec{k}} + 3H^{-1}\dot{Q}_{\vec{k}} + \left(\frac{k}{aH}\right)^2 Q_{\vec{k}} = [6\varepsilon - 3\eta] Q_{\vec{k}}, \quad (7.35)$$

or, in conformal time

$$Q'' + 2\mathcal{H}Q' + k^2Q = \mathcal{H}^2(6\varepsilon - 3\eta_V)Q, \quad (7.36)$$

where we can *take* ε and η_V to be constant (since their time variation is of second order in slow-roll parameters). We use the subscript V in the slow-roll parameter η_V so as not to confuse it with the conformal time η . Defining

$$u \equiv aQ, \quad (7.37)$$

Eq. (7.36) becomes

$$u'' + \left(k^2 - \frac{a''}{a}\right)u = \mathcal{H}^2(6\varepsilon - 3\eta_V)u. \quad (7.38)$$

7.5.2 Background

First we need to solve the background problem, i.e., how do a and \mathcal{H} evolve. From

$$\mathcal{H} = \frac{a'}{a} = aH = \dot{a} \quad (7.39)$$

we find (**exercise**)

$$\frac{a''}{a} = \mathcal{H}^2 \left(1 + \frac{\mathcal{H}'}{\mathcal{H}^2}\right) \quad \text{and} \quad \frac{\mathcal{H}'}{\mathcal{H}^2} = 1 + \frac{\dot{H}}{H^2}. \quad (7.40)$$

Using the slow-roll relations (7.31) we have thus that

$$\frac{\mathcal{H}'}{\mathcal{H}^2} = 1 - \varepsilon \quad (7.41)$$

and

$$\frac{a''}{a} = \mathcal{H}^2(2 - \varepsilon). \quad (7.42)$$

Integrating (7.41), we get

$$\frac{d\mathcal{H}}{\mathcal{H}^2} = (1 - \varepsilon)d\eta \quad \Rightarrow \quad \mathcal{H} = \frac{-1}{(1 - \varepsilon)\eta} = \frac{1}{a} \frac{da}{d\eta}. \quad (7.43)$$

Integrating again,

$$\frac{da}{a} = -\frac{1}{1 - \varepsilon} \frac{d\eta}{\eta} \quad \Rightarrow \quad \ln a = -\frac{1}{1 - \varepsilon} \ln |\eta| + \text{const.} \quad \Rightarrow \quad a \propto (-\eta)^{-\frac{1}{1 - \varepsilon}}. \quad (7.44)$$

Note that here η is negative; as time goes on $\eta \rightarrow 0$ and $a \rightarrow \infty$ (if slow-roll inflation continued forever).

7.5.3 Hankel and Bessel functions

The Hankel functions

$$H_\nu^{(1)}(x) \equiv J_\nu(x) + iN_\nu(x) \quad \text{and} \quad H_\nu^{(2)}(x) \equiv J_\nu(x) - iN_\nu(x), \quad (7.45)$$

where the J_ν and N_ν are Bessel and Neumann functions, are solutions of the Bessel equation

$$x^2 \frac{d^2}{dx^2} Z(x) + x \frac{d}{dx} Z(x) + [x^2 - \nu^2] Z(x) = 0. \quad (7.46)$$

For real x , $J_\nu(x)$ and $N_\nu(x)$ are real, and $H_\nu^{(2)}(x) = H_\nu^{(1)}(x)^*$. Their asymptotic behavior is

$$H_\nu^{(1)}(x) \sim \sqrt{\frac{2}{\pi x}} e^{i\left[x - (\nu + \frac{1}{2})\frac{\pi}{2}\right]} \quad \text{for} \quad x \rightarrow \infty \quad (7.47)$$

$$H_\nu^{(1)}(x) \sim -i \frac{(\nu - 1)!}{\pi} \left(\frac{2}{x}\right)^\nu = \sqrt{\frac{2}{\pi}} e^{-i\frac{\pi}{2}} 2^{\nu - \frac{3}{2}} \frac{\Gamma(\nu)}{\Gamma(\frac{3}{2})} x^{-\nu} \quad \text{for} \quad x \rightarrow 0. \quad (7.48)$$

The latter form will be useful for us, since we'll have ν close to $\frac{3}{2}$.

The solutions of

$$x^2 \frac{d^2}{dx^2} Z(x) + x \frac{d}{dx} Z(x) + [k^2 x^2 - \nu^2] Z(x) = 0 \quad (7.49)$$

are

$$H_\nu^{(1)}(kx) \quad \text{and} \quad H_\nu^{(2)}(kx) \quad (7.50)$$

and for negative x , the solutions are

$$H_\nu^{(1)}(-kx) \quad \text{and} \quad H_\nu^{(2)}(-kx). \quad (7.51)$$

7.5.4 The perturbation equation

Using (7.42), Eq. (7.38) becomes

$$u'' + [k^2 - \mathcal{H}^2(2 + 5\varepsilon - 3\eta_V)] u = 0, \quad (7.52)$$

and using

$$\mathcal{H}^2 = \frac{1}{(1 - \varepsilon)^2 \eta^2} \approx \frac{1 + 2\varepsilon}{\eta^2} \quad (7.53)$$

we get, to 1st order in slow-roll parameters,

$$u'' + \left[k^2 - \frac{1}{\eta^2} (2 + 9\varepsilon - 3\eta_V) \right] u = 0. \quad (7.54)$$

This equation is closely related to the Bessel equation. To see this, write it as

$$u'' + \left[k^2 - \frac{1}{\eta^2} (\nu^2 - \frac{1}{4}) \right] u = 0. \quad (7.55)$$

where

$$\nu^2 = \frac{9}{4} + 9\varepsilon - 3\eta_V \quad \Rightarrow \quad \nu = \frac{3}{2} \sqrt{1 + 4\varepsilon - \frac{4}{3}\eta_V} \approx \frac{3}{2} + 3\varepsilon - \eta_V. \quad (7.56)$$

Define now a new function s so that

$$u \equiv (-\eta)^{1/2} s. \quad (7.57)$$

Eq. (7.55) becomes (**exercise**)

$$\eta^2 s'' + \eta s' + [k^2 \eta^2 - \nu^2] s = 0, \quad (7.58)$$

which we recognize as the Bessel equation (7.49). The solutions are thus

$$s(\eta) \equiv (-\eta)^{-1/2} u(\eta) \equiv (-\eta)^{-1/2} a Q_{\vec{k}}(\eta) = C_{1\vec{k}} H_\nu^{(1)}(-k\eta) + C_{2\vec{k}} H_\nu^{(2)}(-k\eta), \quad (7.59)$$

or

$$Q_{\vec{k}} = C_{\vec{k}} a^{-1} \sqrt{-\eta} H_\nu(-k\eta). \quad (7.60)$$

Early times correspond to $x = -k\eta \rightarrow \infty$ and late times to $x = -k\eta \rightarrow 0$. Thus we have for early times

$$Q_{\vec{k}} = C_{\vec{k}} \sqrt{\frac{2}{\pi}} \frac{1}{a\sqrt{k}} e^{-ik\eta} \quad (7.61)$$

and at late times

$$Q_{\vec{k}} = C_{\vec{k}} a^{-1} \sqrt{-\eta} \sqrt{\frac{2}{\pi}} 2^{\nu-3/2} \frac{\Gamma(\nu)}{\Gamma(\frac{3}{2})} (-k\eta)^{-\nu} \quad (7.62)$$

where we dropped the constant phase factors $\exp[-i(\nu+1/2)\pi/2]$ (early) and $\exp(-i\pi/2)$ (late), since they do not matter in what follows. (One phase factor can be included in the constant $C_{\vec{k}}$, but since this constant is supposed to be the same in both limits, there is in reality an extra phase difference $\exp[i(\nu-1/2)\pi/2]$ between the late and early times, which we did not bother to write in. At early times the phase is rotating rapidly, anyway.) From (7.56),

$$\nu = \frac{3}{2} + 3\varepsilon - \eta_V. \quad (7.63)$$

From (7.44),

$$a^{-1} \propto (-\eta)^{1/(1-\varepsilon)} \approx (-\eta)^{1+\varepsilon} \Rightarrow a^{-1} \sqrt{-\eta} \propto (-\eta)^{3/2+\varepsilon} \quad (7.64)$$

so we have at late times

$$Q_{\vec{k}} \propto (-\eta)^{3/2+\varepsilon-\nu} = (-\eta)^{\eta_V-2\varepsilon}, \quad (7.65)$$

i.e., $Q_{\vec{k}}$ becomes almost constant; that is, it becomes constant to 0th order in slow-roll parameters.

7.6 Generation of Scalar Perturbations

Subhorizon scales during inflation are microscopic⁷ and therefore quantum effects are important. Thus we should study the behavior of scalar fields using quantum field theory. Consider first the quantum field theory of a scalar field in Minkowski space.

7.6.1 Vacuum fluctuations in Minkowski space

The field equation for a massive free (i.e. $V(\varphi) = \frac{1}{2}m^2\varphi^2$) real scalar field in Minkowski space is

$$\ddot{\varphi} - \nabla^2 \varphi + m^2 \varphi = 0, \quad (7.66)$$

or

$$\ddot{\varphi}_{\vec{k}} + E_k^2 \varphi_{\vec{k}} = 0, \quad (7.67)$$

⁷We later give an upper limit to the inflation energy scale, i.e., V at the time cosmological scales exited the horizon, $V^{1/4} < 6.8 \times 10^{16}$ GeV. From $H^2 = V/3M^2$ we have $H < 1.0 \times 10^{15}$ GeV or for the Hubble length $H^{-1} > 1.9 \times 10^{-31}$ m. This is a lower limit to the horizon size, but it is not expected to be very many orders of magnitude larger.

where $E_k^2 = k^2 + m^2$, for Fourier components. We recognize (7.67) as the equation for a harmonic oscillator. Thus each Fourier component of the field behaves as an independent harmonic oscillator. For convenience, we consider the system enclosed in a finite cubic box with volume $V = L^3$ (not to be confused with the field potential), allowing us to do Fourier sums over a discrete set of wave numbers (momenta) \vec{k} , instead of Fourier integrals.

In the quantum mechanical treatment of the harmonic oscillator one introduces the creation and annihilation operators, which raise and lower the energy state of the system. We can do the same here.

Now we have a different pair of creation and annihilation operators $\hat{a}_{\vec{k}}^\dagger, \hat{a}_{\vec{k}}$ for each Fourier mode \vec{k} . We denote the ground state of the system by $|0\rangle$, and call it the *vacuum*. *Particles* are quanta of the oscillations of the field. The vacuum is a state with no particles. Operating on the vacuum with the creation operator $\hat{a}_{\vec{k}}^\dagger$, we add one quantum with momentum \vec{k} and energy E_k to the system, i.e., we create one particle. We denote this state with one particle, whose momentum is \vec{k} by $|1_{\vec{k}}\rangle$. Thus

$$\hat{a}_{\vec{k}}^\dagger|0\rangle = |1_{\vec{k}}\rangle. \quad (7.68)$$

This particle has a well-defined momentum \vec{k} , and therefore it is completely unlocalized (Heisenberg's uncertainty principle). The annihilation operator acting on the vacuum gives zero, i.e., not the vacuum state but the zero element of Hilbert space (the space of all quantum states),

$$\hat{a}_{\vec{k}}|0\rangle = 0. \quad (7.69)$$

We denote the Hermitian conjugate of the vacuum state by $\langle 0|$. Thus

$$\langle 0|\hat{a}_{\vec{k}} = \langle 1_{\vec{k}}| \quad \text{and} \quad \langle 0|\hat{a}_{\vec{k}}^\dagger = 0. \quad (7.70)$$

The commutation relations of the creation and annihilation operators are

$$[\hat{a}_{\vec{k}}^\dagger, \hat{a}_{\vec{k}'}^\dagger] = [\hat{a}_{\vec{k}}, \hat{a}_{\vec{k}'}] = 0, \quad [\hat{a}_{\vec{k}}, \hat{a}_{\vec{k}'}^\dagger] = \delta_{\vec{k}\vec{k}'}. \quad (7.71)$$

When going from classical physics to quantum physics, classical observables are replaced by operators. One can then calculate expectation values for these observables using the operators. Here the classical observable

$$\varphi(t, \vec{x}) = \sum \varphi_{\vec{k}}(t) e^{i\vec{k}\cdot\vec{x}} \quad (7.72)$$

is replaced by the *field operator*

$$\hat{\varphi}(t, \vec{x}) = \sum \hat{\varphi}_{\vec{k}}(t) e^{i\vec{k}\cdot\vec{x}} \quad (7.73)$$

where⁸

$$\hat{\varphi}_{\vec{k}}(t) = w_k(t)\hat{a}_{\vec{k}} + w_k^*(t)\hat{a}_{-\vec{k}}^\dagger \quad (7.74)$$

and

$$w_k(t) = V^{-1/2} \frac{1}{\sqrt{2E_k}} e^{-iE_k t} \quad (7.75)$$

is the mode function, a normalized solution of the field equation (7.67). We are using the Heisenberg picture, i.e. we have time-dependent operators; the quantum states are time-independent.

Classically the ground state would be one where $\varphi = \text{const} = 0$, but we know from the quantum mechanics of a harmonic oscillator, that there are oscillations even in the ground

⁸We skip the detailed derivation of the field operator, which belongs to a course of quantum field theory. See e.g. Peskin & Schroeder, section 2.3 (note different normalizations of operators and states, related to doing Fourier integrals rather than sums and convenience in relation to Lorentz invariance). Our discussion here is rather incomplete, since we do not derive the normalization of the mode function, which is actually the most important part of the result (7.79).

state. Likewise, there are fluctuations of the scalar field, *vacuum fluctuations*, even in the vacuum state.

We shall now calculate the *power spectrum* of these vacuum fluctuations. The power spectrum is defined as the expectation value

$$\mathcal{P}_\varphi(k) = V \frac{k^3}{2\pi^2} \langle |\varphi_{\vec{k}}|^2 \rangle \quad (7.76)$$

and it gives the variance of $\varphi(\vec{x})$ as

$$\langle \varphi(\vec{x})^2 \rangle = \int_0^\infty \frac{dk}{k} \mathcal{P}_\varphi(k). \quad (7.77)$$

For the vacuum state $|0\rangle$ the expectation value of $|\varphi_{\vec{k}}|^2$ is

$$\begin{aligned} \langle 0 | \hat{\varphi}_{\vec{k}} \hat{\varphi}_{\vec{k}}^\dagger | 0 \rangle &= \\ & |w_k|^2 \langle 0 | \hat{a}_{\vec{k}} \hat{a}_{\vec{k}}^\dagger | 0 \rangle + w_k^2 \langle 0 | \hat{a}_{\vec{k}} \hat{a}_{-\vec{k}} | 0 \rangle + (w_k^*)^2 \langle 0 | \hat{a}_{-\vec{k}}^\dagger \hat{a}_{\vec{k}}^\dagger | 0 \rangle + |w_k|^2 \langle 0 | \hat{a}_{-\vec{k}}^\dagger \hat{a}_{-\vec{k}} | 0 \rangle \\ &= |w_k|^2 \langle 1_{\vec{k}} | 1_{\vec{k}} \rangle = |w_k|^2 \end{aligned} \quad (7.78)$$

since all but the first term give 0, and our states are normalized so that $\langle 1_{\vec{k}} | 1_{\vec{k}'} \rangle = \delta_{\vec{k}\vec{k}'}$. From Eq. (7.75) we have that $|w_k|^2 = 1/(2VE_k)$. Our main result is that

$$\mathcal{P}_\varphi(k) = V \frac{k^3}{2\pi^2} |w_k|^2 \quad (7.79)$$

for vacuum fluctuations, which we shall now apply to inflation, where the mode functions $w_k(t)$ are different.

7.6.2 Vacuum fluctuations during inflation

During inflation the field equation (for inflaton perturbations) is, Eq. (7.33). There are oscillations only in the perturbation Q ; the background φ is homogeneous and evolving slowly in time. For the particle point of view, the background solution represents the vacuum,⁹ i.e., particles are quanta of oscillations around that value.

We found that the two independent solutions for $Q_{\vec{k}}(\eta)$ are

$$w_k(\eta) = C_{\vec{k}} a^{-1} \sqrt{-\eta} H_\nu^{(1)}(-k\eta) \quad (7.80)$$

and its complex conjugate $w_k^*(\eta)$, where the time dependence is in $a = a(\eta) \propto (-\eta)^{-1/(1-\epsilon_V)}$.

When the scale k is well inside the horizon, $k \gg \mathcal{H} \sim 1/(-\eta)$,

$$w_k(\eta) \approx C_{\vec{k}} \sqrt{\frac{2}{\pi}} \frac{1}{a\sqrt{k}} e^{-ik\eta} \quad (7.81)$$

oscillates rapidly compared to the Hubble time. If we consider distance and time scales much smaller than the Hubble scale, we can ignore the expansion of the universe, and write $\eta = t/a$. Things should then behave like in Minkowski space and we can equate the above solution with the Minkowski mode function,

$$C_{\vec{k}} \sqrt{\frac{2}{\pi}} \frac{1}{a\sqrt{k}} e^{-ikt/a} = (aL)^{-3/2} \frac{1}{\sqrt{2E_k}} e^{-iE_k t} \quad (7.82)$$

⁹This is not the vacuum state in the sense of being the ground state of the system. The true ground state has φ at the minimum of the potential. However there are no particles related to the background evolution $\varphi(t)$.

(we write $a^3V = (aL)^3$ for the reference volume, so that V represents the comoving volume). From this we identify $E_k = k/a$ (there is no mass here since we are in the large k limit) and

$$C_{\vec{k}} = L^{-3/2} \sqrt{\frac{\pi}{4}}, \quad (7.83)$$

so that the correctly normalized mode function during inflation is

$$w_k(\eta) = L^{-3/2} \sqrt{\frac{\pi}{4}} a^{-1} \sqrt{-\eta} H_\nu(-k\eta), \quad (7.84)$$

which we can now apply at all times during slow-roll inflation.

The field operator for the scalar field perturbations during inflation is

$$\hat{Q}_{\vec{k}}(\eta) = w_k(\eta) \hat{a}_{\vec{k}} + w_k^*(\eta) \hat{a}_{-\vec{k}}^\dagger, \quad (7.85)$$

and the power spectrum of the scalar field fluctuations is

$$\mathcal{P}_Q(k) = L^3 \frac{k^3}{2\pi^2} |w_k|^2. \quad (7.86)$$

Well before horizon exit, $k \gg \mathcal{H}$, observed during timescales much less than the Hubble time, the field operator $\hat{Q}_{\vec{k}}(\eta)$ becomes the Minkowski space field operator and we have standard vacuum fluctuations in φ .

Well after horizon exit, the mode function becomes almost constant in time,

$$w_k(\eta) = L^{-3/2} \frac{1}{\sqrt{2}} 2^{\nu-3/2} \frac{\Gamma(\nu)}{\Gamma(\frac{3}{2})} \frac{1}{a} \sqrt{-\eta} (-k\eta)^{-\nu} \propto (-\eta)^{\eta_V - 2\varepsilon}, \quad (7.87)$$

(use $a \propto (-\eta)^{-1-\varepsilon}$ and $\nu = \frac{3}{2} + 3\varepsilon - \eta_V$) the fluctuations “freeze”. They are now at large scales, and can be treated classically. The power spectrum of Q fluctuations becomes

$$\begin{aligned} \mathcal{P}_Q(\eta, k) &= \frac{2^{2\nu-3}}{(2\pi)^2} \left[\frac{\Gamma(\nu)}{\Gamma(\frac{3}{2})} \right]^2 (-a\eta)^{-2} (-k\eta)^{3-2\nu} \\ &= \frac{2^{6\varepsilon-2\eta_V}}{(2\pi)^2} \left[\frac{\Gamma(\frac{3}{2} + 3\varepsilon - \eta_V)}{\Gamma(\frac{3}{2})} \right]^2 (-a\eta)^{-2} (-k\eta)^{2\eta_V - 6\varepsilon}. \end{aligned} \quad (7.88)$$

We see that the scale dependence of the power spectrum is

$$\mathcal{P}_Q \propto k^{2\eta_V - 6\varepsilon} \equiv k^{n_s} \Rightarrow n_s = 2\eta_V - 6\varepsilon. \quad (7.89)$$

Even at late times, this power spectrum has a weak time dependence $\propto (-\eta)^{2\eta_V - 4\varepsilon}$, and more importantly, our approximation that the slow-roll parameters stay constant will only hold for a number of e-foldings $\ll 1/(\text{slow-roll params})$. Thus we want to switch to other variables after horizon exit.

7.7 The Primordial Power Spectrum

We learned in Sec. 7.3 that the comoving curvature perturbation stays constant at superhorizon scales, and that

$$\mathcal{R} = -\frac{H}{\dot{\varphi}} Q, \quad (7.90)$$

from which follows

$$\mathcal{P}_{\mathcal{R}}(\eta, k) = \left(\frac{H}{\dot{\phi}}\right)^2 \mathcal{P}_Q(\eta, k). \quad (7.91)$$

Thus we have at late times (when $k \ll \mathcal{H}$)

$$\mathcal{P}_{\mathcal{R}}(\eta, k) = \frac{2^{2\nu-3}}{(2\pi)^2} \left[\frac{\Gamma(\nu)}{\Gamma(\frac{3}{2})}\right]^2 \left(\frac{H}{-a\eta\dot{\phi}}\right)^2 (-k\eta)^{3-2\nu} \quad (7.92)$$

The scale dependence is $k^{3-2\nu}$ and the time dependence is given by

$$\left(\frac{H}{a\dot{\phi}}\right)^2 (-\eta)^{1-2\nu}. \quad (7.93)$$

Exercise: Show that this is indeed constant in time in the slow-roll approximation (calculate to 1st order in slow-roll parameters).

Since the factor (7.93) stays constant, we can choose to evaluate it for each k at the time when the scale k exits the horizon, when $k = aH$, although Eq. (7.92) does not give the power spectrum yet at that time (it still has the full Hankel function then, not the late time limit), only later.

From (7.43),

$$\mathcal{H} = aH = \frac{-1}{(1-\varepsilon)\eta} \quad (7.94)$$

we have

$$-a\eta = \frac{1}{(1-\varepsilon)H} \quad \text{and} \quad -k\eta = \frac{k}{(1-\varepsilon)aH} \quad (7.95)$$

so that (7.92) becomes

$$\mathcal{P}_{\mathcal{R}}(\eta, k) = 2^{2\nu-3} \left[\frac{\Gamma(\nu)}{\Gamma(\frac{3}{2})}\right]^2 (1-\varepsilon)^{2\nu-1} \left(\frac{H}{2\pi}\right)^2 \left(\frac{H}{\dot{\phi}}\right)^2 \left(\frac{k}{aH}\right)^{3-2\nu}, \quad (7.96)$$

which we now evaluate for each k at the time of its horizon exit, $k = aH = \mathcal{H}$ to arrive at the final result

$$\boxed{\mathcal{P}_{\mathcal{R}}(k) = \left[2^{2\nu-3} \left(\frac{\Gamma(\nu)}{\Gamma(\frac{3}{2})}\right)^2 (1-\varepsilon)^{2\nu-1} \left(\frac{H}{2\pi}\right)^2 \left(\frac{H}{\dot{\phi}}\right)^2 \right]_{k=aH}}. \quad (7.97)$$

This is the *primordial perturbation spectrum*. Expression (7.97) has now explicitly no time dependence, since by definition, it is to be evaluated for each scale at the time of horizon exit ($k = aH$). Independent of the slow-roll approximation, $\mathcal{P}_{\mathcal{R}}(k)$ stays constant in time for as long as the scales k in question are well outside the horizon. (Note that it does not hold yet at the time of horizon exit of the given scale.)

We can now assume that Eq. (7.97) has been derived separately at each different k , so that for each k the slow-roll parameters were approximated to be constant at that value they had when $k = aH$. Thus Eq. (7.97) is valid for all scales for which the slow-roll approximation was valid around horizon exit, even though the slow-roll parameters may have changed significantly while this whole range of scales exited.

To 1st order in slow-roll parameters (see Sec. 7.7.1), Eq. (7.97) has the same scale dependence as (7.88),

$$n_s \equiv \frac{d \ln \mathcal{P}_R}{d \ln k} = 3 - 2\nu = 2\eta - 6\varepsilon, \quad (7.98)$$

but we can now actually calculate the scale dependence to 2nd order in slow-roll parameters (see Sec. 7.7.2).

7.7.1 Spectral index to first order in slow roll

In Cosmology II we did the preceding calculation for the evolution of perturbations through the horizon to 0th order in slow-roll, i.e., setting $\nu = \frac{3}{2}$ in Eq. (7.55), and got the result

$$\mathcal{P}_{\mathcal{R}}(k) = \left[\left(\frac{H}{2\pi} \right)^2 \left(\frac{H}{\dot{\phi}} \right)^2 \right]_{k=aH} = \frac{1}{24\pi^2} \frac{1}{M^2} \frac{V}{\varepsilon} \Big|_{k=aH}. \quad (7.99)$$

To calculate the *spectral index*¹⁰

$$n_s \equiv \frac{d \ln \mathcal{P}_R}{d \ln k}, \quad (7.100)$$

we first note that

$$\frac{d \ln k}{dt} = \frac{d \ln(aH)}{dt} = \frac{\dot{a}}{a} + \frac{\dot{H}}{H} = (1 - \varepsilon)H, \quad (7.101)$$

where we used $\dot{H} = -\varepsilon H^2$ (in the slow-roll approximation) in the last step. Thus

$$\frac{d}{d \ln k} = \frac{1}{1 - \varepsilon} \frac{1}{H} \frac{d}{dt} = \frac{1}{1 - \varepsilon} \frac{\dot{\phi}}{H} \frac{d}{d\phi} = -\frac{M^2}{1 - \varepsilon} \frac{V'}{V} \frac{d}{d\phi} \quad (7.102)$$

$$\approx \frac{1}{H} \frac{d}{dt} = -M^2 \frac{V'}{V} \frac{d}{d\phi}. \quad (7.103)$$

and

$$\frac{d \ln \mathcal{P}_R}{d \ln k} = \frac{1}{\mathcal{P}_R} \frac{d \mathcal{P}_R}{d \ln k} = \frac{\varepsilon}{V} \frac{d}{d \ln k} \left(\frac{V}{\varepsilon} \right) = \frac{1}{V} \frac{dV}{d \ln k} - \frac{1}{\varepsilon} \frac{d\varepsilon}{d \ln k} \quad (7.104)$$

$$\approx -M^2 \frac{V'}{V^2} \frac{dV}{d\phi} - \frac{1}{\varepsilon} H^{-1} \dot{\varepsilon} = -2\varepsilon - (-2\eta + 4\varepsilon) = -6\varepsilon + 2\eta. \quad (7.105)$$

7.7.2 Spectral index to second order in slow roll

If the 0th order power spectrum was enough to calculate its spectral index to 1st order, then the 1st order spectrum (7.97) should be enough to calculate the spectral index to 2nd order in slow-roll parameters.¹¹ From (7.97),

$$\frac{d \ln \mathcal{P}_R}{d \ln k} = \frac{d}{d \ln k} \ln \left[2^{2\nu-3} \left(\frac{\Gamma(\nu)}{\Gamma(\frac{3}{2})} \right)^2 (1 - \varepsilon)^{2\nu-1} \right] + \frac{d}{d \ln k} \ln \left[\left(\frac{H}{2\pi} \right)^2 \left(\frac{H}{\dot{\phi}} \right)^2 \right]. \quad (7.106)$$

In the first term we must account for the change of ν and ε as different scales k exit. In the calculation we need to derivate the Γ function. The digamma function is defined

$$\psi(x) \equiv \frac{d \ln \Gamma(x)}{dx}$$

and we will need the numerical value $\psi(\frac{3}{2}) = \psi(\frac{1}{2}) + 2 = -\gamma - 2 \ln 2 + 2$, where $\gamma = 0.577 215 664 901 \dots$ is the Euler–Mascheroni constant. Calculation gives (**exercise**)

$$\frac{d}{d \ln k} \ln \left[2^{2\nu-3} \left(\frac{\Gamma(\nu)}{\Gamma(\frac{3}{2})} \right)^2 (1 - \varepsilon)^{2\nu-1} \right] \approx (2 - \ln 2 - \gamma) (-16\varepsilon\eta + 24\varepsilon^2 + 2\xi) + 4\varepsilon\eta - 8\varepsilon^2, \quad (7.107)$$

¹⁰In the literature, especially when discussing observational results, it is common to define the spectral index of scalar perturbations as $n_s \equiv 1 + d \ln \mathcal{P} / d \ln k$, for historical reasons. So take care when comparing different sources.

¹¹This is because the spectral index is a derivative with respect to scale and because of the connection with horizon exit this is essentially a derivative with respect to time, and as we saw in Eq. (7.31) during slow roll these derivatives are typically one order higher in slow-roll parameters.

where $2 - \ln 2 - \gamma \approx 0.729637$.

For the second term, the calculation of Sec. 7.7.1 is not enough, since we want to calculate it now to second order in slow-roll parameters. In particular, we need a higher order result for $H^{-1}\dot{\epsilon}$, since although Eq. (7.31d) that we used was 2nd order, it was used as $(1/\epsilon)H^{-1}\dot{\epsilon}$, which is 1st order.

How do we calculate to higher order in slow-roll? The slow-roll equations of Sec. 7.4, which we may call first-order slow-roll equations are not enough. Go back to the exact background equations (7.22),

$$\ddot{\phi} + 3H\dot{\phi} + V' = 0 \quad \text{and} \quad H^2 = \frac{1}{3M^2} \left(\frac{1}{2}\dot{\phi}^2 + V \right). \quad (7.108)$$

To get the first-order slow-roll equations, we dropped the terms $\ddot{\phi}$ and $\frac{1}{2}\dot{\phi}^2$. To get second-order slow-roll equations, we do not drop these terms, but we replace them by their first-order slow-roll approximations from Eq. (7.31),

$$\ddot{\phi} \approx H\dot{\phi}(\epsilon - \eta) \quad \text{and} \quad \frac{1}{2}\dot{\phi}^2 \approx M^2 H^2 \epsilon \quad (7.109)$$

to arrive at the 2nd order slow-roll equations

$$(3 + \epsilon - \eta)H\dot{\phi} + V' = 0 \Rightarrow H\dot{\phi} = -\frac{V'}{3 + \epsilon - \eta} \quad (7.110)$$

$$H^2 = \frac{1}{3M^2} (M^2 H^2 \epsilon + V) \Rightarrow H^2 = \frac{V}{(3 - \epsilon)M^2},$$

from which we can derive various second-order slow-roll results (**exercise**):

$$\begin{aligned} \epsilon_H &\equiv -\frac{\dot{H}}{H^2} \approx \epsilon - \frac{4}{3}\epsilon^2 + \frac{2}{3}\epsilon\eta \\ (H^{-1}\dot{\phi})^2 &= 2M^2 \frac{(3 - \epsilon)^2}{(3 + \epsilon - \eta)^2} \epsilon \approx 2M^2 \left(1 - \frac{4}{3}\epsilon + \frac{2}{3}\eta\right) \epsilon \approx 2M^2 \epsilon_H \\ H^{-1}\dot{\epsilon} &\approx 4\epsilon^2 - 2\epsilon\eta - \frac{8}{3}\epsilon^3 + \frac{8}{3}\epsilon^2\eta - \frac{2}{3}\epsilon\eta^2 \\ H^{-1}\dot{\epsilon}_H &\approx \left(1 - \frac{8}{3}\epsilon + \frac{2}{3}\eta\right) H^{-1}\dot{\epsilon} + \frac{2}{3}\epsilon H^{-1}\dot{\eta} \\ &\approx \epsilon \left(4\epsilon - 2\eta - \frac{40}{3}\epsilon^2 + \frac{36}{3}\epsilon\eta - 2\eta^2 - \frac{2}{3}\xi\right) \end{aligned} \quad (7.111)$$

We can now attack the second term in Eq. (7.106):

$$\frac{d}{d \ln k} \ln \left[\left(\frac{H}{2\pi} \right)^2 \left(\frac{H}{\dot{\phi}} \right)^2 \right] = \frac{d}{d \ln k} \ln \left(\frac{H^2}{\epsilon_H} \right) = \frac{2}{H} \frac{dH}{d \ln k} - \frac{1}{\epsilon_H} \frac{d\epsilon_H}{d \ln k}. \quad (7.112)$$

Here

$$\frac{d \ln k}{dt} = \frac{d \ln(aH)}{dt} = \frac{\dot{a}}{a} + \frac{\dot{H}}{H} = H \left(1 + \frac{\dot{H}}{H^2} \right) = H(1 - \epsilon_H) \quad (7.113)$$

so that (7.112) becomes (**exercise**)

$$\begin{aligned} -\frac{2\epsilon_H}{1 - \epsilon_H} - \frac{1}{\epsilon_H(1 - \epsilon_H)} H^{-1}\dot{\epsilon}_H &\approx (-2\epsilon + \frac{2}{3}\epsilon^2 - \frac{4}{3}\epsilon\eta) - (4\epsilon - 2\eta - 4\epsilon^2 + \frac{14}{3}\epsilon\eta - \frac{2}{3}\eta^2 - \frac{2}{3}\xi) \\ &= -6\epsilon + 2\eta + \frac{14}{3}\epsilon^2 - 6\epsilon\eta + \frac{2}{3}\eta^2 + \frac{2}{3}\xi. \end{aligned} \quad (7.114)$$

Adding (7.107)+(7.114) we finally get

$$n_s = -6\epsilon + 2\eta - \frac{10}{3}\epsilon^2 - 2\epsilon\eta + \frac{2}{3}\eta^2 + \frac{2}{3}\xi + (2 - \ln 2 - \gamma)(24\epsilon^2 - 16\epsilon\eta + 2\xi). \quad (7.115)$$

According to the Planck 2018 results (assuming adiabatic scalar perturbations and negligible tensor perturbations), the observed value for the spectral index is [13]

$$n_s + 1 = 0.965 \pm 0.004 \quad \Rightarrow \quad n_s = -0.035 \pm 0.004. \quad (7.116)$$

We expect this to be dominated by the first-order contribution, and the order of magnitude of the second-order contribution would be expected to be the square of the first-order contribution, i.e., $\mathcal{O}(10^{-3})$. The exact numbers and the relations between the slow-roll parameters depend on the inflation model. For example, if we suppose for simplicity that $\eta = \xi = 0$, so that $n_s = -6\epsilon - \frac{10}{3}\epsilon^2 + (2 - \ln 2 - \gamma)24\epsilon^2 = -6\epsilon + 14.178\epsilon^2$, the value $n_s = -0.035$ would correspond to $\epsilon = 0.005916$, so that the 1st and 2nd order terms would be $n_s = -0.035496 + 0.000496$. In this case the second order contribution is clearly beyond the reach of Planck. On the other hand, in some inflation models there may be significant cancellation between the first order contributions -6ϵ and $+2\eta$, so that the slow-roll parameters would be larger and the second-order contribution could be more significant.

8 Curvature Perturbation

In the case of N scalar fields, the curvature perturbation is

$$\mathcal{R} \equiv -\psi^C = -\psi - \mathcal{H}\xi^0, \quad (8.1)$$

where, from Eq. (6.15)

$$\xi^0 = \frac{\sum \varphi'_I \delta\varphi_I}{\sum (\varphi'_I)^2} = \frac{\vec{\varphi}' \cdot \vec{\delta\varphi}}{\vec{\varphi}' \cdot \vec{\varphi}'}, \quad (8.2)$$

so that we have

$$\begin{aligned} \mathcal{R} &= -\psi - \mathcal{H} \frac{\sum \varphi'_I \delta\varphi_I}{\sum (\varphi'_I)^2} = -\mathcal{H} \frac{\sum \varphi'_I Q_I}{\sum (\varphi'_I)^2} = -\frac{\mathcal{H}}{(\rho + p)a^2} \sum \varphi'_I Q_I \\ &= 4\pi G \frac{\mathcal{H}}{\mathcal{H}' - \mathcal{H}^2} \sum \varphi'_I Q_I = 4\pi G \frac{\mathcal{H}}{\mathcal{H}' - \mathcal{H}^2} \vec{\varphi}' \cdot \vec{Q}. \end{aligned} \quad (8.3)$$

The derivation of the time evolution \mathcal{R}' is not as simple as in the single-field case ...¹²

9 Slow-Roll Inflation

The exact background equations for many-field inflation, (4.11), are

$$\begin{aligned} \ddot{\varphi}_I + 3H\dot{\varphi}_I &= -V_I \\ H^2 &= \frac{8\pi G}{3}\bar{\rho} = \frac{8\pi G}{3} \left[\frac{1}{2} \sum_I (\dot{\varphi}_I)^2 + V \right]. \end{aligned}$$

We assume that $|\ddot{\varphi}_I| \ll |3H\dot{\varphi}_I|$ and $\sum_I (\dot{\varphi}_I)^2 \ll V$; and make the slow-roll approximation, where (4.11) are replaced with the *slow-roll equations*

$$3H\dot{\varphi}_I + V_I = 0 \quad \Rightarrow \quad V_I = -3H\dot{\varphi}_I \quad \Rightarrow \quad \dot{\varphi}_I = -\frac{V_I}{3H} \quad (9.1)$$

for all fields φ_I , and

$$H^2 = \frac{1}{3M^2}V \quad \Rightarrow \quad V = 3M^2H^2 \quad \Rightarrow \quad 3H^2 = \frac{V}{M^2}. \quad (9.2)$$

¹²We give it, Eq. (10.49), for the case $N = 2$ in Sec. 10.5.

From these

$$H^{-1}\dot{\varphi}_I = \mathcal{H}^{-1}\varphi'_I = -M^2\frac{V_I}{V}, \quad (9.3)$$

or, written in vector form

$$H^{-1}\dot{\vec{\varphi}} = \mathcal{H}^{-1}\vec{\varphi}' = -\frac{M^2}{V}\nabla V. \quad (9.4)$$

That is, in the slow-roll solution, *the background field evolves down along the gradient of the potential.*

Derivating the slow-roll equations, we get (**exercise**)

$$\dot{H} = -\frac{\sum V_I^2}{6V} = -\frac{\nabla V \cdot \nabla V}{6V} \quad (9.5)$$

and

$$\ddot{\varphi}_I = \frac{M^2}{3}\frac{\sum V_{IJ}V_J}{V} - \frac{M^2}{6}\frac{V_I\sum V_J^2}{V^2}, \quad (9.6)$$

in vector form,

$$\ddot{\vec{\varphi}} = \frac{M^2}{3}\frac{\nabla\nabla V \cdot \nabla V}{V} - \frac{M^2}{6}\frac{(\nabla V)^2}{V^2}\nabla V. \quad (9.7)$$

In (9.6), the second term is parallel to $\dot{\vec{\varphi}}$, but the first term may not be.

9.1 Slow-Roll Parameters

We define the slow-roll parameters

$$\varepsilon_{IJ} \equiv \frac{M^2}{2}\frac{V_IV_J}{V^2} \quad \eta_{IJ} \equiv M^2\frac{V_{IJ}}{V} \quad (9.8)$$

and

$$\varepsilon \equiv \text{tr}[\varepsilon_{IJ}] = \sum \varepsilon_{II} = \frac{M^2}{2}\frac{(\nabla V)^2}{V^2}. \quad (9.9)$$

Note that the matrices ε_{IJ} and η_{IJ} are symmetric.

Using the slow-roll equations we obtain the following results (**exercise**)

$$H^{-2}\dot{H} = -\varepsilon \quad (9.10)$$

and

$$\begin{aligned} (H^{-1}\dot{\varphi}_I)^2 &= 2M^2\varepsilon_{II} \quad (\text{no sum}) \\ H^{-2}\ddot{\varphi}_I &= \varepsilon H^{-1}\dot{\varphi}_I - \sum_J \eta_{IJ}H^{-1}\dot{\varphi}_J \\ H^{-1}\dot{\varepsilon}_{IJ} &= 4\varepsilon\varepsilon_{IJ} - \sum_K (\varepsilon_{IK}\eta_{JK} + \eta_{IK}\varepsilon_{JK}) \\ H^{-1}\dot{\varepsilon} &= 4\varepsilon^2 - 2\sum_{IK} \varepsilon_{IK}\eta_{IK} \end{aligned} \quad (9.11)$$

(compare to (7.31)).

During inflation, the slow-roll parameters ε_{IJ} and η_{IJ} are typically small and their time variation is second-order small (in slow-roll parameters). We shall define the ξ parameters only after we have performed a rotation in field space into adiabatic and entropy field perturbations, and therefore we do not give $\dot{\eta}$ equations before (10.68).

9.2 Background Expansion Law

Because the equation (9.10) is the same as in single-field inflation, we obtain the same (see (7.41)–(7.44)) expansion law (but note that now ε is from (9.9)).

$$\frac{\mathcal{H}'}{\mathcal{H}^2} = 1 - \varepsilon \quad \Rightarrow \quad \mathcal{H} = \frac{-1}{(1 - \varepsilon)\eta} \quad \Rightarrow \quad a \propto (-\eta)^{-1/(1-\varepsilon)}, \quad (9.12)$$

and

$$\frac{a''}{a} = \mathcal{H}^2(2 - \varepsilon). \quad (9.13)$$

(These are valid for as long as ε can be approximated as constant.)

9.3 Evolution of Perturbations

From Eq. (6.20), the perturbation equation is

$$\mathcal{H}^{-2}Q_I'' + 2\mathcal{H}^{-1}Q_I' + \left(\frac{k}{\mathcal{H}}\right)^2 Q_I = \mathcal{H}^{-2} \sum_J \left[\frac{8\pi G}{a^2} \left(\frac{a^2}{\mathcal{H}}\varphi_I'\varphi_J'\right)' - a^2 V_{IJ} \right] Q_J, \quad (9.14)$$

or

$$H^{-2}\ddot{Q}_I + 3H^{-1}\dot{Q}_I + \left(\frac{k}{aH}\right)^2 Q_I = H^{-2} \sum_J \left[\frac{8\pi G}{a^3} \frac{d}{dt} \left(\frac{a^3}{H}\dot{\varphi}_I\dot{\varphi}_J\right) - V_{IJ} \right] Q_J. \quad (9.15)$$

Using the slow-roll equations, the rhs is (**exercise**)

$$\left[6\varepsilon_{IJ} - 3\eta_{IJ} + 6\varepsilon\varepsilon_{IJ} - \sum_K (2\varepsilon_{IK}\eta_{JK} + 2\eta_{IK}\varepsilon_{JK}) \right] Q_J \approx (6\varepsilon_{IJ} - 3\eta_{IJ}) Q_J \quad (9.16)$$

(compare to (7.34); we again calculate to 1st order in slow-roll parameters).

Defining

$$u_I \equiv aQ_I, \quad (9.17)$$

Eq. (9.14) becomes (compare to (7.38))

$$u_I'' + \left(k^2 - \frac{a''}{a}\right) u_I = \mathcal{H}^2 \sum_J (6\varepsilon_{IJ} - 3\eta_{IJ}) u_J. \quad (9.18)$$

Using (9.13), this becomes

$$u_I'' + k^2 u_I = \mathcal{H}^2(2 - \varepsilon)u_I + \mathcal{H}^2 \sum_J (6\varepsilon_{IJ} - 3\eta_{IJ}) u_J = \mathcal{H}^2 [(2 - \varepsilon)\delta_{IJ} + (6\varepsilon_{IJ} - 3\eta_{IJ})] u_J. \quad (9.19)$$

Using

$$\mathcal{H}^2 = \frac{1}{(1 - \varepsilon)^2(-\eta)^2} \approx \frac{1 + 2\varepsilon}{\eta^2} \quad (9.20)$$

our slow-roll perturbation evolution equation becomes (compare to (7.54))

$$u_I'' + \left(k^2 - \frac{2}{\eta^2}\right) u_I = \frac{3}{\eta^2} (\varepsilon\delta_{IJ} + 2\varepsilon_{IJ} - \eta_{IJ}) u_J \equiv \frac{3}{\eta^2} M_{IJ} u_J. \quad (9.21)$$

We work in the approximation where the slow-roll parameters, i.e., the matrix M_{IJ} , are assumed constant, which will be valid only for a limited time; our aim is to calculate what happens around horizon exit.

The off-diagonal components of the matrix M_{IJ} couple the evolution of the different field perturbations u_I . However, we can perform a *rotation in field space* to diagonalize M , since it is real and symmetric, to arrive at independently evolving field perturbations v_J . Let us call the rotation matrix U_{IJ} , so that

$$u_I = \sum U_{IJ} v_J \quad \text{and} \quad U^T M U = \text{diag}(\lambda_1, \dots, \lambda_N), \quad (9.22)$$

where the λ_I are the eigenvalues of the matrix M . They are 1st order in slow-roll parameters. Since U is a rotation matrix,

$$U^T U = 1 \quad \text{or} \quad \sum_I U_{IJ} U_{IK} = \delta_{JK}. \quad (9.23)$$

Eq. (9.21) becomes

$$\begin{aligned} v_J'' + \left(k^2 - \frac{2}{\eta^2}\right) v_J &= \frac{3}{\eta^2} \lambda_J v_J \\ \Rightarrow v_J'' + \left[k^2 - \frac{1}{\eta^2}(\nu_J^2 - \frac{1}{4})\right] v_J &= 0, \end{aligned} \quad (9.24)$$

where

$$\nu_J \equiv \sqrt{\frac{9}{4} + 3\lambda_J} \approx \frac{3}{2} + \lambda_J. \quad (9.25)$$

This is the same equation as (7.55), so we have already solved it. The mode functions are

$$\begin{aligned} w_J(\eta) &= L^{-3/2} \sqrt{\frac{\pi}{4}} a^{-1} \sqrt{-\eta} H_{\nu_J}(-k\eta) \\ &\rightarrow L^{-3/2} 2^{\nu_J-3/2} \frac{\Gamma(\nu_J)}{\Gamma(\frac{3}{2})} \frac{1}{a} \frac{1}{\sqrt{2k}} (-k\eta)^{-1-\lambda_J}. \end{aligned} \quad (9.26)$$

Our approximation that slow-roll parameters stay constant will only hold for a number of e -foldings $\ll 1/(\text{slow-roll parameters})$. Thus we want to switch to other variables after horizon exit. One of these will be the comoving curvature perturbation \mathcal{R} . From (8.3), (9.12), and (9.4),

$$\mathcal{R} = 4\pi G \frac{\mathcal{H}}{\mathcal{H}' - \mathcal{H}^2} \vec{\varphi}' \cdot \vec{Q} \approx -\frac{1}{2M^2 \epsilon} \mathcal{H}^{-1} \vec{\varphi}' \cdot \vec{Q} \approx \frac{1}{2\epsilon V} \nabla V \cdot \vec{Q}. \quad (9.27)$$

Thus \mathcal{R} is given by the perturbation component that is in the direction of the background field evolution $\vec{\varphi}'$, which is, in the slow-roll approximation, in the direction of the potential gradient. This is, in general, not in the direction of any of the eigenvectors of the matrix M , so we need to perform *another rotation in field space*.

We rotate from the original field space coordinate basis, where

$$\vec{Q} = \begin{pmatrix} Q_1 \\ Q_2 \\ \cdot \\ \cdot \\ Q_N \end{pmatrix} \quad (9.28)$$

to a new basis for the field perturbations, so that

$$\vec{Q} = \begin{pmatrix} Q_\sigma \\ Q_{s1} \\ \cdot \\ \cdot \\ Q_{s(N-1)} \end{pmatrix} = S^T \begin{pmatrix} Q_1 \\ Q_2 \\ \cdot \\ \cdot \\ Q_N \end{pmatrix}, \quad (9.29)$$

where

$$Q_\sigma \equiv \frac{\vec{\varphi}' \cdot \vec{Q}}{|\vec{\varphi}'|} \quad (9.30)$$

is the component in direction of the background field evolution. The rotation matrix S will, in general, be a function of time, since $\vec{\varphi}'$ may change direction along the background trajectory (the background trajectory in field space may be curved.) We call Q_σ the *adiabatic field perturbation*, and the $N - 1$ orthogonal perturbations Q_{sI} *entropy field perturbations*.

Since the rotation matrix is not constant over the field space, we do not rotate the values of the background fields, but we rotate their time derivatives, defining $\dot{\sigma}$ and \dot{s}_I by

$$\begin{pmatrix} \dot{\sigma} \\ \dot{s}_1 \\ \cdot \\ \cdot \\ \dot{s}_{N-1} \end{pmatrix} \equiv S^T \begin{pmatrix} \dot{\varphi}_1 \\ \dot{\varphi}_2 \\ \cdot \\ \cdot \\ \dot{\varphi}_N \end{pmatrix}. \quad (9.31)$$

That is, the rotation S is not a global rotation of coordinates of the field space, but a local rotation of the vector basis for the field perturbation and time derivative vectors. Note that $\dot{s}_I = 0$ by definition (that was the purpose of the rotation). See Sec. 10 for how σ and s can be defined as coordinates in the field space.

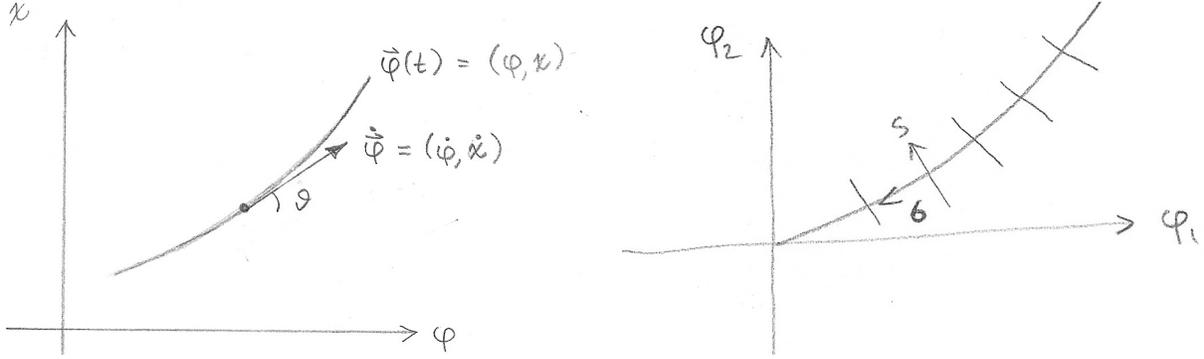


Figure 3: Left: Background trajectory in two-field inflation. Note: In the text we call the two fields φ_1 and φ_2 , but in these figures they are denoted φ and χ . Right: The σ, s coordinate system. (These two graphs correspond to opposite directions of the trajectory.)

10 Two Fields

10.1 Adiabatic and Entropy Field Coordinates

Consider now the case of two fields, $\vec{\varphi} = (\varphi_1, \varphi_2)$. The direction of the background solution is given by $\dot{\vec{\varphi}}$. Its direction angle θ is given by

$$\tan \theta \equiv \frac{\dot{\varphi}_2}{\dot{\varphi}_1}. \quad (10.1)$$

We now define σ , called the *adiabatic field coordinate*, as the integrated path length along the *trajectory* (\equiv *background solution*) from some arbitrary starting point (only changes in σ matter), and s , called the *entropy field coordinate*, as the orthogonal distance from the trajectory. See Fig. 3. Thus for the background solution

$$s = \dot{s} = \ddot{s} = 0 \quad (10.2)$$

by definition.

This defines a coordinate system σ, s in field space, specific to a particular background solution. If the trajectory is curved, this is a curved coordinate system, and is valid only in the vicinity of the trajectory, since further out the s coordinate lines cross. We shall use this coordinate system on the trajectory only, and do not introduce the full machinery of curved coordinate systems; but it is important to keep this in mind to avoid mistakes.

Consider now the background solution $\varphi_1 \equiv \bar{\varphi}_1(t)$, $\varphi_2 \equiv \bar{\varphi}_2(t)$ in terms of the new variables σ, s, θ . The new coordinates σ, s are given by a rotation by θ from the old coordinates φ_1, φ_2 , so that

$$\begin{pmatrix} \dot{\sigma} \\ \dot{s} \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \dot{\varphi}_1 \\ \dot{\varphi}_2 \end{pmatrix} = S^T \begin{pmatrix} \dot{\varphi}_1 \\ \dot{\varphi}_2 \end{pmatrix}, \quad (10.3)$$

where

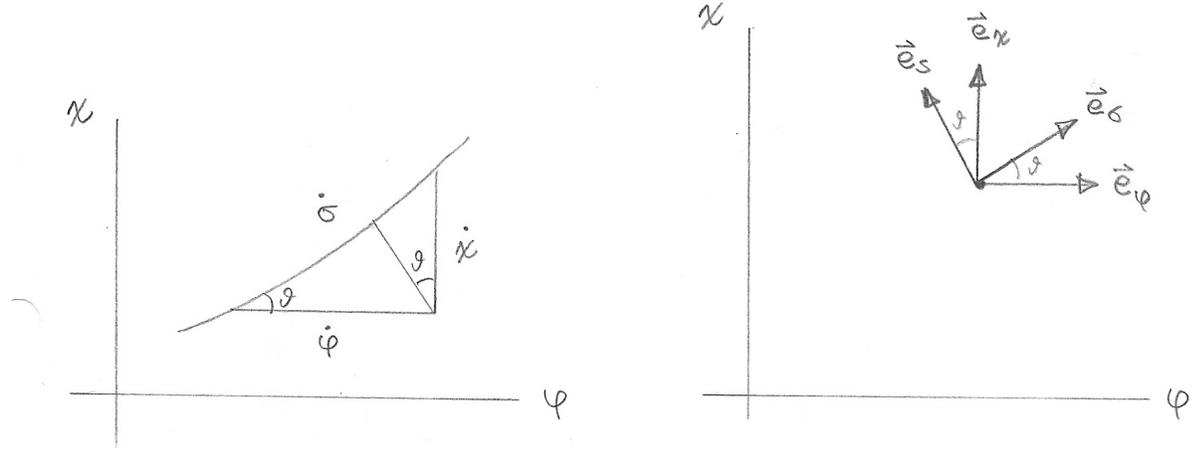
$$S \equiv \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}. \quad (10.4)$$

Thus

$$\dot{\sigma} = \dot{\varphi}_1 \cos \theta + \dot{\varphi}_2 \sin \theta. \quad (10.5)$$

The inverse rotation is

$$\begin{pmatrix} \dot{\varphi}_1 \\ \dot{\varphi}_2 \end{pmatrix} = S \begin{pmatrix} \dot{\sigma} \\ \dot{s} \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \dot{\sigma} \\ \dot{s} \end{pmatrix}. \quad (10.6)$$

Figure 4: Left: $\dot{\sigma}$. Right: The φ_1, φ_2 (φ, χ) basis and the σ, s basis.

Since $\dot{s} = 0$, we have

$$\dot{\varphi}_1 = \dot{\sigma} \cos \theta \quad \text{and} \quad \dot{\varphi}_2 = \dot{\sigma} \sin \theta \quad \Rightarrow \quad \dot{\sigma}^2 = \dot{\varphi}_1^2 + \dot{\varphi}_2^2 \quad (10.7)$$

and (same as (10.1))

$$\dot{\varphi}_1 \sin \theta = \dot{\varphi}_2 \cos \theta. \quad (10.8)$$

The second derivatives are

$$\ddot{\varphi}_1 = \ddot{\sigma} \cos \theta - \dot{\sigma} \dot{\theta} \sin \theta \quad \text{and} \quad \ddot{\varphi}_2 = \ddot{\sigma} \sin \theta + \dot{\sigma} \dot{\theta} \cos \theta. \quad (10.9)$$

The potential $V(\varphi_1, \varphi_2)$ exists everywhere in the (φ_1, φ_2) field space. We can express its gradient either in the φ_1, φ_2 basis or in the σ, s basis (see Fig. 4.) We have

$$\begin{pmatrix} V_\sigma \\ V_s \end{pmatrix} = S^T \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \quad (10.10)$$

and

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = S \begin{pmatrix} V_\sigma \\ V_s \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} V_\sigma \\ V_s \end{pmatrix}. \quad (10.11)$$

The time derivative of V along the trajectory is

$$\dot{V} = V_1 \dot{\varphi}_1 + V_2 \dot{\varphi}_2 = V_\sigma \dot{\sigma}. \quad (10.12)$$

We can similarly rotate the second derivatives V_{IJ} :

$$[V_{I',J'}] \equiv S^T [V_{IJ}] S \quad \text{or} \quad V_{I',J'} \equiv S_{I'}^K S_{J'}^L V_{KL} \quad (10.13)$$

or

$$\begin{pmatrix} V_{\sigma\sigma} & V_{\sigma s} \\ V_{\sigma s} & V_{ss} \end{pmatrix} \equiv \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} V_{11} & V_{12} \\ V_{12} & V_{22} \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \quad (10.14)$$

giving

$$V_{\sigma\sigma} \equiv \cos^2 \theta V_{11} + 2 \cos \theta \sin \theta V_{12} + \sin^2 \theta V_{22} \quad (10.15)$$

$$V_{ss} \equiv \sin^2 \theta V_{11} - 2 \cos \theta \sin \theta V_{12} + \cos^2 \theta V_{22}$$

$$V_{\sigma s} \equiv -\sin \theta \cos \theta V_{11} + (\cos^2 \theta - \sin^2 \theta) V_{12} + \cos \theta \sin \theta V_{22}.$$

We take these equations as the definitions of $V_{\sigma\sigma}$, V_{ss} , and $V_{\sigma s}$. Since they were obtained by rotation, this means that they are *not* partial derivatives, but instead they are *covariant derivatives* in the σ, s coordinates.

Likewise, third derivatives are rotated by

$$V_{I'J'K'} \equiv S_{I'}^L S_{J'}^M S_{K'}^N V_{LMN}, \quad (10.16)$$

giving (**exercise**)

$$\begin{aligned} V_{\sigma\sigma\sigma} &\equiv \cos^3 \theta V_{111} + 3 \cos^2 \theta \sin \theta V_{112} + 3 \cos \theta \sin^2 \theta V_{122} + \sin^3 \theta V_{222} \\ V_{\sigma\sigma s} &\equiv -\cos^2 \theta \sin \theta V_{111} + (\cos^3 \theta - 2 \cos \theta \sin^2 \theta) V_{112} + (2 \cos^2 \theta \sin \theta - \sin^3 \theta) V_{122} + \sin^2 \theta \cos \theta V_{222} \\ V_{\sigma s s} &\equiv \cos \theta \sin^2 \theta V_{111} + (-2 \cos^2 \theta \sin \theta + \sin^3 \theta) V_{112} + (\cos^3 \theta - 2 \sin^2 \theta \cos \theta) V_{122} + \sin \theta \cos^2 \theta V_{222} \\ V_{s s s} &\equiv -\sin^3 \theta V_{111} + 3 \sin^2 \theta \cos \theta V_{112} - 3 \sin \theta \cos^2 \theta V_{122} + \cos^3 \theta V_{222} \end{aligned} \quad (10.17)$$

Derivating Eq. (10.5) again,

$$\ddot{\sigma} = \ddot{\varphi}_1 \cos \theta - \dot{\varphi}_1 \dot{\theta} \sin \theta + \ddot{\varphi}_2 \sin \theta + \dot{\varphi}_2 \dot{\theta} \cos \theta = \ddot{\varphi}_1 \cos \theta + \ddot{\varphi}_2 \sin \theta, \quad (10.18)$$

where we used Eq. (10.8).

10.2 Exact Background Solution

Multiplying the φ_I background field equations

$$\ddot{\varphi}_1 + 3H\dot{\varphi}_1 + V_1 = 0 \quad (10.19)$$

$$\ddot{\varphi}_2 + 3H\dot{\varphi}_2 + V_2 = 0, \quad (10.20)$$

with $\cos \theta$ and $\sin \theta$ and using (10.9) and (10.18) we get the background field equation for the adiabatic field coordinate:

$$\ddot{\sigma} + 3H\dot{\sigma} + V_\sigma = 0. \quad (10.21)$$

Multiplying with $-\sin \theta$ and $\cos \theta$ we get (**exercise**)

$$V_s = -\dot{\theta}\dot{\sigma} \quad \text{or} \quad \dot{\theta} = -\frac{V_s}{\dot{\sigma}}. \quad (10.22)$$

The corresponding equation for the background entropy field coordinate was trivial, $\ddot{s} = 0$. Instead, the role of the other dynamical quantity for the background is taken by θ . Derivating Eq. (10.22a) we get the $\ddot{\theta}$ equation. Here one has to be careful with using the σ, s basis, since it is changing along the trajectory. Thus $\dot{V}_s \neq V_{ss}\dot{s} + V_{\sigma s}\dot{\sigma} = V_{\sigma s}\dot{\sigma}$. To avoid working in a curved coordinate system, one can go back to the original Cartesian φ_1, φ_2 basis for the calculation: write $V_s = -V_1 \sin \theta + V_2 \cos \theta$, derivate this, and pick the σ, s quantities from the result. This gives (**exercise**)

$$\dot{V}_s = V_{\sigma s}\dot{\sigma} - \dot{\theta}V_\sigma \quad (10.23)$$

and we can now derivate (10.22) using (10.23) and (10.21) to get (**exercise**)

$$\ddot{\theta} - 3H\dot{\theta} + V_{\sigma s} - 2\frac{V_\sigma}{\dot{\sigma}}\dot{\theta} = 0. \quad (10.24)$$

If the background trajectory is straight ($\dot{\theta} = 0$), then $V_s = V_{\sigma s} = 0$ along the trajectory.

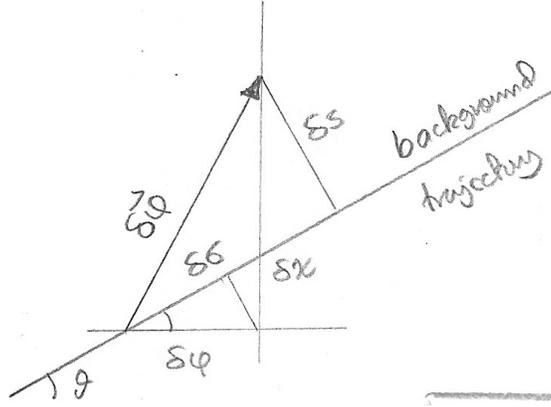


Figure 5: The field perturbation $\delta\vec{\varphi}$ divided into the $\delta\sigma$ and δs components.

10.3 Perturbations

(Note that in Secs. 10.3–10.5 we work with the exact background equations, not the slow-roll equations.)

The field perturbations are rotated likewise into the *adiabatic field perturbation* $\delta\sigma$ and the *entropy field perturbation* δs (see Fig. 5). Thus

$$\begin{pmatrix} \delta\sigma \\ \delta s \end{pmatrix} = S^T \begin{pmatrix} \delta\varphi_1 \\ \delta\varphi_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \delta\varphi_1 \\ \delta\varphi_2 \end{pmatrix} \quad (10.25)$$

and

$$\begin{pmatrix} \delta\varphi_1 \\ \delta\varphi_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \delta\sigma \\ \delta s \end{pmatrix}. \quad (10.26)$$

The perturbation in the potential is

$$\delta V = V_1\delta\varphi_1 + V_2\delta\varphi_2 = V_\sigma\delta\sigma + V_s\delta s. \quad (10.27)$$

Since the rotation angle θ changes with time, we do *not* get the time derivatives $\dot{\delta\sigma}$, $\ddot{\delta\sigma}$, $\dot{\delta s}$, and $\ddot{\delta s}$ by rotating $\dot{\delta\varphi}_I$ and $\ddot{\delta\varphi}_I$. Instead, we must derivate (10.25), giving

$$\begin{aligned} \dot{\delta\sigma} &= \cos\theta(\dot{\delta\varphi}_1 + \dot{\theta}\delta\varphi_2) + \sin\theta(\dot{\delta\varphi}_2 - \dot{\theta}\delta\varphi_1) \\ \dot{\delta s} &= -\sin\theta(\dot{\delta\varphi}_1 + \dot{\theta}\delta\varphi_2) + \cos\theta(\dot{\delta\varphi}_2 - \dot{\theta}\delta\varphi_1) \\ \ddot{\delta\sigma} &= \cos\theta\ddot{\delta\varphi}_1 - 2\dot{\theta}\sin\theta\dot{\delta\varphi}_1 - \left(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta\right)\delta\varphi_1 \\ &\quad + \sin\theta\ddot{\delta\varphi}_2 + 2\dot{\theta}\cos\theta\dot{\delta\varphi}_2 + \left(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta\right)\delta\varphi_2 \\ \ddot{\delta s} &= -\sin\theta\ddot{\delta\varphi}_1 - 2\dot{\theta}\cos\theta\dot{\delta\varphi}_1 - \left(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta\right)\delta\varphi_1 \\ &\quad + \cos\theta\ddot{\delta\varphi}_2 - 2\dot{\theta}\sin\theta\dot{\delta\varphi}_2 - \left(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta\right)\delta\varphi_2. \end{aligned} \quad (10.28)$$

Or, derivating (10.26) one gets $\delta\dot{\varphi}_1, \delta\ddot{\varphi}_1, \delta\dot{\varphi}_2, \delta\ddot{\varphi}_2$ in terms of $\delta\dot{\sigma}, \delta\ddot{\sigma}, \delta\dot{s},$ and $\delta\ddot{s},$

$$\begin{aligned}
\delta\dot{\varphi}_1 &= \cos\theta(\delta\dot{\sigma} - \dot{\theta}\delta s) - \sin\theta(\delta\dot{s} + \dot{\theta}\delta\sigma) \\
\delta\dot{\varphi}_2 &= \sin\theta(\delta\dot{\sigma} - \dot{\theta}\delta s) + \cos\theta(\delta\dot{s} + \dot{\theta}\delta\sigma) \\
\delta\ddot{\varphi}_1 &= \cos\theta\delta\ddot{\sigma} - 2\dot{\theta}\sin\theta\delta\dot{\sigma} - \left(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta\right)\delta\sigma \\
&\quad - \sin\theta\delta\ddot{s} - 2\dot{\theta}\cos\theta\delta\dot{s} - \left(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta\right)\delta s \\
\delta\ddot{\varphi}_2 &= \sin\theta\delta\ddot{\sigma} + 2\dot{\theta}\cos\theta\delta\dot{\sigma} + \left(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta\right)\delta\sigma \\
&\quad + \cos\theta\delta\ddot{s} - 2\dot{\theta}\sin\theta\delta\dot{s} - \left(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta\right)\delta s.
\end{aligned} \tag{10.29}$$

Actually, it is not necessary to derivate (10.26); one gets (10.29) simply by changing the sign of θ in (10.28), since these correspond to opposite rotations. One can write (10.29) as

$$\begin{aligned}
\begin{pmatrix} \delta\dot{\varphi}_1 \\ \delta\dot{\varphi}_2 \end{pmatrix} &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \delta\dot{\sigma} - \dot{\theta}\delta s \\ \delta\dot{s} + \dot{\theta}\delta\sigma \end{pmatrix} \\
\begin{pmatrix} \delta\ddot{\varphi}_1 \\ \delta\ddot{\varphi}_2 \end{pmatrix} &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \delta\ddot{\sigma} - \dot{\theta}^2\delta\sigma - 2\dot{\theta}\delta\dot{s} - \ddot{\theta}\delta s \\ \delta\ddot{s} - \dot{\theta}^2\delta s + 2\dot{\theta}\delta\dot{\sigma} + \ddot{\theta}\delta\sigma \end{pmatrix},
\end{aligned} \tag{10.30}$$

from which one can identify the σ and s components of $\delta\vec{\varphi}$ and $\delta\ddot{\varphi}$.

Multiplying by the background $\dot{\varphi}_1$ and $\dot{\varphi}_2$ we get

$$\dot{\varphi}_1\delta\dot{\varphi}_1 + \dot{\varphi}_2\delta\dot{\varphi}_2 = \dot{\sigma}(\delta\dot{\sigma} - \dot{\theta}\delta s) = \dot{\sigma}\delta\dot{\sigma} + V_s\delta s. \tag{10.31}$$

The perturbation in the potential is

$$\delta V = V_1\delta\varphi_1 + V_2\delta\varphi_2 = V_\sigma\delta\sigma + V_s\delta s. \tag{10.32}$$

The field perturbation equation (5.8) written in vector form and using ordinary time is

$$\delta\ddot{\vec{\varphi}} + 3H\delta\dot{\vec{\varphi}} - \frac{1}{a^2}\nabla_x^2\delta\vec{\varphi} + \nabla_\varphi\nabla_\varphi V \cdot \delta\vec{\varphi} = -2\nabla_\varphi V A + \ddot{\varphi}(\dot{A} + 3\dot{D} + \frac{1}{a}\nabla_x^2 B), \tag{10.33}$$

where ∇_x denotes gradient in coordinate space and ∇_φ gradient in field space (components of $\nabla_\varphi\nabla_\varphi V$ are V_{IJ}). Picking the σ and s components of this vector equation (use 10.30) and using background relations, one finds (**exercise**) the field equations for the adiabatic and entropy field perturbations

$$\begin{aligned}
\delta\ddot{\sigma} + 3H\delta\dot{\sigma} + \left(-\frac{1}{a^2}\nabla^2 + V_{\sigma\sigma} - \dot{\theta}^2\right)\delta\sigma &= -2V_\sigma A + \dot{\sigma}\left(\dot{A} + 3\dot{D} + \frac{1}{a}\nabla^2 B\right) + 2\frac{d}{dt}(\dot{\theta}\delta s) - 2\frac{V_\sigma}{\dot{\sigma}}\dot{\theta}\delta s \\
\delta\ddot{s} + 3H\delta\dot{s} + \left(-\frac{1}{a^2}\nabla^2 + V_{ss} - \dot{\theta}^2\right)\delta s &= -2\frac{\dot{\theta}}{\dot{\sigma}}\left[\dot{\sigma}(\delta\dot{\sigma} - \dot{\sigma}A) - \ddot{\sigma}\delta\sigma\right].
\end{aligned} \tag{10.34}$$

We note that, *if the background trajectory is straight* ($\dot{\theta} = 0$), the adiabatic and entropy perturbations decouple and the $\delta\sigma$ equation is then exactly the earlier single-field equation. So is the δs equation, except that, since $\dot{s} = 0$ and $V_s = -\dot{\theta}\dot{\sigma} = 0$ (we must lie at the bottom of the potential in the s direction for there to be know sideways force, which would curve the trajectory), the rhs, which contains the effect of the metric perturbations, disappears. We also get, when $\dot{\theta} = 0$, that only $\delta\sigma$ contributes to $\delta\rho, \delta p, \delta q$, and thus to Einstein equations and the metric perturbations. And, as we just saw, δs does not then care about the metric perturbations. Thus, indeed, $\delta\sigma$ and δs are then completely decoupled; and $\sigma, \delta\sigma$ behave exactly as the single-field case (to 1st order in perturbation theory).

What are the new features brought by a curved trajectory? On the lhs we note “angular momentum” terms $-\dot{\theta}^2\delta\sigma$ and $-\dot{\theta}^2\delta s$. The rhs now couple δs and $\delta\sigma$ to each other, so that as the trajectory curves, an adiabatic field perturbation can be converted (partially) into an entropy field perturbation and vice versa. The rhs of the δs equation is proportional to the single-field total entropy perturbation, which is proportional to $\delta\rho^C$ (see (7.19) and (10.41)).

10.4 Field Equations for the Adiabatic and Entropy Field Perturbations in the Spatially Flat Gauge

Redo now Sec. 6 to eliminate the metric perturbations from the field perturbation equations.

The energy density, pressure, and “momentum” (define here δq , which appears in δT_i^0) perturbations are

$$\begin{aligned}\delta\rho &= \dot{\varphi}_1\delta\dot{\varphi}_1 + \dot{\varphi}_2\delta\dot{\varphi}_2 - (\dot{\varphi}_1^2 + \dot{\varphi}_2^2)A + \delta V = \dot{\sigma}(\dot{\delta\sigma} - \dot{\theta}\delta s) - \dot{\sigma}^2 A + \delta V \\ \delta p &= \dot{\varphi}_1\delta\dot{\varphi}_1 + \dot{\varphi}_2\delta\dot{\varphi}_2 - (\dot{\varphi}_1^2 + \dot{\varphi}_2^2)A - \delta V = \dot{\sigma}(\dot{\delta\sigma} - \dot{\theta}\delta s) - \dot{\sigma}^2 A - \delta V \\ \delta q &= \dot{\varphi}_1\delta\varphi_1 + \dot{\varphi}_2\delta\varphi_2 = \dot{\sigma}\delta\sigma.\end{aligned}\tag{10.35}$$

The last equation we see directly using vector notation, $\delta q = \dot{\vec{\varphi}} \cdot \delta\vec{\varphi} = \dot{\sigma}\delta\sigma$. This does not work the same for $\dot{\vec{\varphi}} \cdot \dot{\delta\vec{\varphi}}$, since $\dot{\delta\sigma}$ and $\dot{\delta s}$ are not the (σ, s) -base components of $\dot{\delta\vec{\varphi}}$, due to $\dot{\theta} \neq 0$.

We can now write the Einstein equations for the perturbations. We shall need the second Einstein equation

$$\dot{\psi} + HA = 4\pi G a \delta q = 4\pi G \dot{\sigma} \delta\sigma.\tag{10.36}$$

We are so far working in general (arbitrary) gauge. The first Einstein equation we shall not need in general gauge, but later we shall use its gauge-invariant form

$$\nabla^2\Phi = 4\pi G a^2 \delta\rho^C,\tag{10.37}$$

where we already used $\Psi = \Phi$, which holds for scalar field sources.

We can give the gauge transformations either in terms of the ordinary time coordinate, $\tilde{t} = t + \xi_{(t)}^0$, or the conformal time coordinate, $\tilde{\eta} = \eta + \xi_{(\eta)}^0$. These coordinate shifts are related by $\xi_{(t)}^0 = a\xi_{(\eta)}^0$. Below my ξ^0 is $\xi_{(t)}^0$.

The comoving density perturbation is

$$\delta\rho^C = \delta\rho - \dot{\rho}\xi^0,\tag{10.38}$$

where

$$\xi^0 = \frac{\dot{\varphi}_1\delta\varphi_1 + \dot{\varphi}_2\delta\varphi_2}{\dot{\varphi}_1^2 + \dot{\varphi}_2^2} = \frac{\dot{\sigma}\delta\sigma}{\dot{\sigma}^2} = \frac{\delta\sigma}{\dot{\sigma}}\tag{10.39}$$

and

$$\dot{\rho} = \frac{d}{dt} \left(\frac{1}{2}\dot{\varphi}_1^2 + \frac{1}{2}\dot{\varphi}_2^2 + V \right) = \dot{\varphi}_1\ddot{\varphi}_1 + \dot{\varphi}_2\ddot{\varphi}_2 + \dot{V} = \dot{\sigma}\ddot{\sigma} + V_\sigma\dot{\sigma},\tag{10.40}$$

so that

$$\delta\rho^C = \dot{\sigma}\delta\dot{\sigma} - \ddot{\sigma}\delta\sigma - \dot{\sigma}^2 A + 2V_\sigma\delta s.\tag{10.41}$$

Scalar fields gauge transform as

$$\begin{aligned}\delta\tilde{\varphi}_1 &= \delta\varphi_1 - \dot{\varphi}_1\xi^0 \\ \delta\tilde{\varphi}_2 &= \delta\varphi_2 - \dot{\varphi}_2\xi^0,\end{aligned}\tag{10.42}$$

from which we get

$$\begin{aligned}\delta\tilde{\sigma} &= \delta\sigma - \dot{\sigma}\xi^0 \\ \delta\tilde{s} &= \delta s - \dot{s}\xi^0 = \delta s,\end{aligned}\tag{10.43}$$

i.e., δs is gauge invariant.

In the spatially flat gauge the $\delta\sigma$ evolution equation becomes (**exercise**) (Hint: Equations $\dot{H} = -4\pi G\dot{\sigma}^2$, $V_\sigma = -\ddot{\sigma} - 3H\dot{\sigma}$, and $V_s = -\dot{\theta}\dot{\sigma}$ may be useful.)

$$\delta\ddot{\sigma}_Q + 3H\delta\dot{\sigma}_Q + \left[-\frac{1}{a^2}\nabla^2 + V_{\sigma\sigma} - \dot{\theta}^2 - \frac{8\pi G}{a^3} \frac{d}{dt} \left(\frac{a^3}{H} \dot{\sigma}^2 \right) \right] \delta\sigma_Q = 2 \frac{d}{dt} (\dot{\theta}\delta s) - 2 \left(\frac{V_\sigma}{\dot{\sigma}} + \frac{\dot{H}}{H} \right) \dot{\theta}\delta s. \quad (10.44)$$

The δs equation can be written as (**exercise**)

$$\ddot{\delta s} + 3H\dot{\delta s} + \left(-\frac{1}{a^2}\nabla^2 + V_{ss} + 3\dot{\theta}^2 \right) \delta s = -\frac{\dot{\theta}}{\dot{\sigma}} \frac{1}{2\pi G} \frac{1}{a^2} \nabla^2 \Phi, \quad (10.45)$$

which is fully gauge invariant. The source term on its right-hand side is small for superhorizon scales. The effective mass μ_s of the entropy field is given by $\mu_s^2 \equiv V_{ss} + 3\dot{\theta}^2$.

From (10.37) etc.,

$$\nabla^2 \Phi = 4\pi G a^2 \left[\dot{\sigma}\delta\dot{\sigma}_Q + \left(-\ddot{\sigma} + \frac{\dot{H}}{H}\dot{\sigma} \right) \delta\sigma_Q + 2V_s\delta s \right], \quad (10.46)$$

so that in the spatially flat gauge (10.45) can be written (**check this!**)

$$\ddot{\delta s} + 3H\dot{\delta s} + \left(-\frac{1}{a^2}\nabla^2 + V_{ss} - \dot{\theta}^2 \right) \delta s = -2 \frac{d}{dt} (\dot{\theta}\delta\sigma_Q) + 2 \left(\frac{V_\sigma}{\dot{\sigma}} + \frac{\dot{H}}{H} \right) \dot{\theta}\delta\sigma_Q + 2V_{\sigma s}\delta\sigma_Q. \quad (10.47)$$

From hereon we work in the spatially flat gauge and drop the gauge label Q .

We see that for a straight trajectory $\dot{\theta} = 0$, the $\delta\sigma$ equation is just the equation for the single-field case (7.33), but for a curved trajectory, the entropy perturbation δs acts as a source term. The rhs in (10.45) is small for superhorizon scales. This means that if there is no initial δs at superhorizon scales, no (significant) δs is generated while the scales are superhorizon. Thus adiabatic perturbations remain adiabatic at superhorizon scales.

10.5 Curvature and Entropy Perturbations

The comoving curvature perturbation is, from (8.3),

$$\mathcal{R} = -H \frac{\dot{\varphi}_1 \delta\varphi_1 + \dot{\varphi}_2 \delta\varphi_2}{\dot{\varphi}_1^2 + \dot{\varphi}_2^2} = -H \frac{\delta\sigma}{\dot{\sigma}}. \quad (10.48)$$

We obtain for it (**exercise**) an evolution equation

$$\dot{\mathcal{R}} = \frac{H}{\dot{H}} \frac{1}{a^2} \nabla^2 \Phi - 2H \frac{\dot{\theta}}{\dot{\sigma}} \delta s. \quad (10.49)$$

Thus \mathcal{R} stays constant at superhorizon scales, if there are no entropy field perturbations, but δs acts as a source term.

We define an analogous entropy perturbation

$$S \equiv H \frac{\delta s}{\dot{\sigma}}. \quad (10.50)$$

Thus, for superhorizon scales,

$$\dot{\mathcal{R}} = -2\dot{\theta}S \quad (k \ll \mathcal{H}). \quad (10.51)$$

10.6 Slow-Roll Approximation

From Sec. 9 we have that

$$H^2 = \frac{V}{3M^2} \quad (10.52)$$

and

$$H^{-1}\dot{\varphi}_I = -M^2 \frac{V_I}{V}, \quad (10.53)$$

Rotating gives

$$H^{-1}\dot{\sigma} = -M^2 \frac{V_\sigma}{V} \quad (10.54)$$

$$H^{-1}\dot{s} = -M^2 \frac{V_s}{V} = 0 \quad (10.55)$$

so

$$V_s = 0. \quad (10.56)$$

on the slow-roll trajectory. This is just the statement that the trajectory is always in the direction of ∇V , so that there is no sideways component to ∇V . (One might now conclude, from Eq. (10.22), that $\dot{\theta} = 0$, but this would be a mistake—we must now use consistently the slow-roll equations, and not mix them with the exact equations. See below for the slow-roll equation for $\dot{\theta}$.)

This means that

$$\varepsilon \equiv \varepsilon_{11} + \varepsilon_{22} = \frac{M^2 (\nabla V)^2}{2 V^2} = \frac{M^2 V_\sigma^2}{2 V^2} \quad (10.57)$$

and we do not need to define $\varepsilon_{\sigma\sigma}$, ε_{ss} or a separate $\varepsilon_{\sigma\sigma}$. Instead we have the relations

$$\begin{aligned} \varepsilon_{11} &= \varepsilon \cos^2 \theta \\ \varepsilon_{22} &= \varepsilon \sin^2 \theta \\ \varepsilon_{12} &= \varepsilon \cos \theta \sin \theta. \end{aligned} \quad (10.58)$$

But we do define

$$\eta_{\sigma\sigma} \equiv M^2 \frac{V_{\sigma\sigma}}{V} \quad \eta_{\sigma s} \equiv M^2 \frac{V_{\sigma s}}{V} \quad \eta_{ss} \equiv M^2 \frac{V_{ss}}{V} \quad (10.59)$$

that we obtain from η_{IJ} by the same rotation as in (10.15):

$$\begin{aligned} \eta_{\sigma\sigma} &\equiv \cos^2 \theta \eta_{11} + 2 \cos \theta \sin \theta \eta_{12} + \sin^2 \theta \eta_{22} \\ \eta_{ss} &\equiv \sin^2 \theta \eta_{11} - 2 \cos \theta \sin \theta \eta_{12} + \cos^2 \theta \eta_{22} \\ \eta_{\sigma s} &\equiv -\sin \theta \cos \theta \eta_{11} + (\cos^2 \theta - \sin^2 \theta) \eta_{12} + \cos \theta \sin \theta \eta_{22}, \end{aligned} \quad (10.60)$$

from which we see that (trace of the η matrix)

$$\eta_{\sigma\sigma} + \eta_{ss} = \eta_{11} + \eta_{22}. \quad (10.61)$$

We get the opposite rotation by just changing the sign of θ :

$$\begin{aligned} \eta_{11} &\equiv \cos^2 \theta \eta_{\sigma\sigma} - 2 \cos \theta \sin \theta \eta_{\sigma s} + \sin^2 \theta \eta_{ss} \\ \eta_{22} &\equiv \sin^2 \theta \eta_{\sigma\sigma} + 2 \cos \theta \sin \theta \eta_{\sigma s} + \cos^2 \theta \eta_{ss} \\ \eta_{12} &\equiv +\sin \theta \cos \theta \eta_{\sigma\sigma} + (\cos^2 \theta - \sin^2 \theta) \eta_{\sigma s} - \cos \theta \sin \theta \eta_{ss}. \end{aligned} \quad (10.62)$$

No we can also define

$$\xi_{\sigma\sigma\sigma} \equiv M^4 \frac{V_\sigma V_{\sigma\sigma\sigma}}{V^2} \quad \xi_{\sigma\sigma s} \equiv M^4 \frac{V_\sigma V_{\sigma\sigma s}}{V^2} \quad \xi_{\sigma s s} \equiv M^4 \frac{V_\sigma V_{\sigma s s}}{V^2} \quad \xi_{s s s} \equiv M^4 \frac{V_s V_{s s s}}{V^2}. \quad (10.63)$$

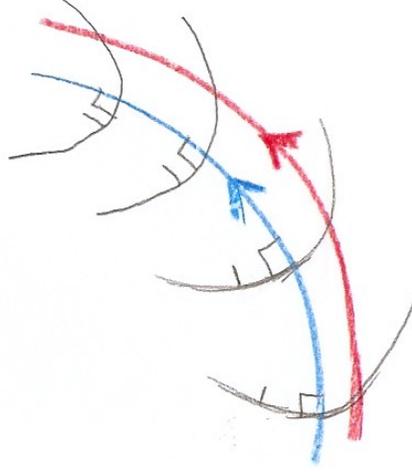


Figure 6: The slow-roll trajectory (blue) and the exact solution (red). The black curves are contours of $V(\vec{\varphi})$. The slow-roll trajectory follows the steepest gradient and is thus orthogonal to the contours. The exact solution requires a sideways push from the potential to force it to bend and therefore “overshoots”.

From Eq. (9.11),

$$H^{-1}\dot{\varepsilon} = 4\varepsilon^2 - 2\varepsilon(\cos^2\eta_{11} + 2\cos\theta\sin\theta\eta_{12} + \sin^2\eta_{22}) = 4\varepsilon^2 - 2\varepsilon\eta_{\sigma\sigma}. \quad (10.64)$$

Also from Eq. (9.11),

$$\begin{aligned} H^{-2}\ddot{\varphi}_1 &= \varepsilon H^{-1}\dot{\varphi}_1 - \eta_{11}H^{-1}\dot{\varphi}_1 - \eta_{12}H^{-1}\dot{\varphi}_2 \\ H^{-2}\ddot{\varphi}_2 &= \varepsilon H^{-1}\dot{\varphi}_2 - \eta_{12}H^{-1}\dot{\varphi}_1 - \eta_{22}H^{-1}\dot{\varphi}_2. \end{aligned} \quad (10.65)$$

Derivating $\dot{\sigma} = \dot{\varphi}_1 \cos\theta + \dot{\varphi}_2 \sin\theta$, the $\dot{\theta}$ part cancels, and we get (**exercise**)

$$H^{-2}\ddot{\sigma} = H^{-1}\dot{\sigma}(\varepsilon - \eta_{\sigma\sigma}) \quad (10.66)$$

(compare to Eq. 7.31c), and derivating $\dot{s} = -\dot{\varphi}_1 \sin\theta + \dot{\varphi}_2 \cos\theta = 0$, we get (**exercise**) the promised slow-roll equation for $\dot{\theta}$:

$$H^{-1}\dot{\theta} = -\eta_{\sigma s} \propto -V_{\sigma s}. \quad (10.67)$$

Note that $V_s = 0$ does not imply $V_{\sigma s} = 0$, since we are working in a curved coordinate system, and in $V_{\sigma s}$ we have covariant derivatives.

It is instructive to contrast the exact equation (10.22) to the slow-roll equation (10.67): In making the slow-roll approximation we eliminate dynamics from the problem by dropping the acceleration terms $\ddot{\varphi}_I$ from the field equation. Then the question of the background solution becomes one of $V(\varphi_1, \varphi_2)$ topography: the trajectories are the paths of steepest descent, so that there is no sideways component V_s to the gradient, and the curvature of the trajectory is determined by the potential: $V_{\sigma s}$ measures how the potential will tilt sideways if you continue straight in the current σ direction. In the exact solution, on the other hand, the field cannot follow the path of steepest descent when that curves, since the “centrifugal effect” pushes it out to where a sideways component V_s of the gradient provides the force needed to make the field turn.

We can now calculate the time derivatives of the other first-order slow-roll parameters (**exercise**):

$$\begin{aligned} H^{-1}\dot{\eta}_{\sigma\sigma} &= 2\varepsilon\eta_{\sigma\sigma} - 2\eta_{\sigma s}^2 - \xi_{\sigma\sigma\sigma} \\ H^{-1}\dot{\eta}_{\sigma s} &= 2\varepsilon\eta_{\sigma s} + \eta_{\sigma s}(\eta_{\sigma\sigma} - \eta_{ss}) - \xi_{\sigma\sigma s} \\ H^{-1}\dot{\eta}_{ss} &= 2\varepsilon\eta_{ss} + 2\eta_{\sigma s}^2 - \xi_{\sigma s s}. \end{aligned} \quad (10.68)$$

10.7 Evolution through Horizon Exit

However, $\delta\sigma$ and δs do not correspond to the independently through the horizon evolving field components discussed in Sec. 9.3. These are obtained by a rotation with a different angle Θ .

For the independent perturbations v_1 and v_2 we obtained

$$\vec{u} = U\vec{v}, \quad \vec{v} = U^T\vec{u}, \quad (10.69)$$

where

$$\vec{u} = \begin{pmatrix} a\delta\varphi_1 \\ a\delta\varphi_2 \end{pmatrix} = \begin{pmatrix} aQ_1 \\ aQ_2 \end{pmatrix} \quad (10.70)$$

and

$$U \equiv \begin{pmatrix} \cos \Theta & -\sin \Theta \\ \sin \Theta & \cos \Theta \end{pmatrix} \quad (10.71)$$

is the rotation matrix that diagonalizes

$$M \equiv \begin{pmatrix} \varepsilon + 2\varepsilon_{11} - \eta_{11} & 2\varepsilon_{12} - \eta_{12} \\ 2\varepsilon_{12} - \eta_{12} & \varepsilon + 2\varepsilon_{22} - \eta_{22} \end{pmatrix}, \quad (10.72)$$

i.e.,

$$U^T M U = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}. \quad (10.73)$$

We can solve Θ from the condition that $U^T M U$ is diagonal, i.e.,

$$(U^T M U)_{12} = (2\varepsilon_{22} - 2\varepsilon_{11} + \eta_{11} - \eta_{22}) \sin \Theta \cos \Theta + (2\varepsilon_{12} - \eta_{12})(\cos^2 \Theta - \sin^2 \Theta) = 0,$$

(**exercise**) where

$$\sin \Theta \cos \Theta = \frac{1}{2} \sin 2\Theta \quad \text{and} \quad \cos^2 \Theta - \sin^2 \Theta = \cos 2\Theta \quad (10.74)$$

so that

$$\tan 2\Theta = 2 \left[\frac{2\varepsilon_{12} - \eta_{12}}{2(\varepsilon_{11} - \varepsilon_{22}) - (\eta_{11} - \eta_{22})} \right]. \quad (10.75)$$

The λ_1, λ_2 are the eigenvalues of matrix M which we solve from

$$\det(M - \lambda I) = 0 \quad (10.76)$$

whose solutions are (**exercise**)

$$\lambda = \frac{1}{2} \left\{ 4\varepsilon - (\eta_{11} + \eta_{22}) \pm \sqrt{[2(\varepsilon_{11} - \varepsilon_{22}) - (\eta_{11} - \eta_{22})]^2 + 4(2\varepsilon_{12} - \eta_{12})^2} \right\}. \quad (10.77)$$

(Hint: Note that since $\varepsilon = \varepsilon_{11} + \varepsilon_{22}$, all three are not independent quantities, and you can, e.g., replace ε_{11} and ε_{22} by $\frac{1}{2}(\varepsilon + x)$ and $\frac{1}{2}(\varepsilon - x)$ where $x \equiv (\varepsilon_{11} - \varepsilon_{22})$.)

This does not say which one is λ_1 and which one is λ_2 , but neither does Eq. (10.75) specify Θ except up to a term $\pi/2$, i.e, we can add $\pi/2$ to Θ , which interchanges λ_1 and λ_2 .

We can use the results of Sec. (10.6) to rewrite the results (10.75) and (10.77) in terms of the rotated slow-roll parameters (**exercise**):

$$\lambda = \frac{1}{2} \left[4\varepsilon - (\eta_{\sigma\sigma} + \eta_{ss}) \pm \sqrt{\omega^2 + 4\eta_{\sigma s}^2} \right] \quad (10.78)$$

$$\begin{aligned} \lambda_1 + \lambda_2 &= 4\varepsilon - (\eta_{\sigma\sigma} + \eta_{ss}) \\ \lambda_1 - \lambda_2 &= \sqrt{\omega^2 + 4\eta_{\sigma s}^2}. \end{aligned} \quad (10.79)$$

and

$$\tan 2\Theta = \frac{\omega \sin 2\theta - 2\eta_{\sigma s} \cos 2\theta}{\omega \cos 2\theta + 2\eta_{\sigma s} \sin 2\theta}, \quad (10.80)$$

where we have defined the short-hand notation

$$\omega \equiv 2\varepsilon - (\eta_{\sigma\sigma} - \eta_{ss}). \quad (10.81)$$

From Eq. (10.80) we see, that $\tan 2\Theta = \tan 2\theta$, if $\eta_{\sigma s} = 0$, i.e., $V_{\sigma s} = 0$ and we have a straight trajectory (we work here with the slow-roll solution). Thus, in this case, the independent field perturbations are the adiabatic and entropy field perturbations, but otherwise they are not.

Combining the two rotations (10.25) and (10.69) we can rotate the independently produced perturbations into the adiabatic and entropy field perturbations

$$\begin{pmatrix} a\delta\sigma \\ a\delta s \end{pmatrix} = S^T U \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \cos(\Theta - \theta) & -\sin(\Theta - \theta) \\ \sin(\Theta - \theta) & \cos(\Theta - \theta) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}. \quad (10.82)$$

(For 2-dimensional rotations, there is no need to perform matrix multiplication, one can just add or subtract rotation angles.)

Using (10.80) we easily find (**exercise**) that

$$\begin{aligned} \tan 2(\Theta - \theta) &= -2 \frac{\eta_{\sigma s}}{\omega} \\ \cos 2(\Theta - \theta) &= \frac{\omega}{\sqrt{\omega^2 + 4\eta_{\sigma s}^2}} \\ \sin 2(\Theta - \theta) &= \frac{-2\eta_{\sigma s}}{\sqrt{\omega^2 + 4\eta_{\sigma s}^2}}. \end{aligned} \quad (10.83)$$

We can now obtain the generated perturbation spectra for $\delta\sigma$ and δs , from those of v_1 and v_2 derived in Sec. 9.3:

$$\begin{aligned} v_J &= aw_J \rightarrow L^{-3/2} 2^{\lambda_J} \frac{\Gamma(\frac{3}{2} + \lambda_J)}{\Gamma(\frac{3}{2})} \frac{1}{\sqrt{2k}} (-k\eta)^{-1-\lambda_J} \\ \langle |v_J|^2 \rangle &= \mathcal{V}^{-1} 2^{2\lambda_J} \left[\frac{\Gamma(\frac{3}{2} + \lambda_J)}{\Gamma(\frac{3}{2})} \right]^2 \frac{1}{2k} (-k\eta)^{-2-2\lambda_J} \\ \langle v_1 v_2^* \rangle &= 0, \end{aligned} \quad (10.84)$$

where $\mathcal{V} \equiv L^3$. For the power spectra we need these in the form

$$\mathcal{V} \frac{k^3}{2\pi^2 a^2} \langle |v_J|^2 \rangle = \frac{1}{4\pi^2} 2^{2\lambda_J} \left[\frac{\Gamma(\frac{3}{2} + \lambda_J)}{\Gamma(\frac{3}{2})} \right]^2 \left(\frac{k}{a} \right)^2 (-k\eta)^{-2-2\lambda_J}. \quad (10.85)$$

For each scale k we evaluate these at horizon exit, where $k = \mathcal{H} = aH$ and $-k\eta = 1/(1 - \varepsilon)$ (from Eq. 9.12). So we have

$$\begin{aligned} \mathcal{V} \frac{k^3}{2\pi^2 a^2} \langle |v_J|^2 \rangle &= \left(\frac{H_*}{2\pi} \right)^2 2^{2\lambda_J} \left[\frac{\Gamma(\frac{3}{2} + \lambda_J)}{\Gamma(\frac{3}{2})} \right]^2 (1 - \varepsilon)^{2+2\lambda_J} \\ &\approx \left(\frac{H_*}{2\pi} \right)^2 (1 + 2C\lambda_J - 2\varepsilon), \end{aligned} \quad (10.86)$$

where $C \equiv 2 - \ln 2 - \gamma \approx 0.729637$ (see Sec. 7.7.2), and H_* signifies that H is to be evaluated at horizon exit (as are ε and λ_J also). (We used $2^{2\lambda} \approx 1 + 2\lambda \ln 2$, $\Gamma(\frac{3}{2} + \lambda)/\Gamma(\frac{3}{2}) \approx 1 + 2\lambda\psi(\frac{3}{2})$, and $(1 - \varepsilon)^{2+2\lambda} \approx 1 - 2\varepsilon$.)

Using (10.82) we have

$$\begin{aligned} a^2 \langle |\delta\sigma|^2 \rangle &= \langle [\cos(\Theta - \theta)v_1 - \sin(\Theta - \theta)v_2][\cos(\Theta - \theta)v_1 - \sin(\Theta - \theta)v_2]^* \rangle \\ &= \cos^2(\Theta - \theta) \langle |v_1|^2 \rangle + \sin^2(\Theta - \theta) \langle |v_2|^2 \rangle \\ &= \frac{1}{2} (\langle |v_1|^2 \rangle + \langle |v_2|^2 \rangle) + \frac{1}{2} \cos 2(\Theta - \theta) (\langle |v_1|^2 \rangle - \langle |v_2|^2 \rangle) \end{aligned} \quad (10.87)$$

$$\begin{aligned} a^2 \langle \delta\sigma \delta s^* \rangle &= \langle [\cos(\Theta - \theta)v_1 - \sin(\Theta - \theta)v_2][\sin(\Theta - \theta)v_1 + \cos(\Theta - \theta)v_2]^* \rangle \\ &= \frac{1}{2} \sin 2(\Theta - \theta) (\langle |v_1|^2 \rangle - \langle |v_2|^2 \rangle) \end{aligned} \quad (10.88)$$

$$a^2 \langle |\delta s|^2 \rangle = \frac{1}{2} (\langle |v_1|^2 \rangle + \langle |v_2|^2 \rangle) - \frac{1}{2} \cos 2(\Theta - \theta) (\langle |v_1|^2 \rangle - \langle |v_2|^2 \rangle). \quad (10.89)$$

(We see that interchanging v_1 and v_2 interchanges the results for $\langle |\delta\sigma|^2 \rangle$ and $\langle |\delta s|^2 \rangle$ and changes the sign of $\langle \delta\sigma \delta s^* \rangle$, as does adding $\pi/2$ to Θ ; so I suppose we should pay attention to how our choice of λ_1 and λ_2 is related to our choice of Θ (we make a choice in Eq. 10.83).)

Finally we get for the generated adiabatic and entropy field perturbation power spectra and their correlation the results (**exercise**):

$$\begin{aligned} \mathcal{P}_{\sigma^*}(k) &\equiv \mathcal{V} \frac{k^3}{2\pi^2} \langle |\delta\sigma_{\vec{k}}|^2 \rangle = \left(\frac{H_*}{2\pi} \right)^2 [1 + (-2 + 6C)\varepsilon - 2C\eta_{\sigma\sigma}] \\ \mathcal{C}_{\sigma s^*}(k) &\equiv \mathcal{V} \frac{k^3}{2\pi^2} \langle \delta\sigma_{\vec{k}} \delta s_{\vec{k}}^* \rangle = -2C\eta_{\sigma s} \left(\frac{H_*}{2\pi} \right)^2 \\ \mathcal{P}_{s^*}(k) &\equiv \mathcal{V} \frac{k^3}{2\pi^2} \langle |\delta s_{\vec{k}}|^2 \rangle = \left(\frac{H_*}{2\pi} \right)^2 [1 - 2(1 - C)\varepsilon - 2C\eta_{ss}]. \end{aligned} \quad (10.90)$$

10.8 Evolution Outside the Horizon

Using Eqs. (10.48) and (10.50), we immediately get from (10.90) the perturbation spectra for \mathcal{R} and S , as they are generated at horizon exit,

$$\mathcal{P}_{\mathcal{R}^*}(k) = \left(\frac{H_*}{\dot{\sigma}_*} \right)^2 \mathcal{P}_{\sigma^*}(k) = \left(\frac{H_*^2}{2\pi\dot{\sigma}_*} \right)^2 [1 + (-2 + 6C)\varepsilon - 2C\eta_{\sigma\sigma}] \quad (10.91)$$

$$\mathcal{C}_{\mathcal{R} S^*}(k) = - \left(\frac{H_*}{\dot{\sigma}_*} \right)^2 \mathcal{C}_{\sigma s^*}(k) = +2C\eta_{\sigma s} \left(\frac{H_*^2}{2\pi\dot{\sigma}_*} \right)^2 \quad (10.92)$$

$$\mathcal{P}_{S^*}(k) = \left(\frac{H_*}{\dot{\sigma}_*} \right)^2 \mathcal{P}_{s^*}(k) = \left(\frac{H_*^2}{2\pi\dot{\sigma}_*} \right)^2 [1 - 2(1 - C)\varepsilon - 2C\eta_{ss}]. \quad (10.93)$$

However, unlike the 1-field case, where $\mathcal{P}_{\mathcal{R}}(k)$ stayed constant in time for as long as $k \ll \mathcal{H}$, now the perturbation spectra may evolve outside the horizon. In first-order perturbation theory, the Fourier components of perturbations at some later time t are proportional to the earlier values at t_* :

$$\begin{pmatrix} \mathcal{R}_{\vec{k}}(t) \\ S_{\vec{k}}(t) \end{pmatrix} = T_k(t, t_*) \begin{pmatrix} \mathcal{R}_{\vec{k}^*} \\ S_{\vec{k}^*} \end{pmatrix} \quad (10.94)$$

(We write T_k , not $T_{\vec{k}}$, since the relevant physics is assumed rotationally invariant.)

For superhorizon scales, the scale dependence of the evolution vanishes, so that $T_k(t, t_*) = T(t, t_*)$. However, Eq. (10.94) remains scale dependent, since t_* depends on k . Thus

$$\begin{pmatrix} \mathcal{R}_{\vec{k}}(t) \\ S_{\vec{k}}(t) \end{pmatrix} = \begin{pmatrix} T_{\mathcal{R}\mathcal{R}}(t, t_*) & T_{\mathcal{R}S}(t, t_*) \\ T_{S\mathcal{R}}(t, t_*) & T_{SS}(t, t_*) \end{pmatrix} \begin{pmatrix} \mathcal{R}_{\vec{k}^*} \\ S_{\vec{k}^*} \end{pmatrix} = \begin{pmatrix} T_{\mathcal{R}\mathcal{R}}(t, t_*)\mathcal{R}_{\vec{k}^*} + T_{\mathcal{R}S}(t, t_*)S_{\vec{k}^*} \\ T_{S\mathcal{R}}(t, t_*)\mathcal{R}_{\vec{k}^*} + T_{SS}(t, t_*)S_{\vec{k}^*} \end{pmatrix}. \quad (10.95)$$

We know two things that apply to superhorizon perturbations in general: 1) adiabatic perturbations remain adiabatic $\Rightarrow T_{S\mathcal{R}} = 0$; 2) for adiabatic perturbations, \mathcal{R} is constant in time $\Rightarrow T_{\mathcal{R}\mathcal{R}} = 1$. Thus Eq. (10.95) becomes

$$\begin{pmatrix} \mathcal{R}_{\vec{k}}(t) \\ S_{\vec{k}}(t) \end{pmatrix} = \begin{pmatrix} 1 & T_{\mathcal{R}S}(t, t_*) \\ 0 & T_{SS}(t, t_*) \end{pmatrix} \begin{pmatrix} \mathcal{R}_{\vec{k}^*} \\ S_{\vec{k}^*} \end{pmatrix} = \begin{pmatrix} \mathcal{R}_{\vec{k}^*} + T_{\mathcal{R}S}(t, t_*)S_{\vec{k}^*} \\ T_{SS}(t, t_*)S_{\vec{k}^*} \end{pmatrix}, \quad (10.96)$$

and we can also write that, in general,

$$H^{-1}\dot{\mathcal{R}} = \alpha(t)S \quad \text{and} \quad H^{-1}\dot{S} = \beta(t)S, \quad (10.97)$$

defining two functions $\alpha(t)$ and $\beta(t)$ related to the transfer functions $T_{\mathcal{R}S}$ and T_{SS} . We can find the relation by integrating (10.97). First, write (10.97b) as

$$\begin{aligned} d \ln S(t') &\equiv \frac{dS(t')}{S(t')} = \beta(t')H(t')dt' \Rightarrow \ln S(t) - \ln S(t_*) = \int_{t_*}^t \beta(t')H(t')dt' \\ \Rightarrow T_{SS}(t, t_*) &\equiv \frac{S(t)}{S(t_*)} = \exp \left\{ \int_{t_*}^t \beta(t')H(t')dt' \right\}. \end{aligned} \quad (10.98)$$

Then

$$\begin{aligned} \mathcal{R}(t) &= \mathcal{R}(t_*) + \int_{t_*}^t \dot{\mathcal{R}} dt = \mathcal{R}(t_*) + \int_{t_*}^t \alpha(t')H(t')S(t')dt' \\ &= \mathcal{R}(t_*) + \int_{t_*}^t \alpha(t')H(t')T_{SS}(t', t_*)S(t_*)dt' \\ \Rightarrow T_{\mathcal{R}S}(t, t_*) &= \int_{t_*}^t \alpha(t')H(t')T_{SS}(t', t_*)dt'. \end{aligned} \quad (10.99)$$

The only quantities that depend on k or \vec{k} in Eqs. (10.98,10.99) are $\mathcal{R} = \mathcal{R}_{\vec{k}}$, $S = S_{\vec{k}}$, and $t_* = t_*(k)$.

In the above, α , β , T_{SS} , and $T_{\mathcal{R}S}$, depend on the inflation model, and $T_{SS}(t, t_*)$, $T_{\mathcal{R}S}(t, t_*)$ may be complicated to calculate. To find the spectral indices, we need the derivatives of T_{SS} and $T_{\mathcal{R}S}$ wrt k , which we get from

$$\frac{\partial T_{SS}(t, t_*)}{\partial t_*} = -\beta(t_*)H(t_*)T_{SS}(t, t_*) \quad (10.100)$$

$$\begin{aligned} \frac{\partial T_{\mathcal{R}S}(t, t_*)}{\partial t_*} &= -\alpha(t_*)H(t_*)T_{SS}(t_*, t_*) + \int_{t_*}^t \alpha(t')H(t') \frac{\partial T_{SS}(t', t_*)}{\partial t_*} dt' \\ &= -\alpha(t_*)H(t_*) - \beta(t_*)H(t_*) \int_{t_*}^t \alpha(t')H(t')T_{SS}(t', t_*)dt' \\ &= -\alpha(t_*)H(t_*) - \beta(t_*)H(t_*)T_{\mathcal{R}S}(t, t_*). \end{aligned} \quad (10.101)$$

Thus we need $\alpha(t_*)$ and $\beta(t_*)$ at the time t_* of horizon exit during inflation, which are more easily accessible in terms of the slow-roll parameters at that time.

The primordial spectra $\mathcal{P}_{\mathcal{R}}(k)$, $\mathcal{C}_{\mathcal{R}S}(k)$, $\mathcal{P}_S(k)$ are defined at some time t after inflation, during the radiation-dominated era, when all cosmological scales are still well outside the horizon:

$$\mathcal{P}_{\mathcal{R}}(k) \equiv \mathcal{V} \frac{k^3}{2\pi^2} \langle |\mathcal{R}_{\vec{k}}|^2 \rangle = \mathcal{P}_{\mathcal{R}^*}(k) + 2T_{\mathcal{R}S}(t, t_*)\mathcal{C}_{\mathcal{R}S^*}(k) + T_{\mathcal{R}S}(t, t_*)^2\mathcal{P}_{S^*}(k) \quad (10.102)$$

$$\mathcal{C}_{\mathcal{R}S}(k) \equiv \mathcal{V} \frac{k^3}{2\pi^2} \langle \mathcal{R}_{\vec{k}} S_{\vec{k}}^* \rangle = T_{SS}(t, t_*)\mathcal{C}_{\mathcal{R}S^*}(k) + T_{\mathcal{R}S}(t, t_*)T_{SS}(t, t_*)\mathcal{P}_{S^*}(k) \quad (10.103)$$

$$\mathcal{P}_S(k) \equiv \mathcal{V} \frac{k^3}{2\pi^2} \langle |S_{\vec{k}}|^2 \rangle = T_{SS}(t, t_*)^2\mathcal{P}_{S^*}(k). \quad (10.104)$$

Since these refer to a time after inflation, when the inflation fields have been replaced by matter and radiation, Eqs. (10.48) and (10.50) no longer apply. Nevertheless, we still have a comoving curvature perturbation \mathcal{R} and quantities called entropy perturbations, that describe the deviation from adiabaticity. Since in this section we assume that two-field slow-roll inflation is the origin of all perturbations, there are just two degrees of freedom for each Fourier mode, leaving only one degree of freedom for the deviation from adiabaticity. (For example, the field φ_2 may decay into CDM particles and field φ_1 into standard model particles, in which case entropy field perturbations may give rise to a CDM isocurvature mode.) Thus all entropy perturbations can be given in terms of one quantity $S_{\vec{k}}$ per Fourier mode, although we may have a choice in what entropy quantity to choose as this $S_{\vec{k}}$. This choice then affects $T_{SS}(t, t_*)$, but does not affect the discussion in the following section.

10.9 Primordial Spectral Indices to First Order

To calculate spectral indices to first order in slow-roll parameters, it is enough to start from the generated spectra calculated to zeroth order. Thus, instead of Eq. (10.93), it is enough to use

$$\mathcal{P}_{\mathcal{R}*}(k) = \left(\frac{H_*^2}{2\pi\dot{\sigma}_*} \right)^2 \equiv \mathcal{P}_*^{(0)}(k) \quad (10.105)$$

$$\mathcal{C}_{\mathcal{R}S*}(k) = 0 \quad (10.106)$$

$$\mathcal{P}_{S*}(k) = \left(\frac{H_*^2}{2\pi\dot{\sigma}_*} \right)^2 = \mathcal{P}_*^{(0)}(k). \quad (10.107)$$

Eqs. (10.102,10.103,10.104) become now

$$\mathcal{P}_{\mathcal{R}}(k) = \mathcal{P}_*^{(0)}(k) + T_{\mathcal{R}S}^2 \mathcal{P}_*^{(0)}(k) \quad (10.108)$$

$$\mathcal{C}_{\mathcal{R}S}(k) = T_{\mathcal{R}S} T_{SS} \mathcal{P}_*^{(0)}(k) \quad (10.109)$$

$$\mathcal{P}_S(k) = T_{SS}^2 \mathcal{P}_*^{(0)}(k). \quad (10.110)$$

To calculate spectral indices, we need $\alpha(t_*)$ and $\beta(t_*)$. From Eq. (10.51), $\dot{\mathcal{R}} = -2\dot{\theta}S$. In the slow-roll approximation, from Eq. (10.67) $\dot{\theta} = -H\eta_{\sigma_s}$. So we have

$$\dot{\mathcal{R}} = 2H\eta_{\sigma_s}S \Rightarrow \alpha(t_*) = 2\eta_{\sigma_s}. \quad (10.111)$$

To get a slow-roll superhorizon value for $\beta(t_*)$, we start from Eq. (10.45). In the superhorizon limit, we can drop the ∇^2 terms, giving

$$\ddot{\delta}_s + 3H\dot{\delta}_s + \left(V_{ss} + 3\dot{\theta}^2 \right) \delta_s = 0. \quad (10.112)$$

In the slow-roll approximation we drop the second time derivative, so we have (writing V_{ss} and $\dot{\theta}$ in terms of the slow-roll parameters),

$$3H\dot{\delta}_s + 3H^2 (\eta_{ss} + \eta_{\sigma_s}^2) \delta_s = 0. \quad (10.113)$$

Dropping the 2nd order term, this becomes

$$H^{-1}\dot{\delta}_s + \eta_{ss}\delta_s = 0, \quad (10.114)$$

(This—slow-roll approximation of perturbation equations at superhorizon scales—should be done in a more systematic manner—perhaps I’ll improve this part later. An essential point is that at superhorizon scales, the different parts of the universe are disconnected, and the inhomogeneity of the perturbation plays no role in local evolution. Therefore $\vec{\varphi} + \delta\vec{\varphi}$ evolves

like the background, just along a different slow-roll trajectory. I have now added a discussion of this in Sec. 12.3.)

For $S \equiv (H/\dot{\sigma})\delta s$ this becomes (**exercise**) the equation

$$H^{-1}\dot{S} = (-2\varepsilon + \eta_{\sigma\sigma} - \eta_{ss})S \quad \Rightarrow \quad \beta(t_*) = -2\varepsilon + \eta_{\sigma\sigma} - \eta_{ss}. \quad (10.115)$$

We are now ready to calculate the spectral indices. From (10.108),

$$\mathcal{P}_{\mathcal{R}}(k) = \mathcal{P}_{\mathcal{R}*}(k) + T_{\mathcal{R}S}(t, t_*)^2 \mathcal{P}_{S*}(k) = \frac{H^4}{4\pi^2 \dot{\sigma}^2} (1 + T_{\mathcal{R}S}^2), \quad (10.116)$$

where for each scale k , H and $\dot{\sigma}$ are evaluated at t_* , when $k = aH$; and $T_{\mathcal{R}S} = T_{\mathcal{R}S}(t, t_*)$. From (10.52,10.55),

$$\frac{H^4}{\dot{\sigma}^2} = H^2(H^{-1}\dot{\sigma})^{-2} = \frac{V}{3M^2} \frac{V^2}{M^4 V_\sigma^2} \propto \frac{V}{\varepsilon}. \quad (10.117)$$

The spectral index $n_{\mathcal{R}}$ is given by

$$n_{\mathcal{R}} \equiv \frac{d \ln \mathcal{P}_{\mathcal{R}}(k)}{d \ln k} = \frac{d \ln V}{d \ln k} - \frac{d \ln \varepsilon}{d \ln k} + \frac{d \ln (1 + T_{\mathcal{R}S}^2)}{d \ln k}. \quad (10.118)$$

The $d \ln k$ can be converted to a time derivative by

$$\frac{d \ln k}{dt} = \frac{d \ln(aH)}{dt} = \frac{\dot{a}}{a} + \frac{\dot{H}}{H} = H \left(1 + \frac{\dot{H}}{H^2} \right) = (1 - \varepsilon)H \quad (10.119)$$

(where we used Eq. (9.10) in the last step). Now, using (10.55) and (10.57),

$$\frac{d \ln V}{d \ln k} = \frac{1}{V} \frac{1}{(1 - \varepsilon)H} \dot{V} = \frac{1}{V} \frac{1}{(1 - \varepsilon)H} V_\sigma \dot{\sigma} = \frac{-M^2}{1 - \varepsilon} \left(\frac{V_\sigma}{V} \right)^2 = \frac{-2}{1 - \varepsilon} \varepsilon \approx -2\varepsilon. \quad (10.120)$$

Using Eq. (10.64),

$$\frac{d \ln \varepsilon}{d \ln k} = \frac{1}{(1 - \varepsilon)H} \dot{\varepsilon} = \frac{4\varepsilon - 2\eta_{\sigma\sigma}}{1 - \varepsilon} \approx 4\varepsilon - 2\eta_{\sigma\sigma}. \quad (10.121)$$

These two contributions add up to the single-field result (7.105).

The new part comes from

$$\begin{aligned} \frac{d \ln (1 + T_{\mathcal{R}S}^2)}{d \ln k} &\approx \frac{1}{1 + T_{\mathcal{R}S}^2} H^{-1} \frac{d}{dt_*} (1 + T_{\mathcal{R}S}^2) = \frac{2T_{\mathcal{R}S}}{1 + T_{\mathcal{R}S}^2} H^{-1} \frac{\partial T_{\mathcal{R}S}}{\partial t_*} \\ &= \frac{2T_{\mathcal{R}S}}{1 + T_{\mathcal{R}S}^2} [-\alpha(t_*) - \beta(t_*)T_{\mathcal{R}S}] \\ &= \frac{T_{\mathcal{R}S}}{1 + T_{\mathcal{R}S}^2} (-4\eta_{\sigma\sigma}) + \frac{T_{\mathcal{R}S}^2}{1 + T_{\mathcal{R}S}^2} (4\varepsilon - 2\eta_{\sigma\sigma} + 2\eta_{ss}). \end{aligned} \quad (10.122)$$

We have no a priori constraint on $T_{\mathcal{R}S}$. It depends on the inflation model, including the reheating process. The combinations $T_{\mathcal{R}S}/(1 + T_{\mathcal{R}S}^2)$ and $T_{\mathcal{R}S}^2/(1 + T_{\mathcal{R}S}^2)$ above, on the other hand, have a limited range—the latter must be between 0 and 1. Its square root must then be between -1 and 1 and it has become customary to define it in terms of an angle Δ , so that

$$\cos \Delta \equiv \frac{T_{\mathcal{R}S}}{\sqrt{1 + T_{\mathcal{R}S}^2}}. \quad (10.123)$$

(The angle Δ is thus related to what part of the primordial \mathcal{R} was generated at horizon exit and what part outside horizon: $\Delta = \pi/2$ means all at exit, none of it outside; $\Delta = 0$ or π means none of it at exit, all outside.) From this we get

$$\frac{T_{\mathcal{R}S}^2}{1 + T_{\mathcal{R}S}^2} = \cos^2 \Delta, \quad \frac{1}{1 + T_{\mathcal{R}S}^2} = \sin^2 \Delta, \quad \frac{1}{\sqrt{1 + T_{\mathcal{R}S}^2}} = \sin \Delta \quad (10.124)$$

(choosing $0 \leq \Delta \leq \pi$), and

$$\frac{1}{T_{\mathcal{R}S}^2} = \tan^2 \Delta, \quad \frac{1}{T_{\mathcal{R}S}} = \tan \Delta. \quad (10.125)$$

This allows us to write Eq. (10.122) as

$$\frac{d \ln (1 + T_{\mathcal{R}S}^2)}{d \ln k} \approx -4\eta_{\sigma\sigma} \cos \Delta \sin \Delta + (4\varepsilon - 2\eta_{\sigma\sigma} + 2\eta_{ss}) \cos^2 \Delta \quad (10.126)$$

and altogether we have the final result

$$n_{\mathcal{R}} = -(6 - 4 \cos^2 \Delta)\varepsilon + 2\eta_{\sigma\sigma} \sin^2 \Delta - 4\eta_{\sigma s} \cos \Delta \sin \Delta + 2\eta_{ss} \cos^2 \Delta. \quad (10.127)$$

(The sign differences when compared to [14] are due to a different sign convention for \mathcal{R} , which changes the sign of $T_{\mathcal{R}S}$ and $\cos \Delta$, but not $\sin \Delta$, which is always nonnegative.)

In a similar way we obtain (**exercise**):

$$n_S = -2\varepsilon + 2\eta_{ss} \quad (10.128)$$

and

$$n_C \equiv \frac{d \ln \mathcal{C}_{\mathcal{R}S}(k)}{d \ln k} = -2\varepsilon - 2\eta_{\sigma s} \tan \Delta + 2\eta_{ss}. \quad (10.129)$$

Derivating these first-order results for the spectral indices we obtain their running

$$q \equiv \frac{dn}{d \ln k} \quad (10.130)$$

to 2nd order in slow-roll parameters (**exercise**):

$$\begin{aligned} q_S &\equiv \frac{dn_S}{d \ln k} \approx H^{-1} \dot{n}_S = -2H^{-1} \dot{\varepsilon} + 2H^{-1} \dot{\eta}_{ss} = -8\varepsilon^2 + 4\varepsilon(\eta_{\sigma\sigma} + \eta_{ss}) + 4\eta_{\sigma s}^2 + 2\xi_{\sigma ss} \\ q_C &\equiv \frac{dn_C}{d \ln k} \approx H^{-1} \dot{n}_C = \dots \\ &= -8\varepsilon^2 + 4\varepsilon(\eta_{\sigma\sigma} + \eta_{ss}) + 4\eta_{\sigma s}^2(1 - \tan^2 \Delta) - 4\eta_{\sigma s}(\eta_{\sigma\sigma} - \eta_{ss}) \tan \Delta + 2\xi_{\sigma\sigma s} \tan \Delta - 2\xi_{\sigma ss} \\ q_{\mathcal{R}} &\equiv \frac{dn_{\mathcal{R}}}{d \ln k} \approx H^{-1} \dot{n}_{\mathcal{R}} = \dots \\ &= 8(-3 + 4 \cos^2 \Delta - 2 \cos^4 \Delta)\varepsilon^2 \\ &\quad + 4(4 - 7 \cos^2 \Delta + 4 \cos^4 \Delta)\varepsilon\eta_{\sigma\sigma} - 32 \sin^3 \Delta \cos \Delta \varepsilon\eta_{\sigma s} + 4(5 \cos^2 \Delta - 4 \cos^4 \Delta)\varepsilon\eta_{\sigma\sigma} \\ &\quad + 4 \sin^2 \Delta \cos^2 \Delta (\eta_{\sigma\sigma}^2 + \eta_{ss}^2) + 4(1 - 4 \sin^2 \Delta \cos^2 \Delta)\eta_{\sigma s}^2 \\ &\quad + 8(\sin \Delta \cos \Delta - 2 \sin \Delta \cos^3 \Delta)\eta_{\sigma s}(\eta_{\sigma\sigma} - \eta_{ss}) - 8 \sin^2 \Delta \cos^2 \Delta \eta_{\sigma\sigma} \eta_{ss} \\ &\quad - 2 \sin^2 \Delta \xi_{\sigma\sigma\sigma} + 4 \sin \Delta \cos \Delta \xi_{\sigma\sigma s} - 2 \cos^2 \Delta \xi_{\sigma ss}. \end{aligned} \quad (10.131)$$

Since the running of the spectral indices is 2nd order in slow-roll parameters, it is usually a good approximation to approximate the power spectra with a power law, i.e., with constant spectral indices. An exception to this is n_C in the case where $T_{\mathcal{R}S}$ is very small, i.e., $\tan \Delta$ is very large, since q_C contains terms with a prefactor $\tan \Delta$ or $\tan^2 \Delta$. These terms can thus be large even though the slow-roll parameters in them are small. The spectral index n_C itself contains the term $-2\eta_{\sigma s} \tan \Delta$, which can make n_C large. Thus the correlation $\mathcal{C}_{\mathcal{R}S}(k)$ is not necessarily well approximated by a power law. However, this happens only when the correlation is small, since the 0th order correlation is proportional to $T_{\mathcal{R}S}$.

10.10 Alternative Parameterization

When comparing the theoretical predictions to observations, the primordial power spectra must be represented in terms of a relatively small number of parameters. For a given inflation model, the model itself may provide these parameters. In our general approach we may assume that the primordial spectra can be approximated by power laws, so that for each spectrum we can take its constant spectral index and its amplitude at some reference “pivot” scale k_p as the parameters to be fitted to the data. We could thus parameterize the three spectra, $\mathcal{P}_{\mathcal{R}}(k)$, $\mathcal{C}_{\mathcal{R}S}(k)$, $\mathcal{P}_S(k)$ with two amplitudes, A and B , a correlation parameter C , and three spectral indices $n_{\mathcal{R}}$, n_C , and n_S , i.e.,

$$\begin{aligned}\mathcal{P}_{\mathcal{R}}(k) &= A^2 \left(\frac{k}{k_p}\right)^{n_{\mathcal{R}}} \\ \mathcal{C}_{\mathcal{R}S}(k) &= CAB \left(\frac{k}{k_p}\right)^{n_C} \\ \mathcal{P}_S(k) &= B^2 \left(\frac{k}{k_p}\right)^{n_S}.\end{aligned}\tag{10.132}$$

However, if we let these 6 parameters vary independently, some parameter combinations lead to an inconsistent description of the perturbations, since in reality $\mathcal{C}_{\mathcal{R}S}(k)^2$ must always be smaller than $\mathcal{P}_{\mathcal{R}}(k)\mathcal{P}_S(k)$, and this parameterization does not guarantee that. Restricting $-1 \leq C \leq 1$ is not enough, since if the three spectral indices are different, $\mathcal{C}_{\mathcal{R}S}(k)$ may still become too large in relation to $\mathcal{P}_{\mathcal{R}}(k)$ and $\mathcal{P}_S(k)$ at some small or large value of k . While there are ways to control this in fitting to the data (the data always covers only a limited range in k), it may be more convenient to parameterize the spectra differently.

We divide the primordial curvature perturbation power spectrum in two parts,

$$\mathcal{P}_{\mathcal{R}}(k) = \mathcal{P}_{ar}(k) + \mathcal{P}_{as}(k),\tag{10.133}$$

where

$$\mathcal{P}_{ar}(k) \equiv \mathcal{P}_{\mathcal{R}*}(k)\tag{10.134}$$

is the part generated by the adiabatic field perturbations $\delta\sigma$ alone, and

$$\mathcal{P}_{as}(k) \equiv 2T_{\mathcal{R}S}(t, t_*)\mathcal{C}_{\mathcal{R}S*}(k) + T_{\mathcal{R}S}(t, t_*)^2\mathcal{P}_{S*}(k)\tag{10.135}$$

is the rest, i.e., the part generated by the entropy field perturbation and its original correlation with the adiabatic field perturbation (this latter part vanishes to lowest order).

To lowest order we have then

$$\mathcal{P}_{ar}(k) = \mathcal{P}_*^{(0)}(k)\tag{10.136}$$

$$\mathcal{P}_{as}(k) = T_{\mathcal{R}S}^2\mathcal{P}_*^{(0)}(k)\tag{10.137}$$

$$\mathcal{P}_S(k) = T_{SS}^2\mathcal{P}_*^{(0)}(k)\tag{10.138}$$

and the correlation spectrum is now

$$\mathcal{C}_{\mathcal{R}S}(k) = T_{\mathcal{R}S}T_{SS}\mathcal{P}_*^{(0)}(k) = \sqrt{\mathcal{P}_{as}(k)\mathcal{P}_S(k)},\tag{10.139}$$

so that

$$n_C = \frac{1}{2}(n_{as} + n_S).\tag{10.140}$$

To the lowest order the spectral indices are (**exercise**)

$$\begin{aligned}n_{ar} &= -6\varepsilon + 2\eta_{\sigma\sigma} \\ n_{as} &= -2\varepsilon - 4\eta_{\sigma s} \tan \Delta + 2\eta_{ss}\end{aligned}\tag{10.141}$$

and their running is

$$q_{ar} = -24\varepsilon^2 + 16\varepsilon\eta_{\sigma\sigma} - 4\eta_{\sigma s}^2 - 2\xi_{\sigma\sigma\sigma} \quad (10.142)$$

$$q_{as} = -8\varepsilon^2 + 4\varepsilon(\eta_{\sigma\sigma} + \eta_{ss}) + 4\eta_{\sigma s}^2(1 - 2\tan^2 \Delta) - 8\eta_{\sigma s}(\eta_{\sigma\sigma} - \eta_{ss})\tan \Delta + 4\xi_{\sigma\sigma s}\tan \Delta - 2\xi_{\sigma ss}$$

(n_C , n_S , q_C , and q_S are as before).

We now approximate the three power spectra with power laws,

$$\mathcal{P}_{ar}(k) \approx A_r^2 \left(\frac{k}{k_p}\right)^{n_{ar}} \quad \text{and} \quad \mathcal{P}_{as}(k) \approx A_s^2 \left(\frac{k}{k_p}\right)^{n_{as}}, \quad (10.143)$$

and

$$\mathcal{P}_S(k) \approx B^2 \left(\frac{k}{k_p}\right)^{n_S}. \quad (10.144)$$

The covariance is then given by

$$\mathcal{C}_{RS}(k) \approx A_s B \left(\frac{k}{k_p}\right)^{\frac{n_{as} + n_S}{2}}. \quad (10.145)$$

Here $A_r^2 \equiv \mathcal{P}_{ar}(k_p)$, $A_s^2 \equiv \mathcal{P}_{as}(k_p)$, and $B^2 \equiv \mathcal{P}_S(k_p)$. Here A_r and B are positive, but we let the sign of A_s represent the sign of the correlation. We have thus 6 independent parameters to describe the primordial perturbations:

$$A_r, A_s, B, n_{ar}, n_{as}, n_S. \quad (10.146)$$

We further define a total amplitude

$$A^2 \equiv A_r^2 + A_s^2 + B^2 = \mathcal{P}_{\mathcal{R}}(k_0) + \mathcal{P}_S(k_0), \quad (10.147)$$

and relative amplitudes

$$\beta_{\text{iso}} \equiv \frac{B^2}{A^2} = \frac{\mathcal{P}_S(k_p)}{\mathcal{P}_{\mathcal{R}}(k_p) + \mathcal{P}_S(k_p)} \quad (10.148)$$

$$\begin{aligned} \gamma &\equiv \text{sign}(A_s B) \frac{A_s^2}{A_r^2 + A_s^2} = \text{sign}(\mathcal{C}_{RS}) \frac{\mathcal{P}_{as}(k_p)}{\mathcal{P}_{\mathcal{R}}(k_p)} = \text{sign}(T_{RS} T_{SS}) \frac{T_{RS}^2}{1 + T_{RS}^2} \\ &= \text{sign}(\cos \Delta) \cos^2 \Delta, \end{aligned} \quad (10.149)$$

so that

$$\begin{aligned} A_s^2 &= |\gamma|(A_r^2 + A_s^2) = |\gamma|(1 - \beta_{\text{iso}})A^2 \\ A_r^2 &= (1 - |\gamma|)(A_r^2 + A_s^2) = (1 - |\gamma|)(1 - \beta_{\text{iso}})A^2 \\ B^2 &= \beta_{\text{iso}}A^2. \end{aligned} \quad (10.150)$$

Now the 6 independent parameters are

$$A, \beta_{\text{iso}}, \gamma, n_{ar}, n_{as}, n_S, \quad (10.151)$$

where $0 \leq \beta_{\text{iso}} \leq 1$ and $-1 \leq \gamma \leq 1$. In the literature, n_S is often called n_{iso} , as \mathcal{P}_S represents the isocurvature mode of the perturbations.

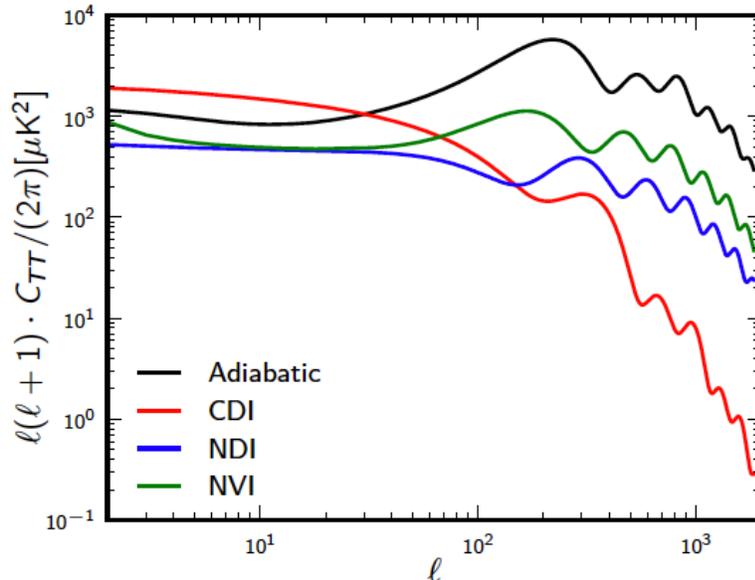


Figure 7: The contribution to the CMB angular power spectrum of the adiabatic mode and CDM (CDI) and neutrino (NDI) entropy perturbations, in the case of scale-invariant ($n = 0$) perturbations. (NVI is the neutrino velocity isocurvature mode, which we do not expect from inflation.) If the modes are correlated, there is an additional contribution that is intermediate between the two modes in shape and can be either positive or negative depending on the sign of the correlation. The same primordial amplitude has been assumed for each mode. From [15].

10.11 Observational Constraints

The tightest constraints on primordial entropy perturbations come from the observations of the cosmic microwave background (CMB) by the Planck satellite. These observations were made in 2009–13 and the final results were published in 2018. The effect on the CMB depends on whether the entropy field perturbations have been converted into CDM/baryon or neutrino entropy perturbations. (The effect of CDM and baryon entropy perturbations is essentially the same, so they cannot be distinguished by CMB observations.) Fig. 7 shows the contribution to the CMB by the adiabatic mode (no primordial entropy perturbation) and the different possibilities for the isocurvature mode (no primordial curvature perturbation, just an entropy perturbation) in the case of scale-invariant primordial perturbations.

Observations agree well with a pure adiabatic mode. A pure isocurvature mode is not allowed, so the data gives upper limits to the relative contribution of the isocurvature mode, which we define as

$$\beta_{\text{iso}}(k) \equiv \frac{\mathcal{P}_S(k)}{\mathcal{P}_{\mathcal{R}}(k) + \mathcal{P}_S(k)}. \quad (10.152)$$

If $\mathcal{P}_{\mathcal{R}}(k)$ and $\mathcal{P}_S(k)$ have different spectral indices, then β_{iso} is scale-dependent, and we get different constraints for it at different scales. Assuming that the running of the spectral indices is negligible in the observational range Planck gives the upper (95% confidence) limits [16]

$$\begin{aligned} \beta_{\text{iso}} &< 0.025 && \text{at } k = 0.002 \text{ Mpc}^{-1} \\ \beta_{\text{iso}} &< 0.26 && \text{at } k = 0.05 \text{ Mpc}^{-1} \\ \beta_{\text{iso}} &< 0.47 && \text{at } k = 0.1 \text{ Mpc}^{-1} \end{aligned} \quad (10.153)$$

in the case of a CDM or baryon isocurvature mode. Why the limit is much tighter for large

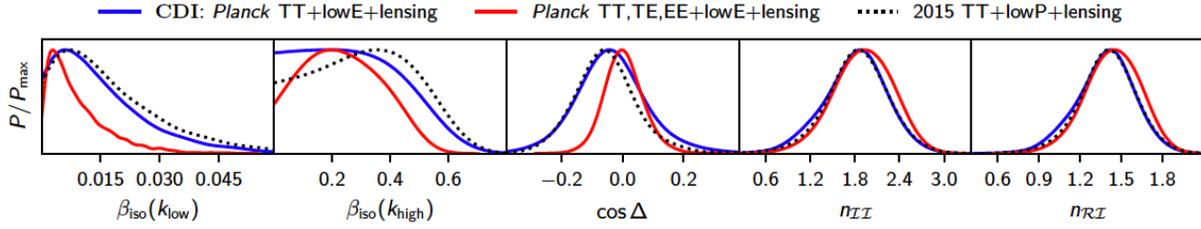


Figure 8: The likelihood distributions of the CDM isocurvature parameters $\beta_{\text{iso}}(0.002 \text{ Mpc}^{-1})$ and $\beta_{\text{iso}}(0.1 \text{ Mpc}^{-1})$, $\cos \Delta$, n_S , and n_C from final (2018) Planck data. From [16]. The dotted curves are the 2015 results. The red and blue curves are essentially with and without using the data on CMB polarization. (The 2015 results did not use polarization data.) Since we do not have a detection of the isocurvature mode, we do not have any constraints on its spectral index. The curve for $1 + n_S \equiv n_{II}$ is just an artifact resulting from priors used for the likelihood estimation reflecting the fact that isocurvature contributions with $1 + n_S \approx 1.8$ can be fit to the data with a larger primordial amplitude than redder or bluer spectra.

scales (low k) is clear from Fig. 7. A given relative amplitude of the isocurvature mode has a much larger relative effect on the CMB at large scales. This also means that it is easier to accommodate an isocurvature mode with a high (blue) spectral index. In Fig. 8 we give the likelihood distributions of the isocurvature parameters β_{iso} (at $k_{\text{low}} = 0.002 \text{ Mpc}^{-1}$ and $k_{\text{high}} = 0.1 \text{ Mpc}^{-1}$), $\cos \Delta$, n_S , and n_C from final Planck data.

These limits on the isocurvature contributions are relatively weak considering the high precision of Planck data. This is due to the large number of isocurvature parameters. One gets tighter limits [16] for specific inflation models or by making some assumptions about the isocurvature parameters. For example, assuming the isocurvature mode is uncorrelated with the adiabatic mode and close to scale invariant, $n_S \approx 0$, (an “axion”) model), the upper limit is $\beta_{\text{iso}} < 0.04$. Assuming the isocurvature mode is fully correlated or anticorrelated with the adiabatic mode, with the same spectral index (“curvaton” models), the upper limit is very tight, $\beta_{\text{iso}} < 0.001$. This is because the correlation has a much bigger effect on the CMB than the direct contribution of the isocurvature mode.

11 Primordial Tensor Perturbations

For tensor perturbations, the metric is

$$g_{\mu\nu} = a^2\eta_{\mu\nu} + \delta g_{\mu\nu}^T = a^2(\eta_{\mu\nu} + h_{\mu\nu}) = a^2 \begin{bmatrix} -1 & & & \\ & 1+h_+ & h_\times & \\ & h_\times & 1-h_+ & \\ & & & 1 \end{bmatrix}. \quad (11.1)$$

There are thus two modes of tensor perturbations, h_+ and h_\times . From the Hilbert action we find (not obvious, this part should be done here) that

$$\psi_{+,\times} \equiv \frac{M}{\sqrt{2}} h_{+,\times} \quad (11.2)$$

appear in the action in the same way as a free massless scalar field. Therefore they acquire the same spectrum¹³ during inflation as perturbations in the scalar fields,

$$\mathcal{P}_\psi(k) \equiv \mathcal{V} \frac{k^3}{2\pi^2} \langle |\psi(\vec{k})|^2 \rangle = \left(\frac{H}{2\pi} \right)_{k=aH}^2 \quad (11.3)$$

to lowest order.¹⁴ Tensor perturbations do not evolve outside the horizon, so this gives the primordial tensor perturbation spectrum. It is commonly defined as

$$\begin{aligned} \mathcal{P}_T(k) &\equiv \mathcal{V} \frac{k^3}{2\pi^2} \langle h_{\mu\nu}(\vec{k}) h^{\mu\nu}(\vec{k})^* \rangle = \mathcal{V} \frac{k^3}{2\pi^2} \langle 2|h_+(\vec{k})|^2 + 2|h_\times(\vec{k})|^2 \rangle \\ &= \frac{8}{M^2} \mathcal{P}_\psi(k) = \frac{8}{M^2} \left(\frac{H}{2\pi} \right)_{k=aH}^2. \end{aligned} \quad (11.4)$$

The two perturbation modes are independent, so

$$\langle h_a(\vec{k}) h_b(\vec{k}')^* \rangle = \frac{2\pi^2}{\mathcal{V} k^3} \delta_{ab} \delta_{\vec{k}\vec{k}'} \frac{1}{4} \mathcal{P}_T(k) \quad \text{where } a, b = +, \times \quad (11.5)$$

Using the slow-roll equations $H^2 = V/3M^2$ etc., we have

$$\mathcal{P}_T = \frac{2}{3\pi^2 M^4} V \quad (11.6)$$

$$n_T \equiv \frac{d \ln \mathcal{P}_T}{d \ln k} = \frac{d \ln V}{d \ln k} = -2\varepsilon \quad (11.7)$$

$$q_T \equiv \frac{dn_T}{d \ln k} = -2 \frac{d\varepsilon}{d \ln k} = -8\varepsilon^2 + 4\varepsilon\eta_{\sigma\sigma}. \quad (11.8)$$

The tensor-to-scalar ratio is defined¹⁵

$$r \equiv \frac{\mathcal{P}_T}{\mathcal{P}_\mathcal{R}}. \quad (11.9)$$

We define

$$r_* \equiv \frac{\mathcal{P}_T}{\mathcal{P}_{\mathcal{R}*}}. \quad (11.10)$$

¹³This generation of primordial tensor perturbations from quantum fluctuations assumes quantum gravity at the perturbation level (quantized linear gravity, which is relatively straightforward to formulate compared to full quantum gravity). The quanta of these perturbations are called *gravitons*.

¹⁴This is calculated to higher (first) order in slow-roll parameters in [22].

¹⁵There are also other definitions of r in the literature. In the older literature it is sometimes defined in terms of its contribution to the CMB quadrupole, $r \equiv C_2^T/C_2^{\mathcal{R}}$, which makes its relation to primordial power spectra depend on the background cosmological parameters.

If $\mathcal{P}_{\mathcal{R}}$ does not evolve outside the horizon, which is the case in single-field inflation, $r = r_*$. But in many-field inflation, $\mathcal{P}_{\mathcal{R}}$ evolves outside the horizon, although \mathcal{P}_T does not, and therefore $r \neq r_*$. Since

$$\mathcal{P}_{\mathcal{R}*} = \mathcal{P}_*^{(0)} = \frac{H^4}{4\pi^2 \dot{\sigma}^2} = \frac{1}{4\pi^2} \frac{V}{3M^2} \frac{V^2}{M^4 V_\sigma^2} = \frac{1}{24\pi^2 M^4} \frac{V}{\varepsilon}, \quad (11.11)$$

we have

$$r_* = \frac{48\pi^2 M^4}{3\pi^2 M^4} \varepsilon = 16\varepsilon. \quad (11.12)$$

From (10.108),

$$\mathcal{P}_R = (1 + T_{RS}^2) \mathcal{P}_*^{(0)} = \frac{\mathcal{P}_*^{(0)}}{\sin^2 \Delta} = \frac{\mathcal{P}_*^{(0)}}{1 - |\gamma|} \quad (11.13)$$

so that

$$r = r_* \sin^2 \Delta = 16\varepsilon \sin^2 \Delta = 16\varepsilon(1 - |\gamma|). \quad (11.14)$$

Since tensor perturbations add new observables (r , n_T , q_T) without adding new slow-roll parameters, we obtain *consistency relations*

$$n_T = -\frac{1}{8} r_* = -\frac{r}{8(1 - |\gamma|)} \quad (11.15)$$

$$q_T = n_T(n_T - n_{ar}) \quad (11.16)$$

that can be used to test inflation without having to assume a particular inflation model.

Primordial tensor perturbations have not been observed so far. The current (2015) upper limit is from combined BICEP2/Keck/Planck data [18],

$$r < 0.07 \quad (11.17)$$

(95 % confidence level). Assuming single-field inflation, this gives an upper limit

$$\varepsilon < 0.0044 \quad (11.18)$$

and, from the consistency relation, a lower limit

$$n_T > -0.009 \quad (11.19)$$

The smaller r is, the less accurately it will be possible to measure n_T (once we have first detected tensor perturbations). While it is estimated that with a dedicated polarization-optimized CMB space mission, we might be able to detect tensor perturbations if $r > 0.0001$, with the current limits it is already questionable whether it is possible to ever detect the deviation of n_T from zero (assuming it satisfies the consistency relation). Thus the hope of verifying the consistency relation observationally, and thus providing quantitative evidence for inflation, is waning. However, if n_T deviates from scale invariance ($n_T = 0$) by much more than the consistency relation implies, we could still detect that with $r < 0.07$, and thus falsify this prediction of inflation.

12 Noninteracting Fields

We now consider the simpler case of N fields, where the fields do not interact with each other, except gravitationally, i.e., the potential is of the form

$$V(\varphi_1, \dots, \varphi_N) = \sum_{I=1}^N V_I(\varphi_I), \quad (12.1)$$

which motivates this **change from our previous notation**: the subscript I now denotes that term in the potential which depends on field φ_I , and thus does not denote a derivative.¹⁶ From (12.1) follows that

$$\frac{\partial^2 V}{\partial \varphi_I \partial \varphi_J} = 0 \quad \text{for} \quad I \neq J, \quad (12.2)$$

i.e., the matrix of second derivatives is diagonal and we can denote the potential derivatives by

$$V'_I \equiv \frac{\partial V}{\partial \varphi_I} = \frac{dV_I}{d\varphi_I}, \quad V''_I \equiv \frac{\partial^2 V}{\partial \varphi_I^2} = \frac{d^2 V_I}{d\varphi_I^2}. \quad (12.3)$$

We can write the background energy density and pressure as

$$\begin{aligned} \rho &= \sum \rho_I, & \text{where} & \quad \rho_I = \frac{1}{2} \dot{\varphi}_I^2 + V_I \\ p &= \sum p_I, & \text{where} & \quad p_I = \frac{1}{2} \dot{\varphi}_I^2 - V_I \end{aligned} \quad (12.4)$$

The background equations are

$$H^2 = \frac{8\pi G}{3} \rho = \frac{1}{3M^2} \sum (\frac{1}{2} \dot{\varphi}_I^2 + V_I) \quad (12.5)$$

$$\ddot{\varphi}_I + 3H\dot{\varphi}_I + V'_I = 0 \quad (12.6)$$

In general, the φ_I will evolve at different rates, so that the background trajectory is curved. To preserve the advantage of field separation we shall mostly calculate in the original basis, i.e., not rotate into adiabatic and entropy field coordinates.

In the slow-roll approximation the background equations become

$$\boxed{H^2 = \frac{V}{3M^2} \quad H^{-1}\dot{\varphi}_I = -M^2 \frac{V'_I}{V} \quad \frac{\dot{H}}{H^2} = -\frac{M^2}{2V^2} \sum (V'_I)^2 = -\varepsilon.} \quad (12.7)$$

We define slow-roll parameters for each field:

$$\varepsilon_I \equiv \frac{M^2}{2} \left(\frac{V'_I}{V} \right)^2 \quad \eta_I \equiv M^2 \frac{V''_I}{V}. \quad (12.8)$$

The ε_{IJ} and η_{IJ} that we defined earlier for the more general case are now

$$\varepsilon_{IJ} = \sqrt{\varepsilon_I \varepsilon_J} \quad \eta_{IJ} = \eta_I \delta_{IJ}. \quad (12.9)$$

¹⁶Sections 12 and 13 were initially written as a single section, and the clean-up after the separation is maybe still incomplete. Originally I had some of the calculations of this section done only for the specific double inflation model of Sec. 13; whereas now I just apply the more general results to it. It is not clear to me which would be pedagogically better: to calculate first in the specific model and then redo it in the more general model, or the current structure where the calculation is done directly in the more general case and the results then applied to the specific model.

The slow-roll conditions may fail for different fields at different times. How does slow-roll fail? It fails when we can no longer drop the $\dot{\varphi}_I^2$ and $\ddot{\varphi}_I$ terms in Eqs. (12.5) and (12.6). In the slow-roll approximation these terms are

$$\frac{1}{2}\dot{\varphi}_I^2 = \frac{1}{3}\varepsilon_I V \quad \text{and} \quad \ddot{\varphi}_I = (\varepsilon - \eta_I)H\dot{\varphi}_I, \quad (12.10)$$

where $V = \sum V_I$ and $\varepsilon = \sum \varepsilon_I$. Thus the slow-roll approximation remains valid for (12.5) while ε is small and for (12.6) while ε and η_I are small. Since $\varepsilon = \sum \varepsilon_I$ it can only remain small when all of the ε_I are small. Thus the way for the slow-roll condition to fail for just one of the fields φ_J is that η_J becomes large, while all the other slow-roll parameters remain small. When that happens, φ_J begins to move rapidly towards the minimum of V_J , where $V_J = 0$, and the energy density ρ_J begins to fall rapidly, while the energy density associated with the other fields keeps changing slowly. Thus the effect of φ_J on the metric and hence on the other fields becomes soon negligible, and we can continue the discussion of the slow-roll solution with the remaining $N - 1$ fields—and so on.

12.1 Perturbations

Following Polarski & Starobinsky [20] and Langlois [21] we work in the **longitudinal (Newtonian) gauge**. We shall need the second Einstein equation (5.15) and the field perturbation equations (5.8). In the Newtonian gauge ($A = D = \Phi$) they become

$$\dot{\Phi} + H\Phi = 4\pi G \sum_I \dot{\varphi}_I \delta\varphi_I^N \quad (12.11)$$

$$\ddot{\delta\varphi}_I^N + 3H\dot{\delta\varphi}_I^N + \frac{k^2}{a^2}\delta\varphi_I^N + V_I''\delta\varphi_I^N = -2V_I'\Phi + 4\dot{\varphi}_I\dot{\Phi}. \quad (12.12)$$

We shall also be interested in the comoving density perturbation (Eq. 6.17)

$$\begin{aligned} \delta\rho^C &= \sum_I \delta\rho_I^C, \quad \text{where} \\ \delta\rho_I^C &= \dot{\varphi}_I\dot{\delta\varphi}_I^N + V_I'\delta\varphi_I^N + 3H\dot{\varphi}_I\delta\varphi_I^N - \dot{\varphi}_I^2\Phi. \end{aligned} \quad (12.13)$$

We define also the total and component comoving relative density perturbations

$$\delta^C \equiv \frac{\delta\rho^C}{\rho} \quad \delta_I^C \equiv \frac{\delta\rho_I^C}{\rho_I}. \quad (12.14)$$

During inflation, $\rho_I = \frac{1}{2}\dot{\varphi}_I^2 + V_I$ is dominated by the potential term, and $\rho_I + p_I = \dot{\varphi}_I^2 \ll \rho_I$. It is useful to define the quantities

$$\Delta_I \equiv \frac{\delta\rho_I^C}{\rho_I + p_I} = \frac{\delta\rho_I^C}{\dot{\varphi}_I^2} = \frac{d}{dt} \left(\frac{\delta\varphi_I^N}{\dot{\varphi}_I} \right) - \Phi, \quad (12.15)$$

which are much larger than the $\delta\rho_I^C$. The last equality follows from

$$\frac{d}{dt} \left(\frac{\delta\varphi_I^N}{\dot{\varphi}_I} \right) - \Phi = \frac{\dot{\delta\varphi}_I^N}{\dot{\varphi}_I} - \frac{\ddot{\varphi}_I\delta\varphi_I^N}{\dot{\varphi}_I^2} - \Phi = \frac{\dot{\varphi}_I\dot{\delta\varphi}_I^N + V_I'\delta\varphi_I^N + 3H\dot{\varphi}_I\delta\varphi_I^N - \dot{\varphi}_I^2\Phi}{\dot{\varphi}_I^2} = \frac{\delta\rho_I^C}{\rho_I + p_I}.$$

Note that $S_{IJ} \equiv \Delta_I - \Delta_J$ is the relative entropy perturbation between components I and J , as defined in part 1 of this course.

12.2 Adiabatic Superhorizon Solution

We know from part I of this course (since the 28.11.2015 version) that, for perfect fluid, the general adiabatic superhorizon solution for the Bardeen potential is

$$\Phi_{\vec{k}}(t) = A_{\vec{k}} \left(1 - \frac{H}{a} \int_0^t a dt \right) + B_{\vec{k}} \frac{H}{a} = \left(A_{\vec{k}} \int_0^t a dt - B_{\vec{k}} \right) \left(-\frac{H}{a} \right) + A_{\vec{k}} \quad (12.16)$$

$$\dot{\Phi}_{\vec{k}}(t) = \left(A_{\vec{k}} \int_0^t a dt - B_{\vec{k}} \right) \left(\frac{-\dot{H} + H^2}{a} \right) - A_{\vec{k}} H, \quad (12.17)$$

where $A_{\vec{k}}$ gives the growing mode and $B_{\vec{k}}$ the decaying mode. Note that

$$\frac{d}{dt} \left(\frac{1}{a} \right) = -\frac{H}{a} \quad \text{and} \quad \frac{d}{dt} \left(\frac{H}{a} \right) = \frac{\dot{H} - H^2}{a}. \quad (12.18)$$

These occur repeatedly in this subsection.

The background field equation is

$$\ddot{\varphi}_I + 3H\dot{\varphi}_I + V'_I = 0. \quad (12.19)$$

For superhorizon scales the field perturbation equation is then

$$\begin{aligned} \ddot{\delta\varphi}_I^N + 3H\dot{\delta\varphi}_I^N + V''_I \delta\varphi_I^N &= -2V'_I \Phi + 4\dot{\varphi}_I \dot{\Phi} \\ &= \left(A_{\vec{k}} \int_0^t a dt - B_{\vec{k}} \right) \left(2\frac{H}{a} V'_I - 4\frac{\dot{H} - H^2}{a} \dot{\varphi}_I \right) - 2A_{\vec{k}} (V'_I + 2H\dot{\varphi}_I). \end{aligned} \quad (12.20)$$

The solution is (**exercise**)

$$\delta\varphi_I^N = \frac{1}{a} \left(A_{\vec{k}} \int_0^t a dt - B_{\vec{k}} \right) \dot{\varphi}_I. \quad (12.21)$$

Note that in the N-dimensional field space the field perturbation is in the direction of the background trajectory, so this indeed represents an adiabatic field perturbation.

Derivating

$$\frac{\delta\varphi_I^N}{\dot{\varphi}_I} = \frac{1}{a} \left(A_{\vec{k}} \int_0^t a dt - B_{\vec{k}} \right) \quad (12.22)$$

and comparing to Eq. 12.15 we find

$$\frac{d}{dt} \left(\frac{\delta\varphi_I^N}{\dot{\varphi}_I} \right) = \Phi \quad \Rightarrow \quad \Delta_I = 0 \quad \Rightarrow \quad \delta\rho_I^C = 0. \quad (12.23)$$

This may look problematic, but remember that we are at the superhorizon limit, and

$$\delta\rho^C = -\frac{2}{3}\rho \left(\frac{k}{\mathcal{H}} \right)^2 \Phi, \quad (12.24)$$

so we are just ignoring $\delta\rho_I^C$ in the superhorizon approximation for the adiabatic case. If we need to know the total $\delta\rho^C$ we can always recover it from Φ .

The key point is that for entropy perturbations (to be discussed later) the individual Δ_I and $\delta\rho_I^C$ are much larger than the total Δ and $\delta\rho^C$, i.e., “nonzero” in the superhorizon limit.

For the comoving curvature perturbation we get (**exercise**) (easy)

$$\mathcal{R}_{\vec{k}} = -\Phi_{\vec{k}} + \frac{H}{\dot{H}} \left(\dot{\Phi}_{\vec{k}} + H\Phi_{\vec{k}} \right) = \dots = -A_{\vec{k}} = \text{const.} \quad (12.25)$$

Thus \mathcal{R} is associated with the growing mode and the decaying mode has $\mathcal{R} = 0$. We can now rewrite the growing mode solution for Φ as

$$\Phi_{\vec{k}}(t) = -\mathcal{R}_{\vec{k}} \left(1 - \frac{H}{a} \int_0^t a dt \right). \quad (12.26)$$

We do not have corresponding “exact” (i.e., no slow-roll approximation, just the superhorizon approximation) results for entropy perturbations. Thus we go to the slow-roll approximation in the next section.

12.3 Slow-Roll Approximation of the Field Perturbation Equations at Superhorizon Scales

In the superhorizon approximation each region of space evolves essentially independently and like a solution for the background universe. The field perturbation equation in the superhorizon approximation is

$$H^{-2} \delta \ddot{\varphi}_I^N + 3H^{-1} \delta \dot{\varphi}_I^N + H^{-2} V_I''(\bar{\varphi}_I) \delta \varphi_I^N = -2H^{-2} V_I'(\bar{\varphi}_I) \Phi + 4H^{-1} \dot{\varphi}_I H^{-1} \dot{\Phi}. \quad (12.27)$$

We temporarily reinstall the overbar to the background field $\bar{\varphi}_I$ to separate it from the total field $\varphi_I = \bar{\varphi}_I + \delta \varphi_I^N$. The background field equation is

$$H^{-2} \ddot{\bar{\varphi}}_I + 3H^{-1} \dot{\bar{\varphi}}_I + H^{-2} V_I'(\bar{\varphi}_I) = 0, \quad (12.28)$$

so that the equation for the total field is

$$H^{-2} \ddot{\varphi}_I + 3H^{-1} \dot{\varphi}_I + H^{-2} V_I'(\varphi_I) = -2H^{-2} V_I'(\bar{\varphi}_I) \Phi + 4H^{-1} \dot{\varphi}_I H^{-1} \dot{\Phi}. \quad (12.29)$$

If we ignore the metric perturbations on the rhs, the equation for the total field is the same as for the background field. Then it will have the same slow-roll trajectories as the background field. It will just choose a slightly different one. The additional effect of the metric perturbations appears as a slightly different expansion rate and a deviation of the coordinate time from the “local time”.¹⁷

We know that in inflation, the slow-roll solutions are attractors. The perturbation equations are second order equations and there will be a growing mode and a decaying mode for each degree of freedom. The decaying of the decaying mode represents the settling of the total field into the attractor solution. Thus the growing modes correspond to the slow-roll solutions. We can then make the slow-roll approximation and drop the $H^{-2} \ddot{\varphi}_I$ term from the total field equation at the same time as from the background field equation, meaning that $H^{-2} \delta \dot{\varphi}_I^N$ drops from the perturbation equation. In the slow-roll approximation the rhs of the perturbation equation is

$$-2H^{-2} V_I'(\bar{\varphi}_I) \Phi + 4H^{-1} \dot{\varphi}_I H^{-1} \dot{\Phi} \approx -2H^{-2} V_I'(\bar{\varphi}_I) \left(\Phi + 2H^{-1} \dot{\Phi} \right) \quad (12.30)$$

Since perturbations evolve slowly in the slow-roll solution, $H^{-1} \dot{\Phi} \ll \Phi$ and we drop it.

Thus the slow-roll superhorizon field perturbation equation is

$$\boxed{3H \delta \dot{\varphi}_I^N + V_I'' \delta \varphi_I^N = -2V_I' \Phi.} \quad (12.31)$$

and the Einstein equation (12.11) becomes

$$\Phi = \frac{1}{2M^2} \sum H^{-1} \dot{\varphi}_I \delta \varphi_I^N = -\frac{1}{2V} \sum V_I' \delta \varphi_I^N. \quad (12.32)$$

¹⁷Should try to formulate this more precisely, to make a more accurate statement about the slow-roll trajectories in the perturbed universe—are they exactly the same as in the background universe? The rhs is proportional to the potential gradient at the background trajectory. The full field is slightly displaced from it, but the difference is a second order perturbation in this term; so we can replace it with the potential gradient at the trajectory of the full field, so this term just pushes the total field slightly faster or slower along the slow-roll trajectory.

12.3.1 Two Fields

To simplify the discussion, we now consider the case of $N = 2$ fields. The Einstein equation (12.11) is now

$$\Phi = \frac{1}{2M^2 H} (\dot{\varphi}_1 \delta\varphi_1^N + \dot{\varphi}_2 \delta\varphi_2^N) = -\frac{1}{2V} (V_1' \delta\varphi_1^N + V_2' \delta\varphi_2^N). \quad (12.33)$$

The general solution to Eqs. (12.31) and (12.33) is (**exercise**)

$$\Phi = -C_1 \frac{\dot{H}}{H^2} + \frac{1}{3} C_3 \frac{V_1 (V_2')^2 - V_2 (V_1')^2}{V^2} \quad (12.34)$$

$$\frac{\delta\varphi_1^N}{\dot{\varphi}_1} = C_1 \frac{1}{H} - 2C_3 \frac{HV_2}{V} \quad (12.35)$$

$$\frac{\delta\varphi_2^N}{\dot{\varphi}_2} = C_1 \frac{1}{H} + 2C_3 \frac{HV_1}{V},$$

where $C_1(\vec{k})$ and $C_3(\vec{k})$ are constants fixed by the initial conditions. (My strategy for showing that this is a solution would be to do the C_1 and C_3 parts separately, and for the more complicated C_3 part use the slow-roll equations to write everything in terms of the potentials and their derivatives.) Constants C_2 and C_4 are reserved for the decaying modes which are killed in the slow-roll approximation.

We see that the C_1 part corresponds to the adiabatic solution, since for it we have

$$\delta\vec{\varphi}^N = \frac{C_1}{H} \dot{\vec{\varphi}}. \quad (12.36)$$

The slow-roll approximation of the superhorizon adiabatic solution (Sec. 12.2) matches this solution (**exercise**) so that $C_1(\vec{k}) = A_{\vec{k}}$.

The new, C_3 , mode is the isocurvature mode. For it we get (**exercise**)

$$\begin{aligned} \Delta_1 &= \frac{d}{dt} \left(\frac{\delta\varphi_1^N}{\dot{\varphi}_1} \right) - \Phi = \frac{1}{3} C_3 \frac{(V_2')^2}{V} \\ \Delta_2 &= \frac{d}{dt} \left(\frac{\delta\varphi_2^N}{\dot{\varphi}_2} \right) - \Phi = -\frac{1}{3} C_3 \frac{(V_1')^2}{V} \\ S_{12} &\equiv \Delta_1 - \Delta_2 = \frac{1}{3} C_3 \frac{(V_1')^2 + (V_2')^2}{V} = \frac{2}{3} C_3 \frac{V}{M^2} \varepsilon \\ \delta\rho_1^C &= \Delta_1 \dot{\varphi}_1^2 = \frac{C_3 M^2}{9V^2} (V_2')^2 (V_1')^2 \\ \delta\rho_2^C &= \Delta_2 \dot{\varphi}_2^2 = \frac{C_3 M^2}{9V^2} (V_1')^2 (V_2')^2 = -\delta\rho_1^C \\ \delta\rho^C &= \delta\rho_1^C + \delta\rho_2^C = 0. \end{aligned} \quad (12.37)$$

Thus in the isocurvature mode we have the opposite comoving density perturbations in the two components so that the total comoving density perturbation vanishes (in the superhorizon limit).

Inverting the equation pair (12.35) we can express the constants as

$$\begin{aligned} C_1 &= \frac{H}{V} \left(V_1 \frac{\delta\varphi_1^N}{\dot{\varphi}_1} + V_2 \frac{\delta\varphi_2^N}{\dot{\varphi}_2} \right) = -\frac{1}{M^2} \left(V_1 \frac{\delta\varphi_1^N}{V_1'} + V_2 \frac{\delta\varphi_2^N}{V_2'} \right) \\ C_3 &= \frac{1}{2H} \left(\frac{\delta\varphi_2^N}{\dot{\varphi}_2} - \frac{\delta\varphi_1^N}{\dot{\varphi}_1} \right) = \frac{3}{2} \left(\frac{\delta\varphi_1^N}{V_1'} - \frac{\delta\varphi_2^N}{V_2'} \right). \end{aligned} \quad (12.38)$$

(Note that C_1 and C_3 have different dimensions.) We can solve the constants C_1 and C_3 from (12.38) at any time they are valid, i.e., in the superhorizon slow-roll regime. The application is that as the perturbations are generated during horizon exit, we get the generated perturbations $\delta\varphi_1$ and $\delta\varphi_2$, from which we solve C_1 and C_3 , and then equations (12.35) tell us how the perturbations evolve until the slow-roll approximation fails.

12.3.2 Generation

In Sec. 10.7 we calculated the generated spectra for the adiabatic and entropy field perturbations in the spatially flat gauge. To first order in slow-roll parameters they were different and correlated, but to zeroth order they were equal and uncorrelated. We learned that the zeroth order spectra are enough for the lowest order results (e.g., spectral indices to first order, and their running to second order). This allows us to apply the results directly to the present discussion, where we use Newtonian gauge and unrotated field coordinates.

From Eq. (6.4) the field perturbations in the two gauges are related by

$$\delta\varphi_I^Q = \delta\varphi_I^N + H^{-1}\dot{\varphi}_I\Phi. \quad (12.39)$$

Since the change in $\vec{\delta\varphi}$ is parallel to the background trajectory, in the rotated field coordinates the gauge transformation changes only $\delta\sigma$,

$$\delta\sigma^Q = \delta\sigma^N + H^{-1}\dot{\sigma}\Phi, \quad (12.40)$$

whereas δs is gauge invariant. In terms of $\delta\sigma^Q$ and δs the constants C_1 and C_3 are

$$\begin{aligned} C_1 &= H \frac{\delta\sigma^N}{\dot{\sigma}} + \left(-\frac{V_1}{V} \tan\theta + \frac{V_2}{V} \cot\theta \right) H \frac{\delta s}{\dot{\sigma}} \\ C_3 &= \frac{1}{2H} \left(\frac{\dot{\varphi}_1}{\dot{\varphi}_2} + \frac{\dot{\varphi}_2}{\dot{\varphi}_1} \right) \frac{\delta s}{\dot{\sigma}} = \frac{1}{2H} (\cot\theta + \tan\theta) \frac{\delta s}{\dot{\sigma}} = \frac{1}{2H} \frac{1}{\cos\theta \sin\theta} \frac{\delta s}{\dot{\sigma}}. \end{aligned} \quad (12.41)$$

The zeroth order spectra for $\delta\sigma^Q$ and δs are, from Eq. (10.90),

$$\begin{aligned} \mathcal{P}_{\sigma^*}(k) &\equiv \mathcal{V} \frac{k^3}{2\pi^2} \langle |\delta\sigma_{\vec{k}}^Q|^2 \rangle = \left(\frac{H_*}{2\pi} \right)^2 \\ \mathcal{C}_{\sigma s^*}(k) &\equiv \mathcal{V} \frac{k^3}{2\pi^2} \langle \delta\sigma_{\vec{k}}^Q \delta s_{\vec{k}}^* \rangle = 0 \\ \mathcal{P}_{s^*}(k) &\equiv \mathcal{V} \frac{k^3}{2\pi^2} \langle |\delta s_{\vec{k}}|^2 \rangle = \left(\frac{H_*}{2\pi} \right)^2, \end{aligned} \quad (12.42)$$

where \mathcal{V} is the reference volume for the Fourier expansion. Since the spectra are equal and uncorrelated, they apply also to field components in an arbitrarily rotated basis, and in particular to the original $\delta\varphi_1^Q$ and $\delta\varphi_2^Q$.

We apply Eq. (12.39) to the perturbation spectra after they were generated, when the superhorizon slow-roll approximation is valid. Thus

$$\begin{aligned} \delta\varphi_1^N &= \delta\varphi_1^Q + M^2 \frac{V_1'}{V} \Phi = \delta\varphi_1^Q - \frac{M^2}{2V^2} V_1' (V_1' \delta\varphi_1^N + V_2' \delta\varphi_2^N) = \delta\varphi_1^Q - \varepsilon_{11} \delta\varphi_1^N - \varepsilon_{12} \delta\varphi_2^N \\ \delta\varphi_2^N &= \dots = \delta\varphi_2^Q - \varepsilon_{12} \delta\varphi_1^N - \varepsilon_{22} \delta\varphi_2^N \\ \delta\sigma^N &= \dots = \delta\sigma^Q - \varepsilon \delta\sigma^N. \end{aligned} \quad (12.43)$$

Thus $\delta\varphi_I^Q$ and $\delta\varphi_I^N$ differ in first slow-roll order and their spectra to zeroth order are the same:

$$\begin{aligned}\mathcal{P}_{1*}(k) &\equiv \mathcal{V} \frac{k^3}{2\pi^2} \langle |\delta\varphi_{1\vec{k}}|^2 \rangle = \left(\frac{H_*}{2\pi} \right)^2 \\ \mathcal{C}_{12*}(k) &\equiv \mathcal{V} \frac{k^3}{2\pi^2} \langle \delta\varphi_{1\vec{k}} \delta\varphi_{2\vec{k}}^* \rangle = 0 \\ \mathcal{P}_{2*}(k) &\equiv \mathcal{V} \frac{k^3}{2\pi^2} \langle |\delta\varphi_{2\vec{k}}|^2 \rangle = \left(\frac{H_*}{2\pi} \right)^2.\end{aligned}\tag{12.44}$$

Another way to write this result is

$$\delta\varphi_{I\vec{k}}(*) = \frac{H_*}{\sqrt{2k^3}} e_{I\vec{k}},\tag{12.45}$$

where the $e_{I\vec{k}}$ are *normalized uncorrelated Gaussian random variables*, i.e.,

$$\langle e_{I\vec{k}} e_{J\vec{k}}^* \rangle = \frac{1}{\mathcal{V}} \delta_{IJ}\tag{12.46}$$

(and their probability distribution has Gaussian shape).

Thus this quantum process generates the values of the constants $C_1(\vec{k})$ and $C_3(\vec{k})$ according to

$$\begin{aligned}C_1(\vec{k}) &= -\frac{H_k}{M^2 \sqrt{2k^3}} \left(\frac{V_1(*)}{V_1'(*)} e_{1\vec{k}} + \frac{V_2(*)}{V_2'(*)} e_{2\vec{k}} \right) \\ C_3(\vec{k}) &= \frac{3H_k}{2\sqrt{2k^3}} \left(\frac{e_{1\vec{k}}}{V_1'(*)} - \frac{e_{2\vec{k}}}{V_2'(*)} \right),\end{aligned}\tag{12.47}$$

where H_k is the Hubble parameter at horizon exit of scale k .

12.3.3 Adiabatic Mode

The adiabatic mode, the one corresponding to the constant C_1 , can be trivially carried from the perturbation generation during inflation through reheating to the primordial epoch, since for it the comoving curvature perturbation stays constant,

$$\mathcal{R}_{\vec{k}}(\text{rad, adi}) = \mathcal{R}_{\vec{k}}(*, \text{adi}) = -C_1(\vec{k}).\tag{12.48}$$

12.3.4 Isocurvature Mode through Reheating

The case of the isocurvature mode is more involved and depends on the details of reheating. It is quite possible that entropy perturbations are erased in reheating. For entropy perturbations to survive reheating and produce a primordial isocurvature mode, they must either be protected by some conserved quantity, or a field must remain decoupled from radiation (or the fields that decay into radiation) so it does not reach thermal equilibrium with it.

Here we assume that the φ_1 field decays into radiation in reheating, whereas the φ_2 field becomes the CDM, remaining decoupled (not interacting with φ_1 or the radiation, except gravitationally) at all times.

Inflation ends when slow-roll conditions fail. With two fields, they are not likely to fail for both at the same time. We have different cases depending on for which field they end first and which field dominates the energy density at that time.

How does slow roll fail? The exact background equations are

$$H^2 = \frac{1}{3M^2} \left(\frac{1}{2}\dot{\varphi}_1^2 + \frac{1}{2}\dot{\varphi}_2^2 + V_1 + V_2 \right) \quad (12.49)$$

$$\ddot{\varphi}_1 + 3H\dot{\varphi}_1 + V_1' = 0 \quad (12.50)$$

$$\ddot{\varphi}_2 + 3H\dot{\varphi}_2 + V_2' = 0. \quad (12.51)$$

The slow-roll approximation is that we drop the $\frac{1}{2}\dot{\varphi}_I^2$ and $\ddot{\varphi}_I$ terms from these equations. In the slow-roll approximation these terms are

$$\frac{1}{2}\dot{\varphi}_I^2 = \frac{1}{3}\varepsilon_I V \quad \text{and} \quad \ddot{\varphi}_I = (\varepsilon - \eta_I)H\dot{\varphi}_I, \quad (12.52)$$

where $V = V_1 + V_2$ and $\varepsilon = \varepsilon_1 + \varepsilon_2$. Thus the slow-roll approximation remains valid for (12.49) while ε is small, for (12.50) while ε and η_1 are small, and for (12.51) while ε and η_2 are small.

We consider now the case, where slow-roll fails first for φ_2 , i.e., we can no longer drop $\ddot{\varphi}_2$ from (12.51), while φ_1 dominates the energy density. Thus

$$V_2 < \rho_2 \ll V_1 \approx V \approx \rho_1. \quad (12.53)$$

The field φ_2 is then “close to the minimum” of V_2 (meaning that is closer to it than it was before). *To proceed we need to know the shape of V_2 in the region around the minimum* that we have now reached. We make the simplest assumption: V_2 can be approximated by the lowest-order term

$$V_2 \approx \frac{1}{2}m^2\varphi_2^2 \quad (12.54)$$

in this region. (We assume no vacuum energy, so there is no constant term—or even if there is, we may consider it as part of V_1 . We choose the origin of the field coordinates at the potential minimum \Rightarrow no first-order term.) Using the approximations (12.53) and (12.54) we have for the slow-roll parameters

$$\varepsilon_1 = \frac{1}{2}M^2 \left(\frac{V_1'}{V} \right)^2 \quad \varepsilon_2 \approx M^2 \frac{m^2 V_2}{V^2} \quad \eta_1 = M^2 \frac{V_1''}{V} \quad \eta_2 = M^2 \frac{m^2}{V}. \quad (12.55)$$

Slow roll fails when one of these parameters is no longer $\ll 1$. We see that $\varepsilon_2 = (V_2/V)\eta_2 \ll \eta_2$, so η_2 becomes large first, while all the other slow-roll parameters remain small. Thus we can keep using the slow-roll approximation for (12.49) and (12.50) but must start using the exact equation (12.51). We will assume that we can start using the approximations (12.53) and (12.54) already a bit earlier than we have to drop the slow-roll approximation for (12.51).

The full equations for φ_2 are now (we are still at superhorizon scales)

$$\begin{aligned} \ddot{\varphi}_2 + 3H\dot{\varphi}_2 + m^2\varphi_2 &= 0 \\ \delta\ddot{\varphi}_2 + 3H\delta\dot{\varphi}_2 + m^2\delta\varphi_2 &= -2m^2\varphi_2\Phi + 4\dot{\varphi}_2\dot{\Phi}. \end{aligned} \quad (12.56)$$

We quote (slightly paraphrased) from Polarski & Starobinsky [20]: “If $\rho_2 \ll \rho$, then irrespective of whether φ_2 is in the slow-roll regime or not, the rhs with the metric perturbations may be ignored for the isocurvature modes”¹⁸ Dropping the rhs, we see that the equations for φ_2 and $\delta\varphi_2$ are the same. This is specific to a quadratic potential, for which $V_2' = V_2''\varphi_2$, and is key to the following discussion.

¹⁸This is obvious, since if only one field has a significant contribution to the energy density, the metric perturbations should depend only on the dominant field, and thus be the same as in the adiabatic solution with the same perturbation in the dominant field. However, I was not able to show this with the equations at hand.

Thus these equations have the same solutions. They are 2nd order equations, so there are two independent solutions, but the slow-roll epoch has already killed the decaying solutions. In the slow-roll regime

$$\frac{\delta\varphi_2^N}{\dot{\varphi}_2} = 2C_3H\frac{V_1}{V} \approx 2C_3H \quad (12.57)$$

(as $V_2 \ll V_1 \Rightarrow V_1 \approx V$), so that

$$\delta\varphi_2^N \approx 2C_3H\dot{\varphi}_2 = -\frac{2}{3}C_3m^2\varphi_2. \quad (12.58)$$

Thus, during the slow-roll regime, $\delta\varphi_2^N$ has picked the same solution as φ_2 (in the sense that $\delta\varphi_2 \propto \varphi_2$) and this relation

$$\boxed{\delta\varphi_2^N = -\frac{2}{3}C_3m^2\varphi_2} \quad (12.59)$$

will hold even after slow roll fails for φ_2 .

After slow roll fails for φ_2 , it begins to oscillate at the bottom of its potential. The slow-roll equation

$$H^2 = \frac{V}{3M^2} \quad (12.60)$$

remains valid for as long as ε remains small. Thus

$$\eta_2 = M^2\frac{m^2}{V} \Rightarrow m^2 = 3\eta_2H^2. \quad (12.61)$$

After slow roll fails for φ_2 , η_2 becomes larger than 1 and H smaller than m . Since m stays constant, while H shrinks (albeit slowly for as long as φ_1 is in the slow-roll regime), after a while $H \ll m$ and the $3H\dot{\varphi}_2$ term becomes subdominant in Eq. (12.56) and the Hubble time becomes longer than the oscillation period.

For time scales shorter than the Hubble time we can ignore the $3H\dot{\varphi}_2$ term, so that we have

$$\ddot{\varphi}_2 + m^2\varphi_2 \approx 0 \quad (12.62)$$

whose solutions are sinusoidal

$$\begin{aligned} \varphi_2 &\approx E \sin m(t - t_1) \\ \dot{\varphi}_2 &\approx Em_2 \cos m(t - t_1) \\ \ddot{\varphi}_2 &\approx -Em_2^2 \sin m(t - t_1) \end{aligned} \quad (12.63)$$

while all the time $\delta\varphi_2^N = \frac{2}{3}C_3m^2\varphi_2$. In this oscillation the background density

$$\rho_2 = \frac{1}{2}(\dot{\varphi}_2^2 + m^2\varphi_2^2) \approx \frac{1}{2}m^2E^2 \quad (12.64)$$

alternates between the kinetic and potential parts.

Over longer time scales the $3H\dot{\varphi}_2$ term damps the amplitude of these oscillations. What happens to ρ_2 ? We now use the full equation (12.56a):

$$\begin{aligned} \dot{\rho}_2 + 3H\rho_2 &= \dot{\varphi}_2(\ddot{\varphi}_2 + m^2\varphi_2) + \frac{3}{2}H(\dot{\varphi}_2^2 + m^2\varphi_2^2) \\ &= \dot{\varphi}_2(-3H\dot{\varphi}_2) + \frac{3}{2}H(\dot{\varphi}_2^2 + m^2\varphi_2^2) = \frac{3}{2}H(m^2\varphi_2^2 - \dot{\varphi}_2^2) \end{aligned} \quad (12.65)$$

which oscillates, but averaged over oscillations is zero, so that the long-time behaviour is

$$\dot{\rho}_2 + 3H\rho_2 = 0 \Rightarrow \rho_2 \propto a^{-3}, \quad (12.66)$$

so the energy density of φ_2 behaves like matter.

At some point it will become appropriate to switch from the field picture to the particle picture. The quanta of the φ_2 field are the CDM particles in our model. Once the φ_1 field has done its thing (reheating) and we have landed in the radiation-dominated universe, where the CDM is the matter contribution (we ignore other matter components), it is time to evaluate the entropy perturbation

$$S \equiv \delta_m - \frac{3}{4}\delta_r = \delta_m^C - \frac{3}{4}\delta_r^C \approx \delta_m^C. \quad (12.67)$$

To justify the approximation, note that in the isocurvature mode during the superhorizon epoch $\delta\rho^C = \delta\rho_m^C + \delta\rho_r^C = 0$, so that $\delta\rho_m^C = -\delta\rho_r^C$. Therefore

$$|\delta_m^C| = \left| \frac{\delta\rho_m^C}{\rho_m} \right| \gg \left| \frac{\delta\rho_r^C}{\rho_r} \right| = |\delta_r^C| \quad (12.68)$$

as $\rho_r \gg \rho_m$ during the primordial epoch.

We have identified $\delta\rho_m^C$ with $\delta\rho_2^C$. From Eq. (12.13)

$$\delta\rho_2^C = \dot{\varphi}_2 \delta\dot{\varphi}_2^N + V_2' \delta\varphi_2^N + 3H\dot{\varphi}_2 \delta\varphi_2^N - \dot{\varphi}_2^2 \Phi. \quad (12.69)$$

We have argued that at this stage we can drop the metric perturbation term with Φ for the isocurvature mode, and that by now $3H\dot{\varphi}_2 \ll V_2' = m^2\varphi_2$, so that, using Eq. (12.59),

$$S = \delta_m^C \approx \frac{\delta\rho_2^C}{\rho_2} \approx 2 \frac{\dot{\varphi}_2 \delta\dot{\varphi}_2^N + m^2\varphi_2 \delta\varphi_2^N}{\dot{\varphi}_2^2 + m^2\varphi_2^2} = -\frac{4}{3}C_3 m^2 \frac{\dot{\varphi}_2^2 + m^2\varphi_2^2}{\dot{\varphi}_2^2 + m^2\varphi_2^2} \quad (12.70)$$

and we have our final result

$$\boxed{S_{\vec{k}}(\text{rad}) = -\frac{4}{3}m^2 C_3(\vec{k})} \quad (12.71)$$

To get the transfer function $T_{SS}(k)$, we compare this to

$$S(*) \equiv H_* \frac{\delta s_*}{\dot{\sigma}_*}. \quad (12.72)$$

Since, from Eq. (12.41),

$$C_3(\vec{k}) = \frac{1}{2H} \left(\frac{\dot{\varphi}_1}{\dot{\varphi}_2} + \frac{\dot{\varphi}_2}{\dot{\varphi}_1} \right) \frac{\delta s_{\vec{k}}}{\dot{\sigma}} = \frac{1}{2H_k^2} \left(\frac{V_1'(*)}{V_2'(*)} + \frac{V_2'(*)}{V_1'(*)} \right) S_{\vec{k}}(*), \quad (12.73)$$

we have

$$T_{SS}(k) \equiv \frac{S_{\vec{k}}(\text{rad})}{S_{\vec{k}}(*)} = -\frac{2}{3} \left(\frac{m}{H_k} \right)^2 \left[\frac{V_1'(*)}{V_2'(*)} + \frac{V_2'(*)}{V_1'(*)} \right] = -\frac{2}{3} \left(\frac{m}{H_k} \right)^2 \frac{1}{\sin\theta_k \cos\theta_k}. \quad (12.74)$$

To calculate $T_{\mathcal{R}S}(k)$ we take note that when the perturbations are generated, there are two contributions to \mathcal{R} :

$$\mathcal{R}_{\vec{k}}(*) = -H_k \frac{\delta\sigma_{\vec{k}}^Q(*)}{\dot{\sigma}_k} = \mathcal{R}_{\vec{k}}(*, \text{adi}) + \mathcal{R}_{\vec{k}}(*, \text{iso}) \quad (12.75)$$

where the first contribution, from the adiabatic mode, stays constant and the second contribution, from the isocurvature mode, decays away as we come to the primordial epoch:

$$\begin{aligned} \mathcal{R}_{\vec{k}}(\text{rad}) &= \mathcal{R}_{\vec{k}}(*, \text{adi}) = -C_1(\vec{k}) \\ &= -H_k \frac{\delta\sigma_{\vec{k}}^N(*)}{\dot{\sigma}_k} + \left(\frac{V_1(*)}{V(*)} \tan\theta_k - \frac{V_2(*)}{V(*)} \cot\theta_k \right) H_k \frac{\delta s_{\vec{k}}(*)}{\dot{\sigma}_k}. \end{aligned} \quad (12.76)$$

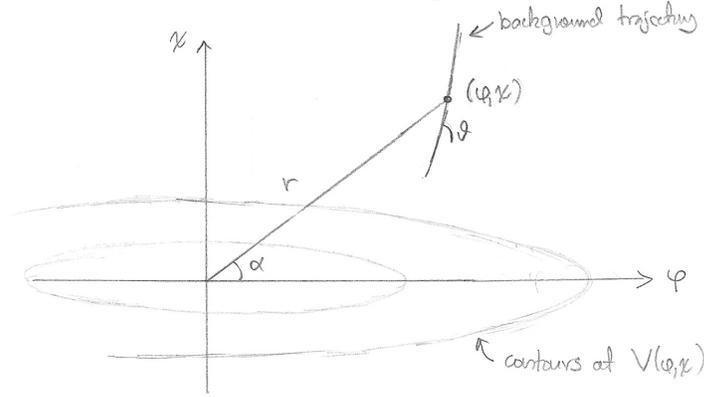


Figure 9: Double inflation field space with cartesian and polar coordinates.

Thus, in the language of Sec. 10,

$$\mathcal{R}_{\vec{k}}(\text{rad}) = \mathcal{R}_{\vec{k}}(*) + T_{\mathcal{R}S}S_{\vec{k}}(*) = \mathcal{R}_{\vec{k}}(*, \text{adi}) \quad (12.77)$$

so that

$$T_{\mathcal{R}S}S_{\vec{k}}(*) = -\mathcal{R}_{\vec{k}}(*, \text{iso}). \quad (12.78)$$

We calculate to lowest slow-roll order, so we can approximate

$$\mathcal{R}_{\vec{k}}(*) = -H_k \frac{\delta\sigma_{\vec{k}}^N(*)}{\dot{\sigma}_{\vec{k}}} (1 + \varepsilon) \approx -H_k \frac{\delta\sigma_{\vec{k}}^N(*)}{\dot{\sigma}_{\vec{k}}} \quad (12.79)$$

Thus we see that the isocurvature mode contribution to the generated $\mathcal{R}_{\vec{k}}(*)$ is

$$\mathcal{R}_{\vec{k}}(*, \text{iso}) = \mathcal{R}_{\vec{k}}(*) - \mathcal{R}_{\vec{k}}(*, \text{adi}) = \left(-\frac{V_1(*)}{V(*)} \tan \theta_k + \frac{V_2(*)}{V(*)} \cot \theta_k \right) S_{\vec{k}}(*), \quad (12.80)$$

and

$$T_{\mathcal{R}S}(k) = \frac{V_1(*)}{V(*)} \tan \theta_k - \frac{V_2(*)}{V(*)} \cot \theta_k. \quad (12.81)$$

To recap, the key feature, or assumption, in the above model was that there were two contributions to the energy density that interact only via gravity, and that the second contribution (φ_2 which decayed into CDM) became negligible compared to the first one during inflation and remained negligible all the way to the primordial epoch, becoming important again only as we approach matter domination. This meant that the second contribution had no effect on spacetime curvature (on Φ or \mathcal{R}) from the end of inflation to the primordial epoch.

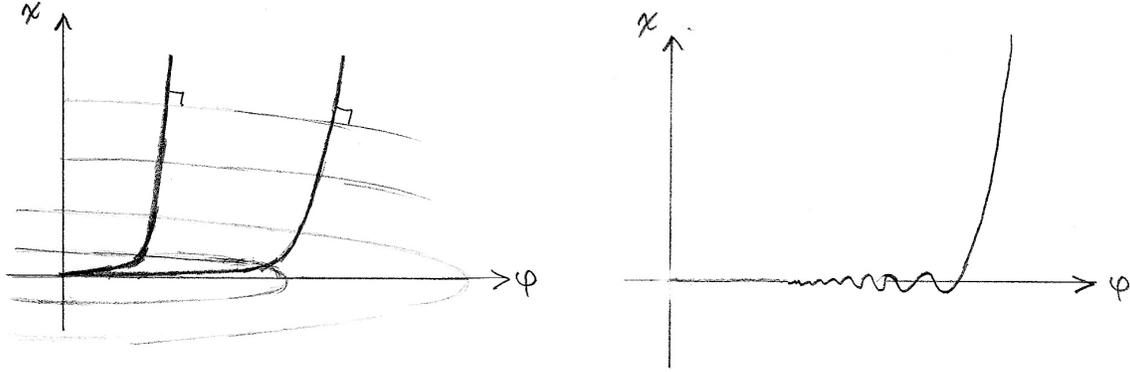
13 Double Inflation

After the earlier more general discussion of two-field inflation, it is good to look at a specific example.

In this section we consider the simplest nontrivial two-field inflation model,

$$V(\varphi, \chi) = \frac{1}{2}m_\varphi^2\varphi^2 + \frac{1}{2}m_\chi^2\chi^2 \quad (13.1)$$

where $m_\varphi < m_\chi$. (The “trivial” case $m_\varphi = m_\chi$ has straight background trajectories.) If $m_\varphi \ll m_\chi$, this model leads to “double inflation”, i.e., there are two periods of inflation: the

Figure 10: Left: Slow-roll trajectories. Right: Slow roll fails for χ .

first one driven by χ , the second by φ . (This is just one example of double inflation. Double inflation models were first discussed in [19].) This is a case of noninteracting fields, so we base our discussion on Sec. 12.

Historical note. Silk and Turner [19] introduced the idea of “double inflation” in 1987 to solve the apparent problem that the standard CDM model of that time predicted less power at large scales than observed, when the power spectrum was normalized to observations at small scales. Their idea was that there were two periods of inflation, the first one responsible for generating structure at large scales and the second one at small scales. This original motivation for double inflation disappeared with the discovery of the acceleration of the expansion of the universe at late times, since this inhibits the growth of structure at large scales.

We have thus

$$V'_\varphi = m_\varphi^2 \varphi, \quad V'_\chi = m_\chi^2 \chi, \quad V''_\varphi = m_\varphi^2, \quad V''_\chi = m_\chi^2. \quad (13.2)$$

We define

$$R \equiv \frac{m_\chi}{m_\varphi} > 1. \quad (13.3)$$

Without loss of generality we assume that initially both $\varphi, \chi > 0$. We also define polar coordinates r, α in field space:

$$\begin{aligned} r &\equiv \sqrt{\varphi^2 + \chi^2} \\ \tan \alpha &\equiv \frac{\chi}{\varphi}, \end{aligned} \quad (13.4)$$

so that

$$\varphi = r \cos \alpha \quad \text{and} \quad \chi = r \sin \alpha. \quad (13.5)$$

Note that $\alpha \neq \theta \equiv \dot{\chi}/\dot{\varphi}$. See Fig. 9.

We shall mostly calculate in this original $\vec{\varphi} = (\varphi, \chi)$ -basis, i.e., not rotating into (σ, s) . The background equations are

$$H^2 = \frac{8\pi G}{3} \rho = \frac{4\pi G}{3} (\dot{\varphi}^2 + \dot{\chi}^2 + m_\varphi^2 \varphi^2 + m_\chi^2 \chi^2) \quad (13.6)$$

$$\ddot{\varphi} + 3H\dot{\varphi} + m_\varphi^2 \varphi = 0 \quad (13.7)$$

$$\ddot{\chi} + 3H\dot{\chi} + m_\chi^2 \chi = 0$$

13.1 Background Equations in Slow-Roll Approximation

We assume that initially both fields are in the slow roll regime. In the slow-roll approximation the background equations are

$$H^2 = \frac{1}{3M^2}V = \frac{1}{6M^2}(m_\varphi^2\varphi^2 + m_\chi^2\chi^2) \Rightarrow V = 3M^2H^2 \quad (13.8)$$

$$3H\dot{\varphi} + m_\varphi^2\varphi = 0 \Rightarrow \dot{\varphi} = -\frac{m_\varphi^2}{3H}\varphi \Rightarrow H^{-1}\dot{\varphi} = -M^2\frac{m_\varphi^2}{V}\varphi \quad (13.9)$$

$$3H\dot{\chi} + m_\chi^2\chi = 0 \Rightarrow \dot{\chi} = -\frac{m_\chi^2}{3H}\chi \Rightarrow H^{-1}\dot{\chi} = -M^2\frac{m_\chi^2}{V}\chi$$

Since $\vec{\dot{\varphi}} \propto -\nabla V$, we can immediately draw the family of slow-roll trajectories: they are everywhere orthogonal to the $V(\varphi, \chi)$ contours (see Fig. 10). In reality, χ may come out of the slow-roll regime when it becomes small, and begin to oscillate, while the slow-roll approximation remains valid for φ (we'll discuss this later).

N of e-foldings for the single-field case. Before giving the two-field slow-roll solution, recall the one-field case $V = V(\varphi) = \frac{1}{2}m^2\varphi^2$:

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{m^2}{6M^2}\varphi^2 \Rightarrow \frac{da}{a} = \frac{1}{\sqrt{6}}\frac{m}{M}\varphi dt \quad \text{and} \quad \frac{\varphi}{H} = \sqrt{6}\frac{M}{m} = \text{const} \quad (13.10)$$

$$\dot{\varphi} = -\frac{m^2}{3}\frac{\varphi}{H} = -\sqrt{\frac{2}{3}}Mm \Rightarrow \varphi = \sqrt{\frac{2}{3}}Mm(t_{\text{end}} - t) \quad (13.11)$$

$$d \ln a = \frac{1}{\sqrt{6}}\frac{m}{M}\varphi dt = -\frac{1}{3}m^2(t - t_{\text{end}})dt, \quad (13.12)$$

and we get for N , the remaining number of e-foldings of inflation,

$$N \equiv -\ln \frac{a}{a_{\text{end}}} = \frac{1}{6}m^2(t - t_{\text{end}})^2 \quad (13.13)$$

and, using N as a “time coordinate”,

$$\varphi = 2M\sqrt{N}. \quad (13.14)$$

Here t_{end} denotes the time when $\varphi = 0$, assuming the slow-roll solution holds until then. In reality, the slow-roll regime, and inflation, ends a little earlier.

The solution of the background slow-roll equations in terms of $r(N)$ and $\alpha(N)$ – or rather $N(\alpha)$ are (**exercise**)

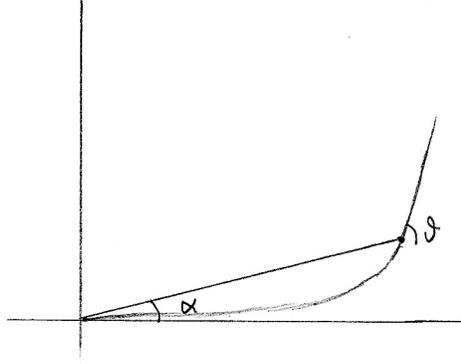
$$\begin{aligned} r &= 2M\sqrt{N} \\ N &= N_0 \frac{(\sin \alpha)^{2/(R^2-1)}}{(\cos \alpha)^{2R^2/(R^2-1)}} = N_0(1 + \tan^2 \alpha)(\tan \alpha)^{2/(R^2-1)}. \end{aligned} \quad (13.15)$$

We see that the solution for $r(N)$ is exactly that of single-field inflation. The constant N_0 corresponds roughly to the value of N when $V_\varphi = V_\chi$, see below. The field “velocity angle” θ and “position angle” α are related by

$$\tan \theta \equiv \frac{\dot{\chi}}{\dot{\varphi}} = \frac{m_\chi^2\chi}{m_\varphi^2\varphi} = R^2 \tan \alpha. \quad (13.16)$$

For all slow-roll trajectories (except the trivial $\varphi = 0$ and $\chi = 0$ trajectories), there is an early χ -dominated part, $V_\chi > V_\varphi$ and a late φ -dominated part, $V_\varphi > V_\chi$. The transition, $V_\varphi = V_\chi$ occurs when

$$m_\varphi^2\varphi^2 = m_\chi^2\chi^2 \Rightarrow \tan \alpha \equiv \frac{\chi}{\varphi} = \frac{m_\varphi}{m_\chi} \equiv \frac{1}{R} \Rightarrow \tan \theta = R \quad (13.17)$$

Figure 11: The transition point $V_\varphi = V_\chi$ for $R = 4$.

(see Fig. 11.) Thus we have that $\chi/\varphi = \dot{\varphi}/\dot{\chi} = 1/R$ at this instant. From Eq. (13.15) we get then

$$N = N_0 \frac{1 + \frac{1}{R^2}}{R^{\frac{2}{R^2-1}}} \Rightarrow \ln \frac{N_0}{N} = \frac{2}{R^2-1} \ln R - \ln \left(1 + \frac{1}{R^2} \right). \quad (13.18)$$

For $R \gg 1$, $\ln(N_0/N) \approx (2 \ln R - 1)/R^2 \ll 1 \Rightarrow N_0 \approx N$.

Note that $\tan \alpha \equiv \chi/\varphi \rightarrow 0$ as $N \rightarrow 0$.

Our model has three parameters: m_ϕ and m_χ , specifying the potential, and N_0 , specifying the trajectory we are on. For the following discussion we want these parameters to have such values that things happen in the following order:

1. The universe becomes φ -dominated, $V_\varphi > V_\chi$, while both fields are still in the slow-roll regime.
2. Some time later, χ comes out of the slow-roll regime, while the slow-roll approximation still remains valid for φ .

The slow-roll parameters are

$$\begin{aligned} \varepsilon_\varphi &\equiv \frac{M^2}{2} \left(\frac{V'_\varphi}{V} \right)^2 = 2M^2 \frac{m_\varphi^4 \varphi^2}{(m_\varphi^2 \varphi^2 + m_\chi^2 \chi^2)^2} = 2 \left(\frac{M}{\varphi} \right)^2 \frac{1}{1 + 2R^2 \left(\frac{\chi}{\varphi} \right)^2 + R^4 \left(\frac{\chi}{\varphi} \right)^4} \\ \varepsilon_\chi &\equiv \frac{M^2}{2} \left(\frac{V'_\chi}{V} \right)^2 = R^4 \left(\frac{\chi}{\varphi} \right)^2 \varepsilon_\varphi \\ \eta_\varphi &\equiv M^2 \frac{V''_\varphi}{V} = 2M^2 \frac{m_\varphi^2}{m_\varphi^2 \varphi^2 + m_\chi^2 \chi^2} = 2 \left(\frac{M}{\varphi} \right)^2 \frac{1}{1 + R^2 \left(\frac{\chi}{\varphi} \right)^2} \\ \eta_\chi &\equiv M^2 \frac{V''_\chi}{V} = R^2 \eta_\varphi. \end{aligned} \quad (13.19)$$

Slow roll is valid for both fields, when $\varepsilon_\varphi, \varepsilon_\chi, |\eta_\varphi|, |\eta_\chi| \ll 1$. The slow-roll equation $V = 3M^2 H^2$ remains valid as long as slow roll is valid for the dominant field.

When $V_\varphi = V_\chi$, we have $\chi/\varphi = 1/R$, so that

$$\begin{aligned} \varepsilon_\varphi &= \frac{1}{2} \left(\frac{M}{\varphi} \right)^2 & \varepsilon_\chi &= \frac{1}{2} \left(\frac{RM}{\varphi} \right)^2 \\ \eta_\varphi &= \left(\frac{M}{\varphi} \right)^2 & \eta_\chi &= \left(\frac{RM}{\varphi} \right)^2. \end{aligned}$$

For slow roll to be valid, the largest of these, η_χ , has to be $< 1 \Rightarrow \varphi > RM$. For $R \gg 1$, we have $r \equiv \sqrt{\varphi^2 + \chi^2} = \varphi\sqrt{1 + 1/R^2} \approx \varphi$ and $N \approx N_0$ then, so that we have the condition

$$\varphi \approx r \approx 2M\sqrt{N_0} > RM \Rightarrow N_0 > \frac{1}{4}R^2 \quad (13.20)$$

for slow roll to be valid when $V_\varphi = V_\chi$. Since χ is then falling more rapidly, $\dot{\chi}/\chi = R^2\dot{\varphi}/\varphi$,

$$\frac{\dot{V}_\chi}{V_\chi} = \frac{V'_\chi \dot{\chi}}{V_\chi \chi} = \frac{m_\chi^2 \chi \dot{\chi}}{m_\varphi^2 \varphi \dot{\varphi}} = R^2 \frac{1}{R} = R^2 \gg 1 \quad (13.21)$$

the universe then becomes φ -dominated, $V_\varphi > V_\chi$ and $\chi/\varphi < 1/R$.

Once $V_\varphi \gg V_\chi$, we have $\chi/\varphi \ll 1/R$, and we can approximate

$$\varepsilon_\varphi \approx \eta_\varphi \approx 2 \left(\frac{M}{\varphi}\right)^2, \quad \varepsilon_\chi \approx 2 \left(\frac{M}{\varphi}\right)^2 R^4 \left(\frac{\chi}{\varphi}\right)^2 \ll 2 \left(\frac{M}{\varphi}\right)^2 R^2, \quad \eta_\chi \approx 2 \left(\frac{M}{\varphi}\right)^2 R^2. \quad (13.22)$$

Of these, η_χ is the largest \Rightarrow slow roll fails first for the χ field. This happens when $\varphi \approx \sqrt{2}RM$. Thus our requirement 2 (χ comes out of the slow-roll regime first) follows from our requirement 1 (slow roll valid when universe becomes φ -dominated). Both are satisfied when the parameters of the model satisfy

$$N_0 > \frac{1}{4}R^2. \quad (13.23)$$

(To clarify this condition, should look at it graphically: draw the line that corresponds to $V_\varphi = V_\chi$, and the $\varphi = RM$ vertical. This N_0 form is useful since it gives information about which regime the cosmologically relevant scales come from. If R is too large, we are led to models where only the second inflation is cosmologically relevant.)

13.2 Perturbations

From Eqs. (12.34) and (12.35) we have the general slow-roll solution for the perturbations:

$$\Phi = -C_1 \frac{\dot{H}}{H^2} + \frac{2}{3} C_3 \frac{(m_\chi^2 - m_\varphi^2) m_\chi^2 m_\varphi^2 \chi^2 \varphi^2}{(m_\chi^2 \chi^2 + m_\varphi^2 \varphi^2)^2} \quad (13.24)$$

$$\frac{\delta\varphi^N}{\dot{\varphi}} = C_1 \frac{1}{H} - 2C_3 H \frac{m_\chi^2 \chi^2}{m_\chi^2 \chi^2 + m_\varphi^2 \varphi^2} \quad (13.25)$$

$$\frac{\delta\chi^N}{\dot{\chi}} = C_1 \frac{1}{H} + 2C_3 H \frac{m_\varphi^2 \varphi^2}{m_\chi^2 \chi^2 + m_\varphi^2 \varphi^2},$$

where

$$-\frac{\dot{H}}{H^2} \approx \varepsilon = 2M^2 \frac{m_\varphi^4 \varphi^2 + m_\chi^4 \chi^2}{(m_\varphi^2 \varphi^2 + m_\chi^2 \chi^2)^2} \quad (13.26)$$

and the coefficients are, from Eq. (12.47),

$$\begin{aligned} C_1(\vec{k}) &= -\frac{H_k}{2M^2 \sqrt{2k^3}} (\varphi_k e_{1\vec{k}} + \chi_k e_{2\vec{k}}) \\ C_3(\vec{k}) &= \frac{3H_k}{2\sqrt{2k^3}} \left(\frac{e_{1\vec{k}}}{m_\varphi^2 \varphi_k} - \frac{e_{2\vec{k}}}{m_\chi^2 \chi_k} \right), \end{aligned} \quad (13.27)$$

where (φ_k, χ_k) is the background field value when scale k exited the horizon.

At late times, when $V_\varphi \gg V_\chi$, the contribution of the isocurvature part (C_3) to Φ shrinks as $(\chi/\varphi)^2$ (as long as the slow-roll approximation is valid) and eventually becomes negligible¹⁹. Likewise, its contribution to \mathcal{R} becomes negligible.

We assume that the χ field decays into CDM whereas the φ field decays into standard model particles, which behave like radiation during the primordial epoch. When the universe eventually becomes matter dominated the CDM isocurvature mode begins again to contribute to Φ and \mathcal{R} . Thus the mode we are calling the isocurvature mode gets its name from not contributing to \mathcal{R} during the primordial epoch, although it did contribute to it earlier, during inflation, and again contributes to it later, when the universe becomes matter dominated.

For the transfer functions we get, using the results of Sec. 12,

$$\begin{aligned} T_{\mathcal{R}S}(k) &= \frac{V_1(*)}{V(*)} \tan \theta_k - \frac{V_2(*)}{V(*)} \cot \theta_k = \frac{(R^2 - 1) \tan \alpha_k}{1 + R^2 \tan \alpha_k} \\ T_{SS}(k) &= -\frac{2}{3} \left(\frac{m_\chi}{H_k} \right)^2 (\cot \theta_k + \tan \theta_k) = -\frac{1}{N_k \cos \alpha_k \sin \alpha_k} \frac{1 + R^4 \tan^2 \alpha_k}{1 + R^2 \tan^2 \alpha_k}. \end{aligned} \quad (13.28)$$

The values α_k and N_k are related by Eq. (13.15b). Using this relation we could obtain the k -dependence of the transfer functions

$$\frac{dT(k)}{d \ln k} = \frac{dT(k)}{d \ln(a_k H_k)} \approx \frac{dT(k)}{d \ln a_k} = -\frac{dT(k)}{d N_k}. \quad (13.29)$$

14 Curvaton

The curvaton scenario is a two-field model, where one field, the inflaton φ , is responsible for inflation, whereas another field, the curvaton χ , is responsible for the primordial perturbations. We follow here Lyth et al.[23].²⁰

We assume that during inflation the energy density in the inflaton field dominates, and that the perturbations in the inflaton field are so small that we can make an approximation where we ignore them. We assume that the perturbations in the curvaton field are larger, but because the energy density in the curvaton is subdominant during inflation, we can make an approximation where we also ignore the effect of the curvaton perturbations on the spacetime curvature *during inflation*. Thus we can approximate that *the spacetime is unperturbed* during inflation.

We assume the curvaton field is practically free during inflation, with $m^2 \ll H^2$, i.e. $|V_{\chi\chi}| \ll H^2$ ($\eta_{\chi\chi} \ll 1$). Therefore, during inflation, Gaussian perturbations in the curvaton field are generated at horizon exit, with spectral index

$$n_\chi = -2\varepsilon + 2\eta_{\chi\chi}. \quad (14.1)$$

During inflation (and as long as we can ignore perturbations in spacetime curvature), the field equations for the curvaton background and perturbation are

$$\ddot{\chi} + 3H\dot{\chi} + V_\chi = 0 \quad (14.2)$$

$$\delta\ddot{\chi} + 3H\delta\dot{\chi} + V_{\chi\chi}\delta\chi = 0. \quad (14.3)$$

¹⁹The slow-roll approximation for χ does not necessarily stay valid long enough for the C_3 contribution to Φ to shrink even faster after slow roll ends for χ . The time evolution of the C_1 part is not so easy to characterize as it is $\propto \varepsilon = \varepsilon_\varphi + \varepsilon_\chi$, where ε_φ grows with time, but what about ε_χ ? Even if the ε_χ part is shrinking, it does not shrink as fast as $(\chi/\varphi)^2$, since it is $\propto (\chi/\varphi)^2 \varepsilon_\varphi$ with ε_φ growing.

²⁰In the literature, the curvaton field is commonly denoted by σ . To avoid confusion with Sec. 10, we denote it by χ .

Thus, if either V is quadratic ($\Rightarrow V_\chi = m\chi$, $V_{\chi\chi} = m$), or sufficiently flat ($V_\chi, V_{\chi\chi} \approx 0$), the ratio $\delta\chi/\chi$ stays constant. Otherwise it will evolve, until the field enters a region of V where it is sufficiently quadratic. Denote by q the factor by which the ratio changed in this evolution. This does not change the spectrum of the inflaton (as long as linear perturbation theory is valid).

During inflation, the curvaton rolls slowly down its potential. Once H falls below the curvaton mass m , the curvaton begins to oscillate. We assume this happens after inflation and reheating (of the inflaton), i.e., the energy of the inflaton field has been converted into radiation, whose energy density we denote ρ_r . Assuming the oscillation stays within a region of the potential where it is sufficiently quadratic, the ratio

$$\frac{\delta\chi}{\chi} = q \left(\frac{\delta\chi}{\chi} \right)_* \quad (14.4)$$

remains constant (in time) in this oscillation (it is of course inhomogeneous, being a perturbation).

The energy density of an oscillating field is determined by the oscillation amplitude, $\chi_{\text{amp}} = \bar{\chi}_{\text{amp}} + \delta\chi_{\text{amp}}$:

$$\rho_\chi = \frac{1}{2}m^2\chi_{\text{amp}}^2 = \frac{1}{2}m^2(\bar{\chi}_{\text{amp}}^2 + 2\bar{\chi}_{\text{amp}}\delta\chi_{\text{amp}} + \delta\chi_{\text{amp}}^2) \quad (14.5)$$

$$\bar{\rho}_\chi = \frac{1}{2}m^2\bar{\chi}_{\text{amp}}^2 \quad (14.6)$$

$$\delta\rho_\chi = \frac{1}{2}m^2(2\bar{\chi}_{\text{amp}}\delta\chi_{\text{amp}} + \delta\chi_{\text{amp}}^2) \quad (14.7)$$

$$\delta_\chi \equiv \frac{\delta\rho_\chi}{\bar{\rho}_\chi} = 2\frac{\delta\chi_{\text{amp}}}{\bar{\chi}_{\text{amp}}} + \left(\frac{\delta\chi_{\text{amp}}}{\bar{\chi}_{\text{amp}}} \right)^2. \quad (14.8)$$

The perturbations $\delta\chi$ are Gaussian, but the second term in the equation for δ_χ is a square of a Gaussian, which is non-Gaussian. Since the perturbations δ_χ have to be small (they are constrained by the observed magnitude of primordial perturbations), the ratio $\delta\chi_{\text{amp}}/\bar{\chi}_{\text{amp}}$ has to be small, and therefore the second term has to be small compared to the first, so that we can approximate

$$\delta_\chi \approx 2\frac{\delta\chi_{\text{amp}}}{\bar{\chi}_{\text{amp}}} \approx 2q \left(\frac{\delta\chi}{\chi} \right)_*. \quad (14.9)$$

Dropping the second term is consistent with sticking with first-order perturbation theory. One might think that keeping the second term would require us to use second-order perturbation theory throughout. However, it may be that $\delta\chi_{\text{amp}}/\bar{\chi}_{\text{amp}}$ is much larger than other perturbation quantities, so that it may make sense to keep the second term here, but still use only first-order perturbation theory for metric perturbations. We return to this later when we discuss non-Gaussianity.

We assume the curvaton oscillations last a long time. In the beginning of this oscillation epoch, $\rho_\chi \ll \rho_r$, but during the oscillation epoch

$$\rho_\chi \propto a^{-3} \quad \text{whereas} \quad \rho_r \propto a^{-4}, \quad (14.10)$$

so that the importance of the curvaton grows. The relevant part of this is, that *the curvaton perturbations become important*—the curvaton background energy density may either become dominant or remain subdominant. At some point the assumption of unperturbed spacetime thus breaks down.

To follow the evolution during the curvaton oscillation epoch, it is useful to define the perturbation quantities

$$\zeta_i \equiv -\psi - \mathcal{H} \frac{\delta\rho_i}{\rho_i} = -\psi + \frac{\delta_i}{3(1+w_i)}, \quad (14.11)$$

i.e., ζ_i is the curvature perturbation of the slice where the component energy density ρ_i is uniform. These are gauge-invariant quantities, so they can be evaluated in any gauge. (Note that until this point we used a gauge where the spacetime was unperturbed, i.e., $\psi = 0$ in this gauge and the ζ_i are proportional to δ_i .)

We already know that the “total” ζ is conserved at superhorizon scales for adiabatic perturbations. The same is true for these “component” ζ_i , if the component is “internally adiabatic”, i.e.,

$$\frac{\delta p_i}{\delta \rho_i} = \frac{\bar{p}'_i}{\bar{\rho}'_i}, \quad (14.12)$$

which is guaranteed when p_i is uniquely determined by ρ_i , and there is no energy transfer between fluid components. (Should show this somewhere.) This is a much more powerful result, since the internal adiabaticity condition is satisfied much more generally than the total adiabaticity condition.

In the curvaton model we assume that during the curvaton oscillation epoch the cosmic fluid consists of two components

$$\rho = \rho_r + \rho_\chi, \quad \text{with} \quad w_r = \frac{1}{3} \quad \text{and} \quad w_\chi = 0 \quad (14.13)$$

that satisfy the above conditions, so that

$$\zeta_r \equiv -\psi + \frac{\delta_r}{4} \quad \text{and} \quad \zeta_\chi \equiv -\psi + \frac{\delta_\chi}{3} \quad (14.14)$$

stay constant, but the total curvature perturbation

$$\zeta = (1 - f_\chi)\zeta_r + f_\chi\zeta_\chi, \quad (14.15)$$

where

$$f_\chi \equiv \frac{\rho_\chi + p_\chi}{\rho + p} = \frac{3\rho_\chi}{4\rho_r + 3\rho_\chi} \quad (14.16)$$

is the curvaton fraction of inertia density, evolves as the curvaton fraction grows,

$$\dot{\zeta} = \dot{f}_\chi(\zeta_\chi - \zeta_r). \quad (14.17)$$

In the curvaton model we assume that $\zeta_r \approx 0$ is negligible, and that initially also the curvaton fraction is negligible, so that in the beginning of the curvaton oscillation period $\zeta \approx 0$, but later it is $\zeta = f_\chi\zeta_\chi$. The constant ζ_χ can be evaluated at the beginning of the oscillation period, using then the gauge where the spacetime was then unperturbed:

$$\zeta_\chi = \frac{1}{3}\delta_\chi = \frac{2q}{3} \left(\frac{\delta\chi}{\chi} \right)_*. \quad (14.18)$$

The curvaton oscillation period ends when the curvaton decays. We assume that the curvaton decay products are eventually thermalized (with the possible exception of CDM), both among themselves and the pre-existing radiation from inflation reheating. (If $f_\chi \approx 1$ when the curvaton decays, we don't have to care about the pre-existing radiation.) This is a second (curvaton) reheating. After curvaton decay there is no more a separate ζ_χ .

In the approximation of sudden decay, there is no time for anything to happen to the curvature perturbation ζ between the end of the curvaton oscillation period and the end of the curvaton reheating, so that we end up with

$$\zeta = f_{\text{dec}}\zeta_\chi = \frac{2}{3}f_{\text{dec}}q \left(\frac{\delta\chi}{\chi} \right)_*. \quad (14.19)$$

More generally, we define r so that ζ after curvaton reheating is given by

$$\zeta = r\zeta_\chi, \quad (14.20)$$

where ζ_χ is from Eq. (14.18), so that

$$\zeta = r\zeta_\chi = \frac{2}{3}rq \left(\frac{\delta\chi}{\chi} \right)_*. \quad (14.21)$$

That is, in the sudden decay approximation, $r = f_{\text{dec}}$. Also otherwise it is (this is presumably backed by numerical simulations) expected to be of the order of ρ_χ/ρ at the time of curvaton decay.

In the curvaton model we assume standard evolution after curvaton reheating, so that ζ is now the primordial curvature perturbation.

An important feature of the curvaton scenario is that it may produce significant non-Gaussianity. A simple kind of non-Gaussianity of primordial perturbations is one where the perturbation is related to a Gaussian perturbation via a simple transformation:

$$\Phi(\vec{x}) = \Phi_G(\vec{x}) - f_{NL}\Phi_G(\vec{x})^2, \quad (14.22)$$

where f_{NL} is called the (local) non-linearity parameter.²¹ Its value gives the level of non-Gaussianity. It is customary to define it in terms of the primordial Bardeen potential Φ . In principle it could have been defined using some other perturbation quantity as well—then it would differ by a numerical factor, and possibly have different sign from this standard definition. Here Φ is the true Bardeen potential, and Φ_G is a quantity with a Gaussian distribution related to Φ in the above way. Since the values of Φ are of the order of 10^{-5} to 10^{-4} , an f_{NL} of the order 1 has a very small effect, unobservable with current methods. A non-Gaussianity of a similar magnitude than the primordial perturbation itself requires f_{NL} of the order 10^4 . The limit from Planck data²² is [17]

$$f_{NL} = -0.9 \pm 5.1 \quad 68\% \text{ CL}. \quad (14.23)$$

In the case of adiabatic perturbations, during the matter-dominated epoch we have at superhorizon scales

$$\Psi = -\frac{3}{5}\zeta \quad (14.24)$$

so Eq. (14.22) becomes

$$\zeta = \zeta_G + \frac{3}{5}f_{NL}\zeta_G^2. \quad (14.25)$$

²¹Note that in the literature there is a lot of confusion about the sign of f_{NL} .

²²An earlier limit from WMAP 7-year data [10] was $-10 < f_{NL} < 74$ (95% CL).

15 Conformal Metric

Often one can simplify the solution of a problem by transforming the problem into different variables.

Given a spacetime with a metric $g_{\mu\nu}$, we may define another metric $\tilde{g}_{\mu\nu}$

$$\tilde{g}_{\mu\nu} \equiv \omega^2 g_{\mu\nu} \quad (15.1)$$

where ω is a function of spacetime position (a scalar field). The factor ω^2 is called the *conformal factor* and $\tilde{g}_{\mu\nu}$ the conformally transformed metric, or, for short, the *conformal metric*. The transformation (15.1) is called a *Weyl transformation*. It may also be called a conformal transformation, but that is a much more general concept; there are many kinds of conformal transformations, in different contexts. It is important not to confuse a Weyl transformation with a coordinate transformation, where we change the coordinate system so that the components of the metric tensor change, but the metric is still the same tensor. A Weyl transformation is not a spacetime coordinate transformation; \tilde{g} and g are really *two different tensor fields*.

From (15.1) we trivially get

$$\tilde{g}^{\mu\nu} \equiv \omega^{-2} g^{\mu\nu}, \quad g_{\mu\nu} \equiv \omega^{-2} \tilde{g}_{\mu\nu}, \quad g^{\mu\nu} \equiv \omega^2 \tilde{g}^{\mu\nu} \quad (15.2)$$

and for the determinant of the metric

$$\tilde{g} = \omega^8 g, \quad g = \omega^{-8} \tilde{g}. \quad (15.3)$$

Note that

$$\tilde{g}^{\alpha\beta} \tilde{g}_{\mu\nu} = g^{\alpha\beta} g_{\mu\nu}. \quad (15.4)$$

We can define connection coefficients and curvature tensors corresponding to the conformal metric. The (Levi–Civita) connection coefficients (in coordinate basis) are defined

$$\Gamma_{\mu\nu}^{\sigma} \equiv \frac{1}{2} g^{\sigma\rho} (\partial_{\mu} g_{\nu\rho} + \partial_{\nu} g_{\mu\rho} - \partial_{\rho} g_{\mu\nu}). \quad (15.5)$$

Thus (**exercise**)

$$\begin{aligned} \tilde{\Gamma}_{\mu\nu}^{\sigma} &\equiv \frac{1}{2} \tilde{g}^{\sigma\rho} (\partial_{\mu} \tilde{g}_{\nu\rho} + \partial_{\nu} \tilde{g}_{\mu\rho} - \partial_{\rho} \tilde{g}_{\mu\nu}) = \dots \\ &= \Gamma_{\mu\nu}^{\sigma} + \frac{1}{\omega} (\delta_{\nu}^{\sigma} \partial_{\mu} \omega + \delta_{\mu}^{\sigma} \partial_{\nu} \omega - g^{\sigma\rho} g_{\mu\nu} \partial_{\rho} \omega). \end{aligned} \quad (15.6)$$

The connection defines a covariant derivative. For a scalar field f we have (**exercise**)

$$\begin{aligned} \tilde{\nabla}_{\mu} f &= \nabla_{\mu} f = \partial_{\mu} f \\ \tilde{\nabla}_{\mu} \tilde{\nabla}_{\nu} f &= \partial_{\mu} \partial_{\nu} f - \tilde{\Gamma}_{\mu\nu}^{\beta} \partial_{\beta} f \\ &= \nabla_{\mu} \nabla_{\nu} f - \frac{1}{\omega} (\delta_{\mu}^{\alpha} \delta_{\nu}^{\beta} + \delta_{\mu}^{\beta} \delta_{\nu}^{\alpha} - g^{\alpha\beta} g_{\mu\nu}) \nabla_{\alpha} \omega \nabla_{\beta} f \\ \nabla_{\mu} \nabla_{\nu} f &= \tilde{\nabla}_{\mu} \tilde{\nabla}_{\nu} f + \frac{1}{\omega} (\delta_{\mu}^{\alpha} \delta_{\nu}^{\beta} + \delta_{\mu}^{\beta} \delta_{\nu}^{\alpha} - \tilde{g}^{\alpha\beta} \tilde{g}_{\mu\nu}) \tilde{\nabla}_{\alpha} \omega \tilde{\nabla}_{\beta} f \\ \square f &\equiv g^{\mu\nu} \nabla_{\mu} \nabla_{\nu} f = \omega^2 \tilde{\square} f - 2\omega \tilde{g}^{\mu\nu} \tilde{\nabla}_{\mu} \omega \tilde{\nabla}_{\nu} f \end{aligned} \quad (15.7)$$

(where $\tilde{\nabla}_{\alpha} \omega = \nabla_{\alpha} \omega = \partial_{\alpha} \omega$).

For the Riemann curvature tensor

$$R^{\rho}_{\sigma\mu\nu} \equiv \partial_{\mu} \Gamma_{\nu\sigma}^{\rho} - \partial_{\nu} \Gamma_{\mu\sigma}^{\rho} + \Gamma_{\mu\lambda}^{\rho} \Gamma_{\nu\sigma}^{\lambda} - \Gamma_{\nu\lambda}^{\rho} \Gamma_{\mu\sigma}^{\lambda} \quad (15.8)$$

we get (**exercise**)

$$\begin{aligned}\tilde{R}^\rho_{\sigma\mu\nu} &\equiv \partial_\mu \tilde{\Gamma}^\rho_{\nu\sigma} - \partial_\nu \tilde{\Gamma}^\rho_{\mu\sigma} + \tilde{\Gamma}^\rho_{\mu\lambda} \tilde{\Gamma}^\lambda_{\nu\sigma} - \tilde{\Gamma}^\rho_{\nu\lambda} \tilde{\Gamma}^\lambda_{\mu\sigma} = \dots \\ &= R^\rho_{\sigma\mu\nu} - \frac{1}{\omega} \left(\delta_\mu^\rho \delta_\nu^\alpha \delta_\sigma^\beta - \delta_\nu^\rho \delta_\mu^\alpha \delta_\sigma^\beta - g_{\sigma\mu} \delta_\nu^\alpha g^{\rho\beta} + g_{\sigma\nu} \delta_\mu^\alpha g^{\rho\beta} \right) \nabla_\alpha \nabla_\beta \omega \\ &\quad + \frac{1}{\omega^2} \left(2\delta_\mu^\rho \delta_\nu^\alpha \delta_\sigma^\beta - 2\delta_\nu^\rho \delta_\mu^\alpha \delta_\sigma^\beta - 2g_{\sigma\mu} \delta_\nu^\alpha g^{\rho\beta} + 2g_{\sigma\nu} \delta_\mu^\alpha g^{\rho\beta} + g_{\mu\sigma} \delta_\nu^\rho g^{\alpha\beta} - g_{\nu\sigma} \delta_\mu^\rho g^{\alpha\beta} \right) \nabla_\alpha \omega \nabla_\beta \omega.\end{aligned}\tag{15.9}$$

Contracting this we get the Ricci tensor (**exercise**)

$$\tilde{R}_{\sigma\nu} \equiv \tilde{R}^\mu_{\sigma\mu\nu} = R_{\sigma\nu} - \frac{1}{\omega} \left(2\delta_\sigma^\alpha \delta_\nu^\beta + g_{\sigma\nu} g^{\alpha\beta} \right) \nabla_\alpha \nabla_\beta \omega + \frac{1}{\omega^2} \left(4\delta_\sigma^\alpha \delta_\nu^\beta - g_{\sigma\nu} g^{\alpha\beta} \right) \nabla_\alpha \omega \nabla_\beta \omega\tag{15.10}$$

and the Ricci scalar (scalar curvature)

$$\tilde{R} \equiv \tilde{g}^{\sigma\nu} \tilde{R}_{\sigma\nu} = \omega^{-2} R - \frac{6}{\omega^3} \square \omega,\tag{15.11}$$

where $\square \equiv g^{\alpha\beta} \nabla_\alpha \nabla_\beta$.

Tools. The calculation of $\tilde{R}^\rho_{\sigma\mu\nu}$ can be shortened by taking advantage of its antisymmetry. For example, one can write

$$\tilde{R}^\rho_{\sigma\mu\nu} \equiv \partial_\mu \tilde{\Gamma}^\rho_{\nu\sigma} + \tilde{\Gamma}^\rho_{\mu\lambda} \tilde{\Gamma}^\lambda_{\nu\sigma} - (\mu \leftrightarrow \nu)\tag{15.12}$$

and calculate just this first half, dropping all terms that are symmetric in $(\mu \leftrightarrow \nu)$. Using this notation, one could shorten (15.9) by half. However, for the contraction (15.10) one has to start from the full (15.9), since the $(\mu \leftrightarrow \nu)$ part contracts differently. Note also that $\delta_\mu^\mu = 4$. (In general, this gives the dimension d of spacetime. There exist generalizations of the above results for arbitrary spacetime dimensions, but we stick here with $d = 4$, since theories with extra dimensions will not be studied in this course.)

In the following sections we will need the inverse transformations, i.e., to express the quantities associated with the original metric in terms of the conformal quantities. One way to derive them would be to replace $\omega \rightarrow 1/\omega$ in the above results, and then do the derivatives of $1/\omega$. However, there is a simpler way: We can just move the “extra parts” in the above equations to the LHS and apply (15.7) to $f = \omega$:

$$\begin{aligned}\tilde{\nabla}_\mu \tilde{\nabla}_\nu \omega &= \partial_\mu \partial_\nu \omega - \tilde{\Gamma}^\beta_{\mu\nu} \partial_\beta \omega \\ &= \nabla_\mu \nabla_\nu \omega - \frac{1}{\omega} \left(\delta_\mu^\alpha \delta_\nu^\beta + \delta_\mu^\beta \delta_\nu^\alpha - g^{\alpha\beta} g_{\mu\nu} \right) \nabla_\alpha \omega \nabla_\beta \omega \\ \nabla_\mu \nabla_\nu \omega &= \tilde{\nabla}_\mu \tilde{\nabla}_\nu \omega + \frac{1}{\omega} \left(\delta_\mu^\alpha \delta_\nu^\beta + \delta_\mu^\beta \delta_\nu^\alpha - \tilde{g}^{\alpha\beta} \tilde{g}_{\mu\nu} \right) \tilde{\nabla}_\alpha \omega \tilde{\nabla}_\beta \omega\end{aligned}\tag{15.13}$$

We get (**exercise**)

$$\begin{aligned}R^\rho_{\sigma\mu\nu} &= \tilde{R}^\rho_{\sigma\mu\nu} + \left[\frac{1}{\omega} \left(\delta_\mu^\rho \delta_\nu^\alpha \delta_\sigma^\beta - \tilde{g}_{\sigma\mu} \delta_\nu^\alpha \tilde{g}^{\rho\beta} \right) \tilde{\nabla}_\alpha \tilde{\nabla}_\beta \omega + \frac{1}{\omega^2} \tilde{g}_{\sigma\mu} \delta_\nu^\rho \tilde{g}^{\alpha\beta} \tilde{\nabla}_\alpha \omega \tilde{\nabla}_\beta \omega - (\mu \leftrightarrow \nu) \right] \\ R_{\sigma\nu} &= \tilde{R}_{\sigma\nu} + \frac{1}{\omega} \left(2\delta_\sigma^\alpha \delta_\nu^\beta + \tilde{g}_{\sigma\nu} \tilde{g}^{\alpha\beta} \right) \tilde{\nabla}_\alpha \tilde{\nabla}_\beta \omega - \frac{3}{\omega^2} \tilde{g}_{\sigma\nu} \tilde{g}^{\alpha\beta} \tilde{\nabla}_\alpha \omega \tilde{\nabla}_\beta \omega \\ R &= \omega^2 \tilde{R} + 6\omega \tilde{g}^{\alpha\beta} \tilde{\nabla}_\alpha \tilde{\nabla}_\beta \omega - 12\tilde{g}^{\alpha\beta} \tilde{\nabla}_\alpha \omega \tilde{\nabla}_\beta \omega\end{aligned}\tag{15.14}$$

and for the Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \tilde{G}_{\mu\nu} + \frac{1}{\omega} \left(2\delta_\mu^\alpha \delta_\nu^\beta - 2\tilde{g}_{\mu\nu} \tilde{g}^{\alpha\beta} \right) \tilde{\nabla}_\alpha \tilde{\nabla}_\beta \omega + \frac{3}{\omega^2} \tilde{g}_{\mu\nu} \tilde{g}^{\alpha\beta} \tilde{\nabla}_\alpha \omega \tilde{\nabla}_\beta \omega.\tag{15.15}$$

By a suitable choice of ω we may convert equations (of a gravity theory) for $g_{\mu\nu}$ into easier equations for $\tilde{g}_{\mu\nu}$.

16 Scalar-Tensor Theories

Scalar-tensor theories are an important class of Modified Gravity theories. They start from an action

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2} f(\varphi) R - \frac{1}{2} g^{\mu\nu} \nabla_\mu \varphi \nabla_\nu \varphi - V(\varphi) + \mathcal{L}_{\text{mat}} \right], \quad (16.1)$$

where R is the scalar curvature, φ a scalar field, and \mathcal{L}_{mat} represents the rest of physics, of which we are now not so interested in. We get ordinary General Relativity with a scalar field, if $f(\varphi) = \text{const}$, since then the first term gives the Hilbert action. This constant determines the strength of gravity, i.e., the gravitational constant G , by $f(\varphi) = 1/(8\pi G) = M_{\text{Pl}}^2$, where

$$M_{\text{Pl}} \equiv \frac{1}{\sqrt{8\pi G}} = 2.4353 \times 10^{18} \text{ GeV} \quad (16.2)$$

is the (reduced) Planck mass.²³

These theories are called scalar-tensor theories, since with the $\frac{1}{2} f(\varphi) R$ in the action the scalar field φ affects gravity in additional ways besides contributing to the energy tensor. One way to look at this is to think of $f(\varphi)$ representing a “gravitational constant” that is not a constant. The “tensor” in “scalar-tensor” is the metric $g_{\mu\nu}$.

Consider now a Weyl transformation $\tilde{g}_{\mu\nu} = \omega^2 g_{\mu\nu}$ with

$$\omega^2 = \frac{f(\varphi)}{M_{\text{Pl}}^2}. \quad (16.3)$$

Using (15.3) and (15.14) we have (**exercise**)

$$\begin{aligned} \frac{1}{2} \sqrt{-g} f(\varphi) R &= \sqrt{-\tilde{g}} \left\{ \frac{1}{2} M_{\text{Pl}}^2 \tilde{R} + \frac{1}{2} M_{\text{Pl}}^2 \left[3 \frac{f''}{f} - \frac{9}{2} \left(\frac{f'}{f} \right)^2 \right] \tilde{g}^{\mu\nu} \tilde{\nabla}_\mu \varphi \tilde{\nabla}_\nu \varphi + \frac{3}{2} M_{\text{Pl}}^2 \frac{f'}{f} \tilde{g}^{\mu\nu} \tilde{\nabla}_\mu \tilde{\nabla}_\nu \varphi \right\} \\ -\frac{1}{2} \sqrt{-g} g^{\mu\nu} \nabla_\mu \varphi \nabla_\nu \varphi &= \sqrt{-\tilde{g}} \left\{ -\frac{1}{2} M_{\text{Pl}}^2 \frac{1}{f} \tilde{g}^{\mu\nu} \tilde{\nabla}_\mu \varphi \tilde{\nabla}_\nu \varphi \right\} \\ -\sqrt{-g} V(\varphi) &= -\sqrt{-\tilde{g}} \frac{M_{\text{Pl}}^4}{f^2} V(\varphi), \end{aligned} \quad (16.4)$$

so that

$$\begin{aligned} S &= \int d^4x \sqrt{-\tilde{g}} \left\{ \frac{1}{2} M_{\text{Pl}}^2 \tilde{R} + \frac{1}{2} M_{\text{Pl}}^2 \left[3 \frac{f''}{f} - \frac{9}{2} \left(\frac{f'}{f} \right)^2 - \frac{1}{f} \right] \tilde{g}^{\mu\nu} \tilde{\nabla}_\mu \varphi \tilde{\nabla}_\nu \varphi \right. \\ &\quad \left. + \frac{3}{2} M_{\text{Pl}}^2 \frac{f'}{f} \tilde{g}^{\mu\nu} \tilde{\nabla}_\mu \tilde{\nabla}_\nu \varphi - \frac{M_{\text{Pl}}^4}{f^2} V(\varphi) + \tilde{\mathcal{L}}_{\text{mat}} \right\}. \end{aligned} \quad (16.5)$$

The term with $\tilde{\nabla}_\mu \tilde{\nabla}_\nu \varphi$ can be converted to 1st derivatives by partial integration

$$\int d^4x \sqrt{-\tilde{g}} \frac{f'}{f} \tilde{g}^{\mu\nu} \tilde{\nabla}_\mu \tilde{\nabla}_\nu \varphi = \int_{\partial\Sigma} \sqrt{|\gamma|} d^3\sigma \frac{f'}{f} \tilde{g}^{\mu\nu} n_\mu \tilde{\nabla}_\nu \varphi - \int d^4x \sqrt{-\tilde{g}} \tilde{g}^{\mu\nu} \tilde{\nabla}_\mu \left(\frac{f'}{f} \right) \tilde{\nabla}_\nu \varphi,$$

where we can ignore the boundary term, since its variation vanishes²⁴, and

$$\tilde{\nabla}_\mu \left(\frac{f'}{f} \right) = \left[\frac{f''}{f} - \left(\frac{f'}{f} \right)^2 \right] \tilde{\nabla}_\mu \varphi, \quad (16.6)$$

²³<https://physics.nist.gov/cgi-bin/cuu/Value?plkmc2gev> | `search_for=Planck+mass` gives $m_{\text{Pl}} \equiv 1/\sqrt{G} = (1.220890 \pm 0.000014) \times 10^{19} \text{ GeV}$. (2018 CODATA recommended value.)

²⁴Should argue this more carefully.

so that we end up with

$$S = \int d^4x \sqrt{-\tilde{g}} \left\{ \frac{1}{2} M_{\text{Pl}}^2 \tilde{R} - \frac{1}{2} M_{\text{Pl}}^2 \left[\frac{3}{2} \left(\frac{f'}{f} \right)^2 + \frac{1}{f} \right] \tilde{g}^{\mu\nu} \tilde{\nabla}_\mu \varphi \tilde{\nabla}_\nu \varphi - \frac{M_{\text{Pl}}^4}{f^2} V(\varphi) + \tilde{\mathcal{L}}_{\text{mat}} \right\}. \quad (16.7)$$

The result of the Weyl transformation was that the gravitational part of the action has now the form of the standard Hilbert action, so that the conformal metric $\tilde{g}_{\mu\nu}$ will obey the standard Einstein equation! The price of this was that the other parts of the action were modified. In particular, the canonical kinetic term $-\frac{1}{2} g^{\mu\nu} \nabla_\mu \varphi \nabla_\nu \varphi$ was replaced by a non-canonical term. We get back to the canonical form by defining a *transformed field* $\tilde{\varphi}$ with

$$d\tilde{\varphi} = M_{\text{Pl}} \sqrt{\frac{3}{2} \left(\frac{f'}{f} \right)^2 + \frac{1}{f}} d\varphi = M_{\text{Pl}} \sqrt{\frac{2f + 3(f')^2}{2f^2}} d\varphi. \quad (16.8)$$

We also define for it a transformed potential

$$\tilde{V}(\tilde{\varphi}) \equiv \frac{M_{\text{Pl}}^4}{f^2} V(\varphi). \quad (16.9)$$

The integration constant for (16.8), which defines when $\tilde{\varphi} = 0$, can be chosen as convenient.

Thus we finally have

$$S = \int d^4x \sqrt{-\tilde{g}} \left\{ \frac{1}{2} M_{\text{Pl}}^2 \tilde{R} - \frac{1}{2} \tilde{g}^{\mu\nu} \tilde{\nabla}_\mu \tilde{\varphi} \tilde{\nabla}_\nu \tilde{\varphi} - \tilde{V}(\tilde{\varphi}) + \tilde{\mathcal{L}}_{\text{mat}} \right\}. \quad (16.10)$$

We have converted the scalar-tensor theory into ordinary Einstein gravity with a canonical minimally coupled scalar field, but for new variables $\tilde{g}_{\mu\nu}$ and $\tilde{\varphi}$, which are not the observed²⁵ metric and field (if we assume that the original formulation of the theory (16.1) holds for those). Thus any solution in terms of $\tilde{g}_{\mu\nu}$ and $\tilde{\varphi}$ should in the end be converted into $g_{\mu\nu}$ and φ – or into whatever observable quantities we are looking for.

Standard terminology is to call the description in terms of $g_{\mu\nu}$ and φ the *Jordan*²⁶ *frame* and in terms of $\tilde{g}_{\mu\nu}$ and $\tilde{\varphi}$ the *Einstein frame*. They describe the same physics, but in terms of different variables.

As an example, we discuss Higgs inflation in Sec. 17.

17 Higgs Inflation

In the Standard Model of particle physics, Higgs is the only fundamental scalar field. Could Higgs be the inflaton? With standard General Relativity and minimal coupling of Higgs to gravity, we do not get viable inflation (the potential is not sufficiently flat). However, by introducing a non-minimal coupling, we get a promising inflation candidate, *Higgs inflation* [24].

In the Standard Model, Higgs field ϕ is an SU(2) doublet and has two complex components

$$\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} \varphi_1 + i\varphi_2 \\ \varphi_3 + i\varphi_4 \end{pmatrix}. \quad (17.1)$$

²⁵This is loose talk. One should consider what are the true observables, and how they are affected by modified gravity. In cosmology observables are, e.g., redshifts, apparent positions, and apparent luminosities of galaxies. These are related to the metric; but one should also consider whether the motions of light and massive particles are affected otherwise in the theory. In Higgs inflation this is not an issue, as we will see in Sec. 17.

²⁶Pascual Jordan (1902–1980), German theoretical physicist.

The potential is

$$V(\phi) = \frac{1}{4} \frac{\mu^4}{\lambda} - \mu^2 \phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2, \quad (17.2)$$

where

$$\phi^\dagger \phi = |\phi|^2 = \frac{1}{2} (\varphi_1^2 + \varphi_2^2 + \varphi_3^2 + \varphi_4^2). \quad (17.3)$$

The situation is spherically symmetric in the internal 4D space $(\varphi_1, \varphi_2, \varphi_3, \varphi_4)$. The potential has a minimum $V(\phi) = 0$ at

$$|\phi|^2 = \frac{\mu^2}{2\lambda} \equiv \frac{1}{2} \sigma^2. \quad (17.4)$$

For inflation dynamics, only the radial degree of freedom in ϕ is important. Rotate the coordinates in the internal space so that only one of the φ_i is nonzero, and call it φ . Then

$$V(\varphi) = \frac{1}{4} \frac{\mu^4}{\lambda} - \frac{1}{2} \mu^2 \varphi^2 + \frac{1}{4} \lambda \varphi^4 = \frac{1}{4} \lambda (\varphi^2 - \sigma^2)^2. \quad (17.5)$$

After the EW transition, the Higgs field has settled at the minimum, $\varphi = \sigma$, and the Higgs mass is given by

$$m_H^2 = V''(\sigma) = 2\mu^2. \quad (17.6)$$

The coupling of the Higgs with the W and Z bosons gives rise to their masses and thus the strength of the weak interaction, which is described by the Fermi constant G_F . We do not review this part of electroweak theory, but the result is [26]

$$G_F = \frac{1}{\sqrt{2}\sigma^2}. \quad (17.7)$$

From the experimental measurements [26],

$$m_H = 125.10 \pm 0.14 \text{ GeV} \quad \text{and} \quad G_F = 1.16638 \times 10^{-5} \text{ GeV}^2, \quad (17.8)$$

we get

$$\mu = \frac{m_H}{\sqrt{2}} = 88.46 \text{ GeV}, \quad \sigma = \frac{1}{(\sqrt{2}G_F)^{1/2}} = 246.22 \text{ GeV}, \quad \lambda = \left(\frac{\mu}{\sigma}\right)^2 = 0.129. \quad (17.9)$$

We get Higgs inflation by letting the Higgs couple non-minimally to gravity so that the action is

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2} M^2 R + \frac{1}{2} \xi \varphi^2 R - \frac{1}{2} g^{\mu\nu} \nabla_\mu \varphi \nabla_\nu \varphi - V(\varphi) + \mathcal{L}_{\text{SM}} \right], \quad (17.10)$$

where \mathcal{L}_{SM} is the standard model Lagrangian except for the $-\frac{1}{2} g^{\mu\nu} \nabla_\mu \varphi \nabla_\nu \varphi - V(\varphi)$ part for the radial component of the Higgs field. The term $\frac{1}{2} \xi \varphi^2 R$ is the nonminimal coupling of Higgs to gravity.²⁷ For $\xi = 0$ we get the minimal coupling. We will find that for Higgs inflation we need $\xi = \mathcal{O}(10^4)$. To begin, we first assume $1 \ll \xi \ll (M_{\text{Pl}}/\sigma)^2 \sim 10^{32}$ (later we adopt a tighter bound $1 \ll \xi \ll M_{\text{Pl}}/\sigma \sim 10^{16}$).

We see that the action has the form (16.1) of scalar-tensor gravity, with

$$f(\varphi) = M^2 + \xi \varphi^2. \quad (17.11)$$

After the EW transition, $\varphi = \sigma$, so that $f(\varphi) = M^2 + \xi \sigma^2 = \text{const}$, which gives the observed strength of gravity, i.e.,

$$M_{\text{Pl}}^2 = M^2 + \xi \sigma^2. \quad (17.12)$$

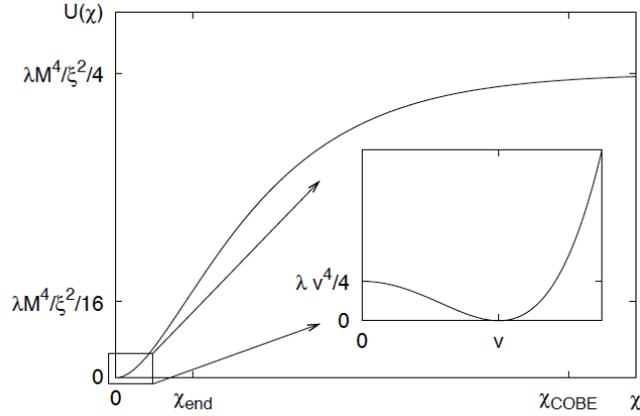


Figure 12: Higgs potential $\tilde{V}(\tilde{\varphi})$ in the Einstein frame. From [24]; here χ , U , v stand for our $\tilde{\varphi}$, \tilde{V} , σ . For $\tilde{\varphi} = \tilde{\varphi}_{\text{end}} = 0$ (end of inflation) (17.18) gives $\lambda M_{\text{Pl}}^4/(16\xi^2)$. There is a range where neither the small-field nor the large-field approximation is valid and we did not make any effort to match then integration constants for $d\tilde{\varphi}/d\varphi$ between these two regimes; matching them would make $\tilde{\varphi}_{\text{end}} > 0$ as in this figure.

Since we assumed $\xi \ll (M_{\text{Pl}}/\sigma)^2$, the difference between M and M_{Pl} is negligible; for $\xi = \mathcal{O}(10^4)$ it is less than 1 eV.²⁸

We now transform to the Einstein frame. From (16.8) and (17.11),

$$d\tilde{\varphi} = M_{\text{Pl}} \frac{\sqrt{M^2 + \xi\varphi^2 + 6\xi^2\varphi^2}}{M^2 + \xi\varphi^2} d\varphi. \quad (17.13)$$

This is difficult to integrate exactly,²⁹ but we can consider various limits.

For small field values $\varphi \ll M/\xi$, we have $\xi\varphi^2 \ll 6\xi^2\varphi^2 \ll M^2$, so that $d\tilde{\varphi} \approx (M_{\text{Pl}}/M)d\varphi \approx d\varphi$, and we can take $\tilde{\varphi} \approx \varphi$. For the Einstein frame potential we get

$$\tilde{V}(\tilde{\varphi}) = \frac{M_{\text{Pl}}^4}{(M^2 + \xi\varphi^2)^2} \frac{1}{4} \lambda (\varphi^2 - \sigma^2)^2 \approx \frac{1}{4} \lambda (\tilde{\varphi}^2 - \sigma^2)^2, \quad (17.15)$$

so that there is no difference between the Einstein frame and the Jordan frame, except that the coupling $\frac{1}{2}\xi\varphi^2 R$ has disappeared (its effect is negligible at these small field values).

We now assume that $\xi \ll M_{\text{Pl}}/\sigma \sim 10^{16}$, so that we are in this small-field limit when $\varphi = \mathcal{O}(\sigma)$ and the non-minimal coupling will not affect the physics of the EW transition. Results from collider experiments at LHC provide an upper limit $\xi \lesssim 10^{15}$ [28, 29].

Inflation will happen at large field values where the Einstein and Jordan frames are quite different, but as inflation ends, φ moves to small field values, and any results we calculated in the Einstein frame will apply as such in the Jordan frame as long as they are about these late times when φ is already small.

²⁷This can be motivated by quantum effects: Even if such a coupling would not exist in the classical level, quantum corrections would introduce such a term.

²⁸One could drop the distinction between M and M_{Pl} at this point. When preparing these notes I kept the distinction to be prepared for the possibility that it might have some role. Anyway, the distinction serves to remind us that we are dealing with modified gravity where the Higgs field affects the observed gravitational constant (in principle, although in practice the effect is negligible).

²⁹With help from WolframAlpha, I get

$$\frac{\tilde{\varphi}}{M_{\text{Pl}}} = \sqrt{\frac{6\xi + 1}{\xi}} \operatorname{arsinh} \left(\sqrt{6\xi + 1} \sqrt{\xi} \frac{\varphi}{M_{\text{Pl}}} \right) - \sqrt{6} \operatorname{arsinh} \left[\frac{\sqrt{6\xi}(\varphi/M)}{\sqrt{\xi(\varphi/M)^2 + 1}} \right] + \text{const}. \quad (17.14)$$

(WolframAlpha gives a formula in terms of \ln and artanh , but I converted it into arsinh to facilitate comparison to Eq. (2.20) of [27].) One can get the same limits from this.

For large field values $\varphi \gg M/\sqrt{\xi} \gg \sigma$, we have $\xi\varphi^2 \gg M^2$, so that

$$\begin{aligned} d\tilde{\varphi} &\approx \sqrt{1 + \frac{1}{6\xi}} \sqrt{6} M_{\text{Pl}} \frac{d\varphi}{\varphi} \approx \sqrt{6} M_{\text{Pl}} \frac{d\varphi}{\varphi} \Rightarrow \tilde{\varphi} \approx \sqrt{6} M_{\text{Pl}} \ln \frac{\varphi}{\varphi_0} \\ &\Rightarrow \varphi = \varphi_0 e^{\tilde{\varphi}/\sqrt{6} M_{\text{Pl}}} \end{aligned} \quad (17.16)$$

where φ_0 is an integration constant. We will find it convenient to choose $\varphi_0 = M/\sqrt{\xi} \Rightarrow \xi\varphi_0^2 = M^2$, so that

$$\varphi = \frac{M}{\sqrt{\xi}} e^{\tilde{\varphi}/\sqrt{6} M_{\text{Pl}}}. \quad (17.17)$$

Now $\varphi \gg M/\sqrt{\xi}$, so that $e^{\tilde{\varphi}/\sqrt{6} M_{\text{Pl}}} \gg 1$. For the Einstein frame potential we get (**exercise**)

$$\begin{aligned} \tilde{V}(\tilde{\varphi}) &= \frac{M_{\text{Pl}}^4}{(M^2 + \xi\varphi^2)^2} \frac{1}{4} \lambda (\varphi^2 - \sigma^2)^2 \approx \frac{M_{\text{Pl}}^4}{(M^2 + \xi\varphi^2)^2} \frac{1}{4} \lambda \varphi^4 \\ &= \frac{\lambda M_{\text{Pl}}^4}{4\xi^2} \frac{e^{(4/\sqrt{6})(\tilde{\varphi}/M_{\text{Pl}})}}{\left[1 + e^{(2/\sqrt{6})(\tilde{\varphi}/M_{\text{Pl}})}\right]^2} = \frac{\lambda M_{\text{Pl}}^4}{4\xi^2} \left(1 + e^{-2\tilde{\varphi}/\sqrt{6} M_{\text{Pl}}}\right)^{-2}. \end{aligned} \quad (17.18)$$

There is a temptation here to approximate $M^2 + \xi\varphi^2 \approx \xi\varphi^2$; but this would lead to $\tilde{V}(\tilde{\varphi}) \approx \lambda M_{\text{Pl}}^4/4\xi^2 = \text{const}$; although $e^{-2\tilde{\varphi}/\sqrt{6} M_{\text{Pl}}} \ll 1$, it will be important: it gives the small slope for slow-roll inflation. See Fig. 12.

In the Einstein frame, physics behaves just as in the standard treatment of inflation, so all our old results apply in it. We can follow $\tilde{\varphi}$ slowly rolling down the potential \tilde{V} and generating primordial perturbations as different scales exit the horizon, all concepts defined in the Einstein frame. We find for the slow-roll parameters³⁰ (**exercise**)

$$\begin{aligned} \tilde{\epsilon} &\equiv \frac{M_{\text{Pl}}^2}{2} \left(\frac{\tilde{V}'}{\tilde{V}} \right)^2 = \frac{4}{3} \frac{1}{\left(e^{2\tilde{\varphi}/\sqrt{6} M_{\text{Pl}}} + 1\right)^2} = \frac{4}{3} \frac{1}{\left(\frac{\xi\varphi^2}{M^2} + 1\right)^2} \approx \frac{4}{3\xi^2} \left(\frac{\varphi}{M}\right)^{-4} \\ \tilde{\eta} &\equiv M_{\text{Pl}}^2 \frac{\tilde{V}''}{\tilde{V}} = -\frac{4}{3} \frac{e^{-2\tilde{\varphi}/\sqrt{6} M_{\text{Pl}}} \left(1 - 2e^{-2\tilde{\varphi}/\sqrt{6} M_{\text{Pl}}}\right)}{\left(1 + e^{-2\tilde{\varphi}/\sqrt{6} M_{\text{Pl}}}\right)^2} = -\frac{4}{3} \frac{M^2}{\xi\varphi^2} \frac{1 - \frac{2M^2}{\xi\varphi^2}}{\left(1 + \frac{M^2}{\xi\varphi^2}\right)^2} \approx -\frac{4}{3\xi} \left(\frac{\varphi}{M}\right)^{-2} \\ \tilde{\xi} &\equiv M_{\text{Pl}}^4 \frac{\tilde{V}''' \tilde{V}'}{\tilde{V}^2} \approx \frac{16}{9} e^{-4\tilde{\varphi}/\sqrt{6} M_{\text{Pl}}} \approx \frac{16}{9\xi^2} \left(\frac{\varphi}{M}\right)^{-4} \end{aligned} \quad (17.20)$$

expressed both in terms of the transformed $\tilde{\varphi}$ and original φ . We see that inflation ends (at least one of the slow-roll parameters becomes $\mathcal{O}(1)$), when $\varphi \approx M/\sqrt{\xi}$, i.e., when our large-field approximation breaks down.

³⁰For the calculation of $\tilde{\xi}$ we made first a further approximation in (17.18),

$$\frac{\lambda M_{\text{Pl}}^4}{4\xi^2} \left(1 + e^{-2\tilde{\varphi}/\sqrt{6} M_{\text{Pl}}}\right)^{-2} \approx \frac{\lambda M_{\text{Pl}}^4}{4\xi^2} \left(1 - 2e^{-2\tilde{\varphi}/\sqrt{6} M_{\text{Pl}}}\right). \quad (17.19)$$

(I was too lazy to do the third derivative of the original form.) This approximation will also lead directly to the final approximate forms for $\tilde{\epsilon}$ and $\tilde{\eta}$ we give in (17.20). This is not such a good approximation, since during the “observable” part of inflation $e^{-\tilde{\varphi}/\sqrt{6} M_{\text{Pl}}}$ is not that small; it leads to $\mathcal{O}(10\%)$ errors in the slow-roll parameters and \tilde{N} .

The number of remaining e-foldings of inflation is given (Cosmology II) by (**exercise**)

$$\begin{aligned}
\tilde{N}(\tilde{\varphi}) &= \frac{1}{M_{\text{Pl}}^2} \int_{\tilde{\varphi}_{\text{end}}}^{\tilde{\varphi}} \frac{\tilde{V}}{\tilde{V}'} d\tilde{\varphi} \\
&= \frac{3}{4} \left(e^{2\tilde{\varphi}/\sqrt{6}M_{\text{Pl}}} - e^{2\tilde{\varphi}_{\text{end}}/\sqrt{6}M_{\text{Pl}}} \right) + \frac{\sqrt{6}}{4} \frac{1}{M_{\text{Pl}}} (\tilde{\varphi} - \tilde{\varphi}_{\text{end}}) \\
&\approx \frac{3}{4} e^{2\tilde{\varphi}/\sqrt{6}M_{\text{Pl}}} \approx \frac{3}{4} \xi \left(\frac{\varphi}{M} \right)^2
\end{aligned} \tag{17.21}$$

so that we have (do not confuse the slow-roll parameter $\tilde{\xi}$ with the coupling ξ)

$$\tilde{\varepsilon} \approx \frac{3}{4\tilde{N}^2}, \quad \tilde{\eta} \approx -\frac{1}{\tilde{N}}, \quad \tilde{\xi} \approx \frac{1}{\tilde{N}^2}. \tag{17.22}$$

The Einstein and Jordan frame scale factors are related by $\tilde{a} = \bar{\omega}a$, so the numbers of e-foldings will be related by $\tilde{N} = N + \ln \bar{\omega}_{\text{end}}/\bar{\omega}$ (where $\bar{\omega}$ is the background value of the conformal factor). (**Exercise:** find N for $\tilde{N} = 60$.) However, we don't care here about N – we haven't even considered whether we have inflation in the Jordan frame.³¹ The point is that the physics of the Einstein frame is the familiar inflation physics, where we produce a spectrum of primordial perturbations, and once this Einstein frame inflation is over the difference between the two frames disappears and we are left with primordial scalar and tensor perturbation spectra

$$\begin{aligned}
\mathcal{P}_{\mathcal{R}}(k) &= \frac{1}{24\pi^2} \frac{1}{M_{\text{Pl}}^4} \frac{\tilde{V}}{\tilde{\varepsilon}} \approx \frac{\lambda\tilde{N}^2}{72\pi^2\xi^2} \\
\mathcal{P}_T(k) &= \frac{2}{3\pi^2 M_{\text{Pl}}^4} \tilde{V},
\end{aligned} \tag{17.23}$$

whose spectral indices and ratio are given by

$$\begin{aligned}
n_s &= 1 - 6\tilde{\varepsilon} + 2\tilde{\eta} \approx 1 + 2\tilde{\eta} = 1 - \frac{2}{\tilde{N}} \\
n_T &= -2\tilde{\varepsilon} = -\frac{3}{2\tilde{N}^2} \\
r &\equiv \frac{\mathcal{P}_T}{\mathcal{P}_{\mathcal{R}}} = 16\tilde{\varepsilon} = \frac{12}{\tilde{N}^2} \\
\frac{dn_s}{d \ln k} &= 16\tilde{\varepsilon}\tilde{\eta} - 24\tilde{\varepsilon}^2 - 2\tilde{\xi} \approx -2\tilde{\xi} = -\frac{2}{\tilde{N}^2}
\end{aligned} \tag{17.24}$$

to lowest order in slow-roll parameters.

The observed magnitude of primordial perturbations, $\mathcal{P}_{\mathcal{R}} \approx 2.1 \times 10^{-9}$ [13] requires

$$\xi \approx 816\sqrt{\lambda}\tilde{N} \approx 293\tilde{N}. \tag{17.25}$$

For $\tilde{N} = 50$ – 60 this means $\xi \approx 1.47$ – 1.76×10^4 . Since this is the standard model, reheating physics is known: interactions between the Higgs and other particles after end of inflation are strong so that reheating is not delayed³², which means that the scales observable in the CMB correspond to $\tilde{N} \approx 60$, which gives

$$n_s \approx 0.967, \quad n_T \approx -0.00042, \quad r \approx 0.0033, \quad \frac{dn_s}{d \ln k} \approx -0.00056, \tag{17.26}$$

³¹We must have; the coupling of Higgs to gravity does not affect, e.g., the ‘‘horizon problem’’, which inflation solves. The above exercise will show that the difference between N and \tilde{N} is not that large.

³²Citing[25]: ‘‘the Higgs-inflaton field rapidly produces weak bosons, which subsequently decay into all other standard model particles and reheat the Universe, leading to a . . . temperature $T_{\text{reh}} \approx 6 \times 10^{13}$ GeV, with uncertainty factor about two’’. [25] gives $\tilde{N} = 57.66$ for the WMAP pivot scale $k = 0.002/\text{Mpc}$.

which are all in agreement with observations [13]

$$n_s = 0.965 \pm 0.004, \quad r < 0.065, \quad \frac{dn_s}{d \ln k} = -0.004 \pm 0.013. \quad (17.27)$$

There is no observation for n_T , since primordial tensor perturbations have not been observed. A future CMB satellite optimized for polarization observations should be able to detect tensor perturbations corresponding to this value of r , verifying this r prediction (the accuracy of n_T measurement would still be poor).

The non-minimal coupling of Higgs to R can be motivated by quantum effects, i.e., even if it did not exist at the classical level, quantum effects would introduce such coupling. There would be also other quantum effects that could modify the above “tree-level” calculation.

More accurate spectral index. We noted already in the footnote that $e^{-\tilde{\varphi}/\sqrt{6}M_{\text{Pl}}}$ is not that small during the part of inflation that generates the perturbations at observable scales. One sees the results (17.22) given in literature, but they are not that accurate. We can perhaps do better by using the second-to-last forms for $\tilde{\varepsilon}$ and $\tilde{\eta}$ in (17.20) and not dropping the $\frac{\sqrt{6}}{4} \frac{1}{M_{\text{Pl}}} \tilde{\varphi}$ in (17.21). The contribution from the lower limit of the integral in (17.21) is an approximation, since the large-field limit breaks down there, but if we take it at face value and define end of inflation to correspond to $\varphi \approx M/\sqrt{\xi}$, then this gives $\tilde{\varphi}_{\text{end}} \approx 0$ and the lower-limit contribution is $\mathcal{O}(1)$, which we keep ignoring. Thus we have

$$\tilde{N} = \frac{3}{4} \left[e^{2\tilde{\varphi}/\sqrt{6}M_{\text{Pl}}} + \frac{2\tilde{\varphi}}{\sqrt{6}M_{\text{Pl}}} \right] = \frac{3}{4} \left[\xi \left(\frac{\varphi}{M_{\text{Pl}}} \right)^2 + \ln \xi \left(\frac{\varphi}{M_{\text{Pl}}} \right)^2 \right] \equiv \frac{3}{4}(x + \ln x). \quad (17.28)$$

For a given \tilde{N} , we can solve $x = \frac{4}{3}\tilde{N} - \ln x$ by iteration. For $\tilde{N} = 60$,

$$x = 80 - \ln x = 75.62, 75.67, 75.67. \quad (17.29)$$

With $\xi(\varphi/M)^2 = 75.67$, we get $\tilde{\varepsilon} = 0.000227$, $\tilde{\eta} = -0.017610$, and

$$n_s = 1 - 6\tilde{\varepsilon} + 2\tilde{\eta} = 0.9634. \quad (17.30)$$

One should not take all the digits seriously; there is room for further improvement in the accuracy of $d\tilde{\varphi}/d\varphi$ towards the end of inflation. In particular, we have still used the $\xi\varphi^2 \gg M^2$ approximation in going from (17.13) to (17.16). This might cause an error of similar order of magnitude so it is not clear that (17.30) is better than (17.26). For more accurate results one should use (17.14). [25] says “Using exact formulas gives numerically . . . $n_s = 0.967$, $r = 0.0032$ ”. Also it is possible that quantum corrections change these results further.

18 Palatini Variation

In General Relativity the metric defines the connection (2.9). We call this the Christoffel or Levi–Civita connection. In the *Palatini formalism* one assumes that the metric $g_{\mu\nu}$ and the connection $\Gamma_{\beta\gamma}^\alpha$ are independent degrees of freedom and one uses the action principle to derive field equations for both.

What does it mean that the metric and connection are independent?

The metric defines (infinitesimal) distances; it determines the length of spacelike curves and proper time of timelike curves. It defines the light cone, i.e., separates timelike, spacelike, and lightlike directions and separates past from the future. We can use the metric to define a *geodesic* between two spacetime events as the path that extremises the path length or proper time.

The connection defines the covariant derivative $\nabla_\beta v^\alpha = \partial_\beta v^\alpha + \Gamma_{\beta\gamma}^\alpha v^\gamma$ and parallel transport

$$\frac{D}{d\lambda} v^\alpha \equiv \frac{dx^\beta}{d\lambda} \nabla_\beta v^\alpha = \frac{dv^\alpha}{d\lambda} + \Gamma_{\beta\gamma}^\alpha \frac{dx^\beta}{d\lambda} v^\gamma = 0. \quad (18.1)$$

It defines the Riemann and Ricci curvature tensors (2.3). We can use the connection to define a *geodesic* as a curve that parallel transports its tangent vector, i.e., satisfies the *geodesic equation*

$$\frac{D}{d\lambda} \frac{dx^\alpha}{d\lambda} \equiv \frac{d^2 x^\alpha}{d\lambda^2} + \Gamma_{\beta\gamma}^\alpha \frac{dx^\beta}{d\lambda} \frac{dx^\gamma}{d\lambda} = 0. \quad (18.2)$$

For the definition of the scalar (Ricci) curvature

$$R \equiv g^{\mu\nu} R_{\mu\nu} \quad (18.3)$$

both the metric and the connection are needed.

If the connection is the Christoffel connection, the metric and parallel transport definitions of geodesics agree. But if the connection is something else, they do not. This leaves open the question, which geodesics give the paths of freely falling particles. In GR we don't face this question, since the field equation one finds for the connection when one applies the Palatini formalism to the action (2.1) is exactly the one that makes it equal to the Christoffel connection (see Sec. 18.1). But if the action is something else (modified gravity) the Palatini formalism typically leads to a non-Christoffel connection.

Thus for a given modified gravity action we have the possibility for two different modified gravity theories; the one obtained when the connection is assumed to be the Christoffel one (*metric formalism*) and the one where the connection is an independent dynamical variable (*Palatini formalism*).

The Riemann tensor is by definition antisymmetric in the last two indices. If the connection is Christoffel, then there are additional symmetries in Riemann, and from these follow that the Ricci tensor is symmetric. *Metric compatibility and the Stokes theorem are true only for the Christoffel connection.*

18.1 Palatini formulation of GR

(This was done first by Einstein, in 1925. The association with Palatini comes from the use of the Palatini identity, which Palatini derived in 1919.) The Palatini–Hilbert action for empty spacetime is

$$S = \int d^4x \sqrt{-g} g^{\mu\nu} R_{\mu\nu}(\Gamma), \quad (18.4)$$

where $R_{\mu\nu}(\Gamma)$ signifies that $R_{\mu\nu}$ is defined by the independent connection, not by the metric. We assume the connection is torsion-free, i.e., symmetric in the lower indices, $\Gamma_{\mu\nu}^\lambda = \Gamma_{\nu\mu}^\lambda$. Varying

the metric gives

$$\delta S = \int d^4x \sqrt{-g} R_{\mu\nu} \delta g^{\mu\nu} + \int d^4x R \delta \sqrt{-g} = \int d^4x \sqrt{-g} (R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R) \delta g^{\mu\nu}, \quad (18.5)$$

where $R = g^{\mu\nu} R_{\mu\nu}(\Gamma)$, leading to the field equation for the metric

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0. \quad (18.6)$$

Varying the connection will lead to the Christoffel connection. This is easiest to show by writing the connection as

$$\Gamma_{\mu\nu}^\lambda = \tilde{\Gamma}_{\mu\nu}^\lambda + C_{\mu\nu}^\lambda, \quad (18.7)$$

where $\tilde{\Gamma}_{\mu\nu}^\lambda$ is the Christoffel connection (sorry about using notation I elsewhere use for Weyl-transformed quantities). Being a difference between two connections, $C_{\mu\nu}^\lambda$ is a tensor field. We get, using the Palatini identity (2.7),

$$\delta S = \int d^4x \sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu} = \int d^4x \sqrt{-g} g^{\mu\nu} \left[\nabla_\sigma (\delta \Gamma_{\nu\mu}^\sigma) - \nabla_\nu (\delta \Gamma_{\lambda\mu}^\lambda) \right], \quad (18.8)$$

where now

$$\begin{aligned} \nabla_\sigma (\delta \Gamma_{\nu\mu}^\sigma) &= \tilde{\nabla}_\sigma (\delta \Gamma_{\nu\mu}^\sigma) + C_{\sigma\lambda}^\sigma \delta \Gamma_{\nu\mu}^\lambda - C_{\sigma\nu}^\lambda \delta \Gamma_{\lambda\mu}^\sigma - C_{\sigma\mu}^\lambda \delta \Gamma_{\nu\lambda}^\sigma \\ -\nabla_\nu (\delta \Gamma_{\lambda\mu}^\lambda) &= -\tilde{\nabla}_\nu (\delta \Gamma_{\lambda\mu}^\lambda) - C_{\nu\sigma}^\lambda \delta \Gamma_{\lambda\mu}^\sigma + C_{\nu\lambda}^\sigma \delta \Gamma_{\sigma\mu}^\lambda + C_{\nu\mu}^\sigma \delta \Gamma_{\lambda\sigma}^\lambda, \end{aligned} \quad (18.9)$$

where $\tilde{\nabla}$ is the covariant derivative defined by the Christoffel connection. The part of the variation with $\tilde{\nabla}$ vanishes as before. Noting that two terms with C cancel and that other two become equal after contracting with $g^{\mu\nu}$, we are left with the condition

$$\begin{aligned} \delta S &= \int d^4x \sqrt{-g} g^{\mu\nu} \left(C_{\sigma\lambda}^\sigma \delta \Gamma_{\nu\mu}^\lambda + C_{\nu\mu}^\sigma \delta \Gamma_{\lambda\sigma}^\lambda - 2C_{\nu\sigma}^\lambda \delta \Gamma_{\lambda\mu}^\sigma \right) \\ &= \int d^4x \sqrt{-g} \left(g^{\mu\nu} C_{\sigma\lambda}^\sigma + g^{\rho\sigma} \delta_\lambda^\mu C_{\rho\sigma}^\nu - 2g^{\mu\sigma} C_{\sigma\lambda}^\nu \right) \delta \Gamma_{\mu\nu}^\lambda = 0 \end{aligned} \quad (18.10)$$

for any variation $\delta \Gamma_{\mu\nu}^\lambda$. Noting that by assumption $\delta \Gamma_{\mu\nu}^\lambda$ is symmetric in $\mu\nu$, this leads to the condition that the symmetric part of the expression in parenthesis must vanish, i.e., that

$$2g^{\mu\nu} C_{\sigma\lambda}^\sigma + \delta_\lambda^\mu C^{\nu\sigma}{}_\sigma + \delta_\lambda^\nu C^{\mu\sigma}{}_\sigma - 2C_{\lambda}^{\nu\mu}{}_\lambda - 2C_{\lambda}^{\mu\nu}{}_\lambda = 0. \quad (18.11)$$

Multiplying this with δ_ν^λ gives $C^{\mu\sigma}{}_\sigma = 0$, eliminating two terms. Multiplying then with $g_{\mu\nu}$ gives $C_{\sigma\lambda}^\sigma = 0$. We are left with the field equation

$$C_{\lambda}^{\mu\nu}{}_\lambda + C_{\lambda}^{\nu\mu}{}_\lambda = 0. \quad (18.12)$$

Since this is a tensor equation, we can lower the indices. The rank 3 tensor $C_{\mu\nu\lambda}$ is thus antisymmetric in the first two indices and symmetric in the last two indices. Such a tensor must vanish. To see this, subtract the equation $C_{\mu\nu\lambda} + C_{\nu\mu\lambda} = 0$ twice from itself, with different naming of indices:

$$C_{\mu\nu\lambda} + C_{\nu\mu\lambda} - C_{\nu\lambda\mu} - C_{\lambda\nu\mu} - C_{\lambda\mu\nu} - C_{\mu\lambda\nu} = -2C_{\lambda\mu\nu} = 0. \quad (18.13)$$

Therefore the connection is the Christoffel connection,

$$\Gamma_{\mu\nu}^\lambda = \tilde{\Gamma}_{\mu\nu}^\lambda \equiv \frac{1}{2} g^{\lambda\rho} (\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}). \quad (18.14)$$

The matter Lagrangian, at least the kind based on scalar fields that we have discussed, involves the metric and partial derivatives of the scalar fields. For scalar fields there is no difference between the partial and covariant derivative; in derivation of the field equations we replace the partial derivatives with the Christoffel covariant derivatives so that we can use the Stokes theorem. Thus the independent connection has no role in the matter Lagrangian, so that matter will not affect the field equation for the connection. We get the contribution of the matter Lagrangian to the field equation of the metric (the Einstein equation) and the effect of the metric on the field equations of the matter fields the same way as before.

Thus if the gravity part of the Lagrangian is R , we get the same field equations, i.e., we get General Relativity, both using the metric and using the Palatini formalism.

Palatini $f(R)$ gravity. Consider now the Palatini version of $f(R)$ gravity. The (vacuum) action is

$$S = \int \sqrt{-g} d^4x f(\hat{R}), \quad \text{where } \hat{R} = g^{\mu\nu} \hat{R}_{\mu\nu}(\hat{\Gamma}), \quad (18.15)$$

$\hat{\Gamma}$ is a connection independent of the metric, and $\hat{R}_{\mu\nu}(\hat{\Gamma})$ is the Ricci tensor obtained from this connection. We write $\hat{\Gamma}$ and \hat{R} , since in the following discussion we will have a role also for the Christoffel connection associated with the metric $g_{\mu\nu}$ and the curvature tensors obtained from it, which we will write as Γ and R .

Varying $g^{\mu\nu}$ gives the condition

$$\delta S = \int \sqrt{-g} d^4x \left[F(\hat{R}) \hat{R}_{\mu\nu} - \frac{1}{2} f(\hat{R}) g_{\mu\nu} \right] \delta g^{\mu\nu} = 0, \quad (18.16)$$

where

$$F(\hat{R}) \equiv \frac{df(\hat{R})}{d\hat{R}}, \quad (18.17)$$

leading to the field equation

$$F(\hat{R}) \hat{R}_{\mu\nu} - \frac{1}{2} f(\hat{R}) g_{\mu\nu} = 0 \quad (18.18)$$

for the metric. We still need to relate $\hat{R}_{\mu\nu}$ and \hat{R} to the metric.

Varying $\hat{\Gamma} \rightarrow \hat{\Gamma} + \delta\Gamma$ gives the condition

$$\delta S = \int d^4x \sqrt{-g} F(\hat{R}) g^{\mu\nu} \delta \hat{R}_{\mu\nu} = \int d^4x \sqrt{-g} F(\hat{R}) g^{\mu\nu} \left[\hat{\nabla}_\sigma (\delta\Gamma^\sigma_{\nu\mu}) - \hat{\nabla}_\nu (\delta\Gamma^\lambda_{\lambda\mu}) \right] = 0, \quad (18.19)$$

The trick of (18.7) does not work as such, since the presence of $F(\hat{R})$ prevents the conversion of the Christoffel part of the covariant derivatives into a total derivative.

However, if we introduce a conformal metric

$$\tilde{g}_{\mu\nu} = F(\hat{R}) g_{\mu\nu} \quad \Rightarrow \quad g^{\mu\nu} = F \tilde{g}^{\mu\nu} \quad \text{and} \quad \sqrt{-g} = F^{-2} \sqrt{-\tilde{g}} \quad (18.20)$$

the variation becomes

$$\delta S = \int d^4x \sqrt{-\tilde{g}} \tilde{g}^{\mu\nu} \left[\hat{\nabla}_\sigma (\delta\Gamma^\sigma_{\nu\mu}) - \hat{\nabla}_\nu (\delta\Gamma^\lambda_{\lambda\mu}) \right], \quad (18.21)$$

and if we write

$$\hat{\Gamma}^\lambda_{\mu\nu} = \tilde{\Gamma}^\lambda_{\mu\nu} + C^\lambda_{\mu\nu}, \quad (18.22)$$

where $\tilde{\Gamma}^\lambda_{\mu\nu}$ is the Christoffel connection for $\tilde{g}_{\mu\nu}$, then the same calculation as we did for the Palatini-Hilbert action shows that $C^\lambda_{\mu\nu} = 0$. Thus

$$\hat{\Gamma}^\lambda_{\mu\nu} = \tilde{\Gamma}^\lambda_{\mu\nu} \quad \Rightarrow \quad \hat{R}^\rho_{\sigma\mu\nu} = \tilde{R}^\rho_{\sigma\mu\nu}, \quad \hat{R}_{\mu\nu} = \tilde{R}_{\mu\nu} \quad \text{and} \quad \hat{R} = g^{\mu\nu} \tilde{R}_{\mu\nu} \quad (18.23)$$

and we can use the results from Sec. 15 to relate them to the $\Gamma^\lambda_{\mu\nu}$, $R^\rho_{\sigma\mu\nu}$, $R_{\mu\nu}$ and R associated with the metric $g_{\mu\nu}$. Note that $\hat{R} = F\tilde{R}$, not \tilde{R} .

If the matter Lagrangian involves just the metric, not the independent connection, then adding matter just adds the energy tensor in the field equation for the metric in the usual way,

$$F(\hat{R})\tilde{R}_{\mu\nu} - \frac{1}{2}f(\hat{R})g_{\mu\nu} = 8\pi GT_{\mu\nu}. \quad (18.24)$$

Compared to (2.28) we got rid of the term $(\nabla_\mu \nabla_\nu - g_{\mu\nu} \square)F(R)$, which contains fourth derivatives of the metric. However, as we see from (15.10), now $\tilde{R}_{\mu\nu}$ will contain first and second derivatives of the conformal factor F (or \sqrt{F}).

Taking the trace of (18.24), i.e., multiplying it with $g^{\mu\nu}$ gives

$$F(\hat{R})\hat{R} - 2f(\hat{R}) = 8\pi GT, \quad (18.25)$$

which gives an algebraic relation between \hat{R} and T . For example, if $f(\hat{R}) = c\hat{R}^\alpha$ and the matter is nonrelativistic so that $T = -\rho$, we get $\hat{R} = [8\pi G\rho/c(2-\alpha)]^{1/\alpha}$, $f(\hat{R}) = 8\pi G\rho/(2-\alpha)$, and $F = c^{-1/\alpha}\alpha[8\pi G\rho/(2-\alpha)]^{1-1/\alpha}$ for $\alpha \neq 2$.

18.2 Palatini Scalar-Tensor Theories

The Palatini action for scalar-tensor theories is

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2}f(\varphi)g^{\mu\nu}\hat{R}_{\mu\nu}(\hat{\Gamma}) - \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - V(\varphi) + \mathcal{L}_{\text{mat}} \right], \quad (18.26)$$

where $\hat{R}_{\mu\nu}(\hat{\Gamma})$ is the Ricci tensor obtained from the independent connection $\hat{\Gamma}$. We will write $\hat{\nabla}_\mu$ for the covariant derivative defined by the independent connection, and ∇_μ for the covariant derivative defined by the Christoffel connection compatible with the metric $g_{\mu\nu}$. Since φ is a scalar field, its covariant derivative is just the partial derivative and does not depend on the connection, so

$$\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi = \frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi = \frac{1}{2}g^{\mu\nu}\hat{\nabla}_\mu\varphi\hat{\nabla}_\nu\varphi = \frac{1}{2}g^{\mu\nu}\nabla_\mu\varphi\nabla_\nu\varphi. \quad (18.27)$$

We assume here that \mathcal{L}_{mat} does not depend on the connection.

Varying $g^{\mu\nu} \rightarrow g^{\mu\nu} + \delta g^{\mu\nu}$ gives the condition

$$\delta S = \int \sqrt{-g}d^4x \left\{ \frac{1}{2}f(\varphi) \left(\hat{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\hat{R} \right) - \frac{1}{2}\partial_\mu\varphi\partial_\nu\varphi + \frac{1}{2}g_{\mu\nu} \left[\frac{1}{2}\partial_\sigma\varphi\partial^\sigma\varphi + V(\varphi) \right] \right\} \delta g^{\mu\nu} + \delta S_{\text{mat}} = 0 \quad (18.28)$$

leading to the field equation

$$f(\varphi)\hat{R}_{\mu\nu} - \frac{1}{2}f(\varphi)\hat{R}g_{\mu\nu} = \partial_\mu\varphi\partial_\nu\varphi - g_{\mu\nu} \left[\frac{1}{2}\partial_\sigma\varphi\partial^\sigma\varphi + V(\varphi) \right] + T_{\mu\nu}^{\text{mat}}. \quad (18.29)$$

We still need to relate $\hat{R}_{\mu\nu}$ and \hat{R} to the metric.

Varying $\hat{\Gamma} \rightarrow \hat{\Gamma} + \delta\Gamma$ gives the condition

$$\delta S = \int d^4x \sqrt{-g} \frac{1}{2}f(\varphi)g^{\mu\nu}\delta\hat{R}_{\mu\nu} = \int d^4x \sqrt{-g} \frac{1}{2}f(\varphi)g^{\mu\nu} \left[\hat{\nabla}_\sigma(\delta\Gamma_{\nu\mu}^\sigma) - \hat{\nabla}_\nu(\delta\Gamma_{\lambda\mu}^\lambda) \right] = 0, \quad (18.30)$$

The trick of (18.7) does not work as such, since the presence of $f(\varphi)$ prevents the conversion of the Christoffel part of the covariant derivatives into a total derivative.

However, if we introduce a conformal metric

$$\tilde{g}_{\mu\nu} = \omega^2 g_{\mu\nu}, \quad \text{where } \omega^2 = \frac{f(\varphi)}{M_{\text{Pl}}^2} \quad (18.31)$$

we have

$$g^{\mu\nu} = \frac{f(\varphi)}{M_{\text{Pl}}^2} \tilde{g}^{\mu\nu} \quad \text{and} \quad \sqrt{-g} = \frac{M_{\text{Pl}}^4}{f(\varphi)^2} \sqrt{-\tilde{g}} \quad (18.32)$$

and the variation becomes

$$\delta S = \frac{1}{2} M_{\text{Pl}}^2 \int d^4x \sqrt{-\tilde{g}} \tilde{g}^{\mu\nu} \left[\hat{\nabla}_\sigma (\delta \Gamma_{\nu\mu}^\sigma) - \hat{\nabla}_\nu (\delta \Gamma_{\lambda\mu}^\lambda) \right], \quad (18.33)$$

and if we write

$$\hat{\Gamma}_{\mu\nu}^\lambda = \tilde{\Gamma}_{\mu\nu}^\lambda + C_{\mu\nu}^\lambda, \quad (18.34)$$

where $\tilde{\Gamma}_{\mu\nu}^\lambda$ is the Christoffel connection for $\tilde{g}_{\mu\nu}$, then the same calculation as we did for the Palatini–Hilbert action shows that $C_{\mu\nu}^\lambda = 0$. Thus

$$\hat{\Gamma}_{\mu\nu}^\lambda = \tilde{\Gamma}_{\mu\nu}^\lambda \quad \Rightarrow \quad \hat{R}^\rho_{\sigma\mu\nu} = \tilde{R}^\rho_{\sigma\mu\nu}, \quad \hat{R}_{\mu\nu} = \tilde{R}_{\mu\nu} \quad \text{and} \quad \hat{R} = g^{\mu\nu} \tilde{R}_{\mu\nu} \quad (18.35)$$

and we can use the results from Sec. 15 to relate them to the $\Gamma_{\mu\nu}^\lambda$, $R^\rho_{\sigma\mu\nu}$, $R_{\mu\nu}$ and R associated with the metric $g_{\mu\nu}$. This also means that the Riemann tensor has all its usual symmetries and the Ricci tensor is symmetric, as they are derived from a Christoffel connection of some metric. Note that $\hat{R} = \omega^2 \tilde{R}$, not \tilde{R} .

Now the connection is not the Christoffel connection and we have the issue of two different definitions of geodesics. (We could also say that the theory has two metrics, $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$, and two different connections $\tilde{\Gamma}$ and Γ .) That freely falling particles follow geodesics is not an independent assumption in GR, but can be derived from the Einstein equation. Thus the behavior of test particles in this theory should also be determined by the field equation we derived (but we will not try to do this here). The answer may also depend on whether \mathcal{L}_{mat} depends on the connection (we assumed here that it does not), and if it does, which of the two connections appears there.

The field equation (18.29) is likely to be difficult to handle because of the complicated φ -dependent relation $\tilde{R}_{\mu\nu}$ has with the metric $g_{\mu\nu}$. It will be easier in the Einstein frame. To rewrite (18.26) in the Einstein frame the other parts go as in Sec. 16, except the $\frac{1}{2} \sqrt{-g} f(\varphi) g^{\mu\nu} \hat{R}_{\mu\nu}$, where $\hat{R}_{\mu\nu}$ is now independent of the metric at this stage and is not affected by the Weyl transformation. Instead we just convert the factor

$$\frac{1}{2} \sqrt{-g} g^{\mu\nu} = \frac{1}{2} M_{\text{Pl}}^2 \frac{1}{f} \sqrt{-\tilde{g}} \tilde{g}^{\mu\nu}. \quad (18.36)$$

The Einstein frame action is thus

$$S = \int d^4x \sqrt{-\tilde{g}} \left[\frac{1}{2} M_{\text{Pl}}^2 \tilde{g}^{\mu\nu} \hat{R}_{\mu\nu}(\hat{\Gamma}) - \frac{1}{2} M_{\text{Pl}}^2 \frac{1}{f} \tilde{g}^{\mu\nu} \tilde{\nabla}_\mu \varphi \tilde{\nabla}_\nu \varphi - \frac{M_{\text{Pl}}^4}{f^2} V(\varphi) + \tilde{\mathcal{L}}_{\text{mat}} \right]. \quad (18.37)$$

The kinetic term is now simpler than in the metric formalism (16.7) and to convert it to the canonical form requires the simpler field transformation

$$d\tilde{\varphi} = \frac{M_{\text{Pl}}}{\sqrt{f}} d\varphi. \quad (18.38)$$

Finally we have

$$S = \int d^4x \sqrt{-\tilde{g}} \left\{ \frac{1}{2} M_{\text{Pl}}^2 \tilde{g}^{\mu\nu} \hat{R}_{\mu\nu}(\hat{\Gamma}) - \frac{1}{2} \tilde{g}^{\mu\nu} \tilde{\nabla}_\mu \tilde{\varphi} \tilde{\nabla}_\nu \tilde{\varphi} - \tilde{V}(\tilde{\varphi}) + \tilde{\mathcal{L}}_{\text{mat}} \right\}, \quad (18.39)$$

where

$$\tilde{V}(\tilde{\varphi}) \equiv \frac{M_{\text{Pl}}^4}{f^2} V(\varphi). \quad (18.40)$$

This is the Palatini–Hilbert action with a scalar field and matter. Varying $\hat{\Gamma}$ gives $\hat{\Gamma}_{\mu\nu}^\lambda = \tilde{\Gamma}_{\mu\nu}^\lambda$ and varying $\tilde{g}^{\mu\nu}$ and $\tilde{\varphi}$ give the Einstein equation for $\tilde{g}_{\mu\nu}$ and the usual field equation for $\tilde{\varphi}$.

18.3 Palatini Higgs Inflation

The Palatini version of Higgs inflation has the action (18.26) with

$$f(\varphi) = M^2 + \xi\varphi^2 \quad \text{and} \quad V(\varphi) = \frac{1}{4}\lambda(\varphi^2 - \sigma^2)^2. \quad (18.41)$$

Again $\varphi = \sigma$ after the EW transition, so that $M_{\text{Pl}}^2 = M^2 + \xi\sigma^2$ and we assume $\xi \ll (M_{\text{Pl}}/\sigma)^2$ so that $M \approx M_{\text{Pl}}$, although the distinction between them is kept in the following. Compared to Sec. 17 the difference is that the transformed field $\tilde{\varphi}$ is obtained from (18.38) instead of (16.8). We have now

$$d\tilde{\varphi} = \frac{M_{\text{Pl}}}{M} \frac{d\varphi}{\sqrt{1 + \xi(\varphi/M)^2}} \Rightarrow \varphi = \frac{M}{\sqrt{\xi}} \sinh \sqrt{\xi} \frac{\tilde{\varphi}}{M_{\text{Pl}}} \quad (18.42)$$

and

$$\tilde{V}(\tilde{\varphi}) = \frac{1}{4} \frac{\lambda M_{\text{Pl}}^4}{(M^2 + \xi\varphi^2)^2} \left(\frac{M^2}{\xi} \sinh^2 \sqrt{\xi} \frac{\tilde{\varphi}}{M_{\text{Pl}}} - \sigma^2 \right)^2. \quad (18.43)$$

For small field values $\varphi \ll M/\sqrt{\xi}$, we have $\tilde{\varphi} \approx \varphi$ and $\tilde{V} \approx V$ and the difference between the Einstein and Jordan frame (and the coupling of Higgs to curvature scalar) disappears. Since we assumed $\xi \ll (M_{\text{Pl}}/\sigma)^2$, we are already in this small-field limit at the EW transition. (Note that compared to the metric version of Higgs inflation, $\sqrt{\xi}$ appears here instead of ξ .)

In the large-field limit, $\tilde{\varphi} \gg M_{\text{Pl}}/\sqrt{\xi}$ we can ignore the σ^2 term in the potential, so that

$$\tilde{V}(\tilde{\varphi}) \approx \frac{\lambda M_{\text{Pl}}^4}{4\xi^2} \frac{\sinh^4 \sqrt{\xi}(\tilde{\varphi}/M_{\text{Pl}})}{[1 + \sinh^2 \sqrt{\xi}(\tilde{\varphi}/M_{\text{Pl}})]^2} = \frac{\lambda M_{\text{Pl}}^4}{4\xi^2} \tanh^4 \sqrt{\xi}(\tilde{\varphi}/M_{\text{Pl}}). \quad (18.44)$$

Approximating

$$\sinh \sqrt{\xi}(\tilde{\varphi}/M_{\text{Pl}}) \approx \frac{1}{2} \exp \left[\sqrt{\xi}(\tilde{\varphi}/M_{\text{Pl}}) \right] \quad \text{or} \quad \varphi \approx \frac{M}{2\sqrt{\xi}} e^{\sqrt{\xi}\tilde{\varphi}/M_{\text{Pl}}}, \quad (18.45)$$

we have

$$\tilde{V}(\tilde{\varphi}) \approx \frac{\lambda M_{\text{Pl}}^4}{4\xi^2} \frac{1}{1 + 8e^{-2\sqrt{\xi}(\tilde{\varphi}/M_{\text{Pl}})} + 16e^{-4\sqrt{\xi}(\tilde{\varphi}/M_{\text{Pl}})}} \approx \frac{\lambda M_{\text{Pl}}^4}{4\xi^2} \left[1 - 8e^{-2\sqrt{\xi}(\tilde{\varphi}/M_{\text{Pl}})} \right]. \quad (18.46)$$

From this we get the slow-roll parameters (**exercise**)

$$\tilde{\epsilon} \approx \frac{8}{\xi} \left(\frac{M}{\varphi} \right)^4, \quad \tilde{\eta} \approx -8 \left(\frac{M}{\varphi} \right)^2, \quad \tilde{\xi} \approx 64 \left(\frac{M}{\varphi} \right)^4. \quad (18.47)$$

For the remaining number of inflation e -foldings we get (**exercise**)

$$\tilde{N}(\tilde{\varphi}) \approx \frac{1}{8} \left(\frac{\varphi}{M} \right)^2 \quad (18.48)$$

(ignoring the contribution from $\tilde{\varphi}_{\text{end}}$) so that

$$\tilde{\epsilon} \approx \frac{1}{8\xi\tilde{N}^2}, \quad \tilde{\eta} \approx -\frac{1}{\tilde{N}}, \quad \tilde{\xi} \approx \frac{1}{\tilde{N}^2}. \quad (18.49)$$

The results for $\tilde{\eta}$ and $\tilde{\xi}$ in terms of \tilde{N} are the same as in the metric Higgs inflation, but $\tilde{\epsilon}$ is smaller by the factor $1/6\xi$. The same factor (or its square for $\tilde{\xi}$) appears in (18.47) and (18.48) compared to the metric case and it can be traced to the $\sqrt{6\xi}$ difference in the exponent in (18.45).

For the primordial power spectrum we get

$$\mathcal{P}_{\mathcal{R}} = \frac{1}{24\pi^2} \frac{1}{M_{\text{Pl}}^4} \frac{\tilde{V}}{\tilde{\epsilon}} \approx \frac{\lambda \tilde{N}^2}{12\pi^2 \xi}. \quad (18.50)$$

The observed value $\mathcal{P}_{\mathcal{R}} \approx 2.1 \times 10^{-9}$ requires

$$\xi \approx \frac{\lambda}{12\pi^2} \frac{\tilde{N}^2}{2.1 \times 10^{-9}} = 4.0 \times 10^6 \lambda \tilde{N}^2 \approx 5.2 \times 10^5 \tilde{N}^2 \approx 1.9 \times 10^9, \quad (18.51)$$

where the last number is for $\tilde{N} = 60$. The spectral indices and tensor/scalar ratio are

$$\begin{aligned} n_s &= 1 - 6\tilde{\epsilon} + 2\tilde{\eta} \approx 1 + 2\tilde{\eta} = 1 - \frac{2}{\tilde{N}} \approx 0.967 \\ n_T &= -2\tilde{\epsilon} = -\frac{1}{4\xi\tilde{N}^2} \approx 4 \times 10^{-14} \\ r &\equiv \frac{\mathcal{P}_T}{\mathcal{P}_{\mathcal{R}}} = 16\tilde{\epsilon} = \frac{2}{\xi\tilde{N}^2} = \frac{3.8 \times 10^{-6}}{\tilde{N}^4} \approx 3 \times 10^{-13} \\ \frac{dn_s}{d \ln k} &= 16\tilde{\epsilon}\tilde{\eta} - 24\tilde{\epsilon}^2 - 2\tilde{\xi} \approx -2\tilde{\xi} = -\frac{2}{\tilde{N}^2} \approx -0.00056, \end{aligned} \quad (18.52)$$

where the last numbers are for $\tilde{N} = 60$. The main difference from metric Higgs inflation is the much smaller amplitude of tensor perturbations (primordial gravitational waves). For metric Higgs inflation they should be observable with a future polarization-optimized CMB satellite; for Palatini Higgs inflation they are completely unobservable.

19 $f(R)$ gravity in the Einstein frame

Consider a theory given by the action

$$S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2} [f(\chi) + f'(\chi)(R - \chi)] + \mathcal{L}_m \right\}, \quad (19.1)$$

where the degrees of freedom are the metric $g_{\mu\nu}$, a scalar field χ , and matter of degrees of freedom in \mathcal{L}_m ; $R = g^{\mu\nu} R_{\mu\nu}$ is the curvature scalar; and $f(\chi)$ is a scalar function of χ , which is at this point unspecified. We require $f''(\chi) \neq 0$, because otherwise $f(\chi) \propto \chi$ and the χ -dependence disappears in (19.1) leaving just the Hilbert action. The following discussion puts some additional constraints on the form of $f(\chi)$. We now find the field equations for this theory using the principle of stationary action.

Varying χ ,

$$\delta S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2} [f'(\chi) + f''(\chi)(R - \chi) - f'(\chi)] \right\} \delta\chi, \quad (19.2)$$

the stationary requirement $\delta S = 0$ gives the field equation for χ ,

$$\chi = R. \quad (19.3)$$

Varying $g^{\mu\nu}$,

$$\begin{aligned} \delta S &= \int d^4x \sqrt{-g} \frac{1}{2} f'(\chi) g^{\mu\nu} \delta R_{\mu\nu} + \int d^4x \sqrt{-g} \frac{1}{2} f'(\chi) R_{\mu\nu} \delta g^{\mu\nu} \\ &\quad + \int d^4x \left\{ \frac{1}{2} [f(\chi) + f'(\chi)(R - \chi)] \right\} \delta \sqrt{-g} + \delta S_m \\ &\equiv \delta S_1 + \delta S_2 + \delta S_3 + \delta S_m. \end{aligned} \quad (19.4)$$

We did similar variations in Sec. 2.2 on $f(R)$ gravity. Following this example, we get

$$\begin{aligned} \delta S_1 &= \int d^4x \sqrt{-g} \frac{1}{2} \left[g_{\mu\nu} \nabla_\lambda \nabla^\lambda f'(\chi) - \nabla_\nu \nabla_\mu f'(\chi) \right] \delta g^{\mu\nu} \\ \delta S_2 &= \int d^4x \sqrt{-g} \frac{1}{2} f'(\chi) R_{\mu\nu} \delta g^{\mu\nu} \\ \delta S_3 &= - \int d^4x \sqrt{-g} \frac{1}{4} [f(\chi) + f'(\chi)(R - \chi)] g_{\mu\nu} \delta g^{\mu\nu}. \end{aligned} \quad (19.5)$$

The stationary requirement gives the field equation for $g_{\mu\nu}$,

$$f'(\chi) R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} [f(\chi) + f'(\chi)(R - \chi)] - \nabla_\mu \nabla_\nu f'(\chi) + g_{\mu\nu} \square f'(\chi) = - \frac{\delta S_m}{\frac{1}{2} \sqrt{-g} \delta g^{\mu\nu}}, \quad (19.6)$$

where we identify the rhs as the energy tensor $T_{\mu\nu}$ due to matter degrees of freedom. Solving χ from (19.3) this becomes

$$f'(R) R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} f(R) = (\nabla_\mu \nabla_\nu - g_{\mu\nu} \square) f'(R) + T_{\mu\nu}, \quad (19.7)$$

which is the same equation as (2.28) (where we denoted $F(R) \equiv f'(R)$; I need to decide where I want to put the $8\pi G = 1/M_{\text{Pl}}^2$).

Thus the theory (19.1) is equivalent to $f(R)$ gravity³³ (in the metric formalism; one could also do a similar operation in the Palatini formalism).

³³I haven't seen a pedagogical discussion about how one comes up with the idea of trying out (19.1) to attack $f(R)$ gravity. I traced the idea in literature back to [30]. See also [31]. Somewhere I saw the (19.1) Lagrangian referred to as the Legendre transform of the $f(R)$ Lagrangian, but I have lost the reference. One may wonder how a theory with just the metric as a variable is equivalent to a theory with both metric and a scalar field. But note that $f(R)$ gravity has a fourth-order field equation and therefore it has more degrees of freedom than Einstein gravity: to specify initial conditions one must specify also second and third derivatives. Now these additional degrees of freedom are transferred into the scalar field.

Go back to (19.1). Define

$$\begin{aligned}\varphi &\equiv f'(\chi) \\ W(\varphi) &\equiv \chi(\varphi)\varphi - f(\chi(\varphi)).\end{aligned}\tag{19.8}$$

Here we have to assume that $f'(\chi)$ is monotonic, so that we can invert it to $\chi(\varphi)$. With these definitions, (19.1) becomes

$$S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2} [\varphi R - W(\varphi)] + \mathcal{L}_m \right\}.\tag{19.9}$$

This looks like a scalar-tensor theory, except that the kinetic term is missing.

Consider now a Weyl transformation $\tilde{g}_{\mu\nu} = \omega^2 g_{\mu\nu}$ with

$$\omega^2 = \frac{\varphi}{M_{\text{Pl}}^2}.\tag{19.10}$$

Following Sec. 16,

$$\begin{aligned}\frac{1}{2}\sqrt{-g}\varphi R &= \sqrt{-\tilde{g}} \left\{ \frac{1}{2}M_{\text{Pl}}^2\tilde{R} + \frac{1}{2}M_{\text{Pl}}^2 \left[3 \cdot 0 - \frac{9}{2} \left(\frac{1}{\varphi} \right)^2 \right] \tilde{g}^{\mu\nu} \tilde{\nabla}_\mu \varphi \tilde{\nabla}_\nu \varphi + \frac{3}{2}M_{\text{Pl}}^2 \frac{1}{\varphi} \tilde{g}^{\mu\nu} \tilde{\nabla}_\mu \tilde{\nabla}_\nu \varphi \right\} \\ -\frac{1}{2}\sqrt{-g}W(\varphi) &= -\frac{1}{2}\sqrt{-\tilde{g}} \frac{M_{\text{Pl}}^4}{\varphi^2} W(\varphi),\end{aligned}\tag{19.11}$$

so that, after partial integration of the term with $\tilde{\nabla}_\mu \tilde{\nabla}_\nu \varphi$,

$$S = \int d^4x \sqrt{-\tilde{g}} \left\{ \frac{1}{2}M_{\text{Pl}}^2\tilde{R} - \frac{3}{4}M_{\text{Pl}}^2 \frac{1}{\varphi^2} \tilde{g}^{\mu\nu} \tilde{\nabla}_\mu \varphi \tilde{\nabla}_\nu \varphi - \frac{1}{2} \frac{M_{\text{Pl}}^2}{\varphi^2} W(\varphi) + \tilde{\mathcal{L}}_m \right\}.\tag{19.12}$$

We see that a kinetic term has appeared! We convert it to the canonical form by defining a transformed field $\tilde{\varphi}$ by

$$d\tilde{\varphi} = M_{\text{Pl}} \sqrt{\frac{3}{2}} \frac{d\varphi}{\varphi} \quad \Rightarrow \quad \varphi = e^{\sqrt{2/3}(\tilde{\varphi}/M_{\text{Pl}})}\tag{19.13}$$

(with a convenient choice of integration constant). We also define a transformed potential

$$\tilde{V}(\tilde{\varphi}) = \frac{1}{2} \frac{M_{\text{Pl}}^2}{\varphi^2} W(\varphi),\tag{19.14}$$

and we have the action in the Einstein frame,

$$S = \int d^4x \sqrt{-\tilde{g}} \left\{ \frac{1}{2}M_{\text{Pl}}^2\tilde{R} - \frac{1}{2}\tilde{g}^{\mu\nu} \tilde{\nabla}_\mu \tilde{\varphi} \tilde{\nabla}_\nu \tilde{\varphi} - \tilde{V}(\tilde{\varphi}) + \tilde{\mathcal{L}}_m \right\}.\tag{19.15}$$

Thus, by going to the Einstein frame, we have converted $f(R)$ gravity to ordinary Einstein gravity with a scalar field. The question is what is the Einstein frame potential $\tilde{V}(\tilde{\varphi})$. It depends on $f(\chi)$, i.e. of the function $f(R)$ defining the $f(R)$ gravity theory.

19.1 Starobinsky inflation

Starobinsky inflation[1], also known as R^2 inflation, was already introduced in Sec. 2.2. For

$$f(R) = R + \frac{1}{6M^2}R^2 \quad (19.16)$$

we get

$$\varphi \equiv f'(\chi) = 1 + \frac{\chi}{3M^2} \quad \Rightarrow \quad \chi = 3M^2(\varphi - 1) \quad (19.17)$$

and (**exercise**)

$$\begin{aligned} W(\varphi) &= \frac{3}{2}M^2(\varphi - 1)^2 \\ \tilde{V}(\tilde{\varphi}) &= \frac{3}{2}M^2M_{\text{Pl}}^2 \left(1 - e^{-\sqrt{2/3}(\tilde{\varphi}/M_{\text{Pl}})}\right)^2. \end{aligned} \quad (19.18)$$

We note that the potential $\tilde{V}(\tilde{\varphi})$ is approximately the same as in Higgs inflation, if we replace

$$\frac{\lambda}{\xi^2} \rightarrow 3 \left(\frac{M}{M_{\text{Pl}}}\right)^2. \quad (19.19)$$

Thus the inflation predictions are the same,

$$n_s - 1 \approx \frac{2}{\tilde{N}}, \quad r \approx \frac{12}{\tilde{N}^2} \quad (19.20)$$

and comparing the predicted scalar perturbation spectrum to its observed magnitude,

$$\mathcal{P}_{\mathcal{R}}(k) \approx \frac{1}{24\pi^2} \left(\frac{M}{M_{\text{Pl}}}\right)^2 \tilde{N}^2 \approx 2.1 \times 10^{-9}, \quad (19.21)$$

we get, for $\tilde{N} = 50\text{--}60$,

$$M = 1.2\text{--}1.4 \times 10^{-5}M_{\text{Pl}} \quad \Rightarrow \quad \alpha \equiv \frac{1}{6M^2} = 8.3\text{--}11.9 \times 10^8 M_{\text{Pl}}^{-2}, \quad (19.22)$$

a somewhat small M and large α compared to our natural expectation $M = \mathcal{O}(M_{\text{Pl}})$.

From the literature[25], where reheating is discussed in Higgs and R^2 inflation, the latter has a lower reheating temperature, $T_{\text{reh}} \approx 10^9$ GeV, leading to a smaller $\tilde{N} \approx 55$ and thus a slightly smaller (redder) spectral index and higher tensor/scalar ratio. [25] gives $\tilde{N} = 54.37$ for the WMAP pivot scale $k = 0.002/\text{Mpc}$ and says: ‘‘Using exact formulas gives numerically . . . $n_s = 0.965$, $r = 0.0036$ ’’. Both Higgs and R^2 inflation are consistent with Planck data ([16], p. 15) but could be distinguished from each other by a more accurate future CMB experiment.

A Rules to Convert between Different Time Coordinates

The cosmic time t and the conformal time η are related by

$$dt = a d\eta \Rightarrow \frac{d}{dt} = \frac{1}{a} \frac{d}{d\eta} \Rightarrow (\dot{}) = \frac{1}{a} ()' \quad \text{or} \quad ()' = a(\dot{}) \quad (\text{A.1})$$

For any two functions of time, f and g , we have

$$\frac{df}{dg} = \frac{\dot{f}}{\dot{g}} = \frac{f'}{g'} \quad (\text{A.2})$$

The ordinary Hubble parameter H and the conformal (or comoving) Hubble parameter are related by

$$\mathcal{H} = aH = \dot{a} = \frac{a'}{a} \quad (\text{A.3})$$

Sometimes it is convenient to use the scale factor a or its logarithm $\ln a$ as the time coordinate. We have the following relations between the derivatives wrt these time coordinates:

$$\mathcal{H}^{-1} f' = H^{-1} \dot{f} = a \frac{df}{da} = \frac{df}{d \ln a} \quad (\text{A.4})$$

$$\mathcal{H}^{-2} f'' = H^{-2} \ddot{f} + H^{-1} \dot{f} = a^2 \frac{d^2 f}{da^2} + \frac{1-3w}{2} a \frac{df}{da} = \left(\frac{d}{d \ln a} \right)^2 f - \frac{1+3w}{2} \frac{df}{d \ln a} \quad (\text{A.5})$$

In many equations the combination

$$\mathcal{H}^{-2} f'' + 2\mathcal{H}^{-1} f' = H^{-2} \ddot{f} + 3H^{-1} \dot{f} \quad (\text{A.6})$$

appears. We also have that

$$\mathcal{H}' = a\ddot{a} = a^2 (\dot{H} + H^2). \quad (\text{A.7})$$

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