

Cosmological Perturbation Theory I

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About these lecture notes:

I have lectured a course on cosmological perturbation theory at the University of Helsinki in the spring of 2003, in the fall of 2010, in the fall of 2015, and in spring 2020, 2022, and 2024.

This course cover similar topics as Cosmology II: structure formation, inflation, and a bit of CMB; but on a deeper level and more widely. In Cosmology II we accepted many results as given; now they are derived rigorously from general relativity.

Topics not covered: Quantum field theory is not used, except for a brief discussion of generation of primordial perturbations during inflation in CPT II. Non-linear effects or higher-order perturbation theory are not covered. Open and closed cosmologies are not covered, i.e., the background (unperturbed) spacetime is assumed flat. Of CMB we discuss only the Sachs–Wolfe effect, in Section 26. I have lectured a couple of times a separate course on CMB Physics, but it is not currently in the teaching program.

There is an unfortunate variety in the notation employed by various authors. My notation is the result of first learning perturbation theory from Mukhanov, Feldman, and Brandenberger (Phys. Rep. **215**, 213 (1992)) and then from Liddle and Lyth (*Cosmological Inflation and Large-Scale Structure*, Cambridge University Press 2000, Chapters 14 and 15), and represents a compromise between their notations.

After 2022, the course was divided into two parts: Cosmological Perturbation Theory I and Cosmological Perturbation Theory II. In CPT I, we cover the evolution of perturbations from the early radiation-dominated epoch to the present time—to the extent that it can be discussed using first-order (linear) perturbation theory; the energy content of the universe is discussed in terms of fluid quantities. In CPT II we discuss generation and evolution of perturbations during inflation; the energy content of the universe is discussed in terms of scalar fields; also modified gravity is discussed, as some inflation models are based on modified gravity. These are the lecture notes for CPT I.

Over the years I have added material, so that there is now too much in these CPT I lecture notes to cover in the seven weeks of the CPT I course. In 2024 I skipped Sections 17, 19.5, 20, 22.4, 24.1, 27.1–3 (except figures), and 28; and left Sections 29 and 30 for CPT II.

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1 Perturbative General Relativity

In the perturbation theory of general relativity one considers a spacetime, the *perturbed spacetime*, that is *close* to a simple, symmetric, spacetime, the *background spacetime*, that we already know. In the development of perturbation theory we keep referring to these *two different spacetimes* (see Fig. 1).

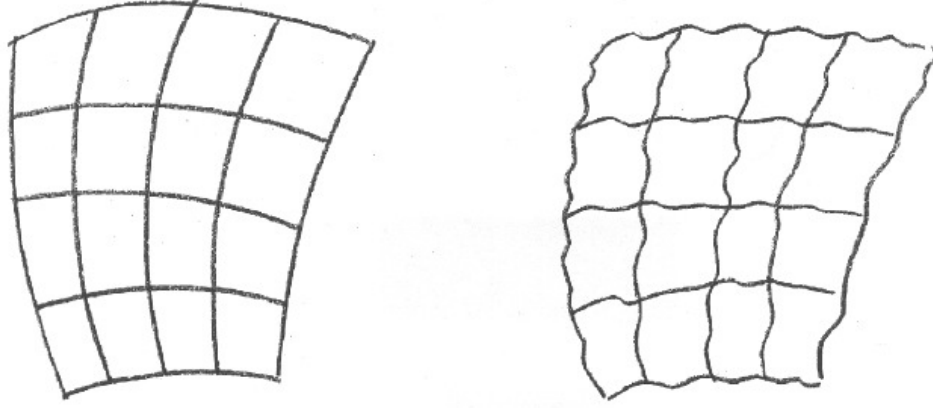


Figure 1: The background spacetime and the perturbed spacetime.

This means that there exists a coordinate system on the perturbed spacetime, where its metric can be written as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}, \quad (1.1)$$

where $\bar{g}_{\mu\nu}$ is the metric of the background spacetime (we shall refer to the *background* quantities with the overbar¹), and $\delta g_{\mu\nu}$ is small. We also require that the first and second partial derivatives, $\delta g_{\mu\nu,\rho}$ and $\delta g_{\mu\nu,\rho\sigma}$ are small².

The curvature tensors and the energy tensor of the perturbed spacetime can then be written as

$$G^\mu_\nu = \bar{G}^\mu_\nu + \delta G^\mu_\nu \quad (1.2)$$

$$T^\mu_\nu = \bar{T}^\mu_\nu + \delta T^\mu_\nu, \quad (1.3)$$

where δG^μ_ν and δT^μ_ν are small. Subtracting the Einstein equations of the two spacetimes,

$$G^\mu_\nu = 8\pi G T^\mu_\nu \quad \text{and} \quad \bar{G}^\mu_\nu = 8\pi G \bar{T}^\mu_\nu, \quad (1.4)$$

from each other, we get the field equation for the perturbations

$$\delta G^\mu_\nu = 8\pi G \delta T^\mu_\nu. \quad (1.5)$$

The above discussion requires a pointwise correspondence between the two spacetimes, so that we can perform the comparisons and the subtractions. This correspondence is given by the coordinate system (x^0, x^1, x^2, x^3) : the point \bar{P} in the background spacetime and the point P in the perturbed spacetime which have the same coordinate values, correspond to each other. Now, given a coordinate system on the background spacetime, there exist many coordinate systems,

¹Beware: the overbars will be dropped eventually.

²Actually it is not always necessary to require the second derivatives of the metric perturbation to be small; but then our development would require more care.

all close to each other, for the perturbed spacetime, for which (1.1) holds. The choice among these coordinate systems is called the *gauge choice*, to be discussed a little later.

In *first-order* (or *linear*) perturbation theory, we drop all terms from our equations which contain products of the small quantities $\delta g_{\mu\nu}$, $\delta g_{\mu\nu,\rho}$ and $\delta g_{\mu\nu,\rho\sigma}$. The field equation (1.5) becomes then a linear differential equation for $\delta g_{\mu\nu}$, making things much easier than in full GR. In *second-order* perturbation theory, one keeps also those terms with a product of two (but no more) small quantities. In these lectures we only discuss first-order perturbation theory.

The simplest case is the one where the background is the Minkowski space. Then $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$, and $\bar{G}^\mu_\nu = \bar{T}^\mu_\nu = 0$.

In *cosmological perturbation theory* the background spacetime is the Friedmann–Robertson–Walker universe. Now the background spacetime is curved, and is not empty. While it is homogeneous and isotropic, it is *time-dependent*. In these lectures we shall only consider the case where the background is the *flat* FRW universe. This case is much simpler than the open and closed ones³, since now the $t = \text{const}$ time slices (“space at time t ”) have Euclidean geometry. This will allow us to do 3-dimensional Fourier transformations in space.

The separation into the background and perturbation is always done so that the spatial average of the perturbation is zero, i.e., the background value (at time t) is the spatial average of the full quantity over the time slice $t = \text{const}$.

Backreaction. An important question is, whether the \bar{G}^μ_ν and \bar{T}^μ_ν defined as spatial averages of G^μ_ν and T^μ_ν indeed satisfy $\bar{G}^\mu_\nu = 8\pi G \bar{T}^\mu_\nu$, i.e., correspond to a (homogeneous and isotropic) solution of General Relativity, i.e., an FRW universe. We expect the exact answer to be negative, since GR is a nonlinear theory, so that perturbations affect the evolution of the mean. This effect is called *backreaction*. However, in *first-order (linear) perturbation theory* the answer is yes. As the second name implies, the theory is linear in the perturbations, meaning that the effect of overdensities cancel the effect of underdensities on, e.g., the average expansion rate. Thus the mean values evolve just like they would in the absence of perturbations.

While the perturbations at large scales have remained small, during the later history of the universe the perturbations have grown large at smaller scales; and may, via nonlinear effects, affect evolution of the background universe. How big is the effect of this backreaction, is an open research question in cosmology, since the calculations are difficult, but a common view is that the effect is small compared to the present accuracy of observations. For this course, we adopt this view, and assume that the background universe simultaneously represents a FRW universe (“the universe we would have if we did not have the perturbations”) and the mean values of the quantities in the true universe at each time t ; and that we can apply linear perturbation theory at large scales while perturbations at these scales are small, even though they are already large at small scales.

2 The Background Universe

Our background spacetime is the flat Friedmann–Robertson–Walker universe FRW(0). The background metric in *comoving* coordinates (t, x, y, z) is

$$ds^2 = \bar{g}_{\mu\nu} dx^\mu dx^\nu = -dt^2 + a^2(t) \delta_{ij} dx^i dx^j = -dt^2 + a^2(t) (dx^2 + dy^2 + dz^2), \quad (2.1)$$

where $a(t)$ is the *scale factor*, to be solved from the (flat universe) Friedmann equations,

$$H^2 = \left(\frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \bar{\rho} \quad (2.2)$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} (\bar{\rho} + 3\bar{p}) \quad (2.3)$$

³For cosmological perturbation theory for open or closed FRW universe, see e.g. Mukhanov, Feldman, Brandenberger, Phys. Rep. **215**, 203 (1992).

where $H \equiv \dot{a}/a$ is the Hubble parameter,

$$\cdot \equiv \frac{d}{dt},$$

and the $\bar{\rho}$ and \bar{p} are the homogeneous background energy density and pressure. Another version of the second Friedmann equation is

$$\dot{H} \equiv \frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 = -4\pi G(\bar{\rho} + \bar{p}). \quad (2.4)$$

We shall find it more convenient to use as a time coordinate the *conformal time* η , defined by

$$d\eta = \frac{dt}{a(t)} \quad (2.5)$$

so that the background metric is

$$ds^2 = \bar{g}_{\mu\nu} dx^\mu dx^\nu = a^2(\eta) [-d\eta^2 + \delta_{ij} dx^i dx^j] = a^2(\eta) (-d\eta^2 + dx^2 + dy^2 + dz^2). \quad (2.6)$$

That is,

$$\bar{g}_{\mu\nu} = a^2(\eta) \eta_{\mu\nu} \quad \Rightarrow \quad \bar{g}^{\mu\nu} = a^{-2}(\eta) \eta^{\mu\nu}, \quad (2.7)$$

where

$$[\eta_{\mu\nu}] \equiv [\eta^{\mu\nu}] \equiv \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \quad (2.8)$$

(the “Minkowski metric”).

Using the conformal time, the Friedmann equations are (**exercise**)

$$\mathcal{H}^2 = \left(\frac{a'}{a}\right)^2 = \frac{8\pi G}{3} \bar{\rho} a^2 \quad (2.9)$$

$$\mathcal{H}' = -\frac{4\pi G}{3} (\bar{\rho} + 3\bar{p}) a^2, \quad (2.10)$$

where

$$' \equiv \frac{d}{d\eta} = a \frac{d}{dt} = a(\cdot) \quad (2.11)$$

and

$$\mathcal{H} \equiv \frac{a'}{a} = aH = \dot{a} \quad (2.12)$$

is the *conformal*, or *comoving*, Hubble parameter. Note that

$$\mathcal{H}' = \left(\frac{a'}{a}\right)' = \frac{a''}{a} - \left(\frac{a'}{a}\right)^2 = (a\dot{a})' - \dot{a}^2 = a\ddot{a} = a^2 \frac{\ddot{a}}{a} = a^2(\dot{H} + H^2). \quad (2.13)$$

The energy continuity equation

$$\dot{\bar{\rho}} = -3H(\bar{\rho} + \bar{p}) \quad (2.14)$$

becomes just

$$\bar{\rho}' = -3\mathcal{H}(\bar{\rho} + \bar{p}) \equiv -3\mathcal{H}(1 + w)\bar{\rho}. \quad (2.15)$$

For later convenience we define the equation-of-state parameter

$$w \equiv \frac{\bar{p}}{\bar{\rho}} \quad (2.16)$$

and the speed-of-sound parameter⁴

$$c_s^2 \equiv \frac{\dot{\bar{p}}}{\dot{\bar{\rho}}} \equiv \frac{\bar{p}'}{\bar{\rho}'}. \quad (2.17)$$

These two quantities always refer to the background values.

From the Friedmann equations (2.9,2.10) and the continuity equation (2.15) one easily derives additional useful *background relations*, like (**exercise**)

$$\mathcal{H}' = -\frac{1}{2}(1+3w)\mathcal{H}^2, \quad (2.18)$$

$$2\mathcal{H}' + \mathcal{H}^2 = -8\pi G\bar{\rho}a^2 = -3w\mathcal{H}^2, \quad (2.19)$$

$$\frac{w'}{1+w} = 3\mathcal{H}(w - c_s^2), \quad (2.20)$$

and

$$\bar{p}' = w\bar{\rho}' + w'\bar{\rho} = -3\mathcal{H}(1+w)c_s^2\bar{\rho}. \quad (2.21)$$

Eq. (2.18) shows that $w = -\frac{1}{3}$ corresponds to constant comoving Hubble length $\mathcal{H}^{-1} = \text{const.}$ For $w < -\frac{1}{3}$ the comoving Hubble length shrinks with time (“inflation”), whereas for $w > -\frac{1}{3}$ it grows with time (“normal” expansion). When $w = \text{const.}$, we have $c_s^2 = w$.

From these one can derive (**exercise**) further background relations, useful⁵ for converting from equation-of-state quantities w and c_s^2 to Hubble quantities:

$$\begin{aligned} w &= -\frac{1}{3} \left(1 + \frac{2\mathcal{H}'}{\mathcal{H}^2} \right) = -1 - \frac{2\dot{H}}{3H^2} \Rightarrow 1+w = \frac{2}{3} \left(1 - \frac{\mathcal{H}'}{\mathcal{H}^2} \right) = -\frac{2\dot{H}}{3H^2} \\ c_s^2 - w &= \frac{1}{3\mathcal{H}^2} \left[\frac{\mathcal{H}\mathcal{H}'' - 2(\mathcal{H}')^2}{\mathcal{H}^2 - \mathcal{H}'} \right] = -\frac{1}{3H\dot{H}} \left(\ddot{H} - \frac{2\dot{H}^2}{H} \right) \\ c_s^2 &= \frac{1}{3\mathcal{H}} \left(\frac{\mathcal{H}'' - \mathcal{H}\mathcal{H}' - \mathcal{H}^3}{\mathcal{H}^2 - \mathcal{H}'} \right) = -1 - \frac{\ddot{H}}{3H\dot{H}} \\ 1 + c_s^2 &= \frac{\mathcal{H}'' - 4\mathcal{H}\mathcal{H}' + 2\mathcal{H}^3}{3\mathcal{H}(\mathcal{H}^2 - \mathcal{H}')} = -\frac{\ddot{H}}{3H\dot{H}}. \end{aligned} \quad (2.22)$$

We see that the Hubble parameter H decreases with time ($\dot{H} < 0$) for normal forms of matter and energy ($w > -1$) and is constant for a universe with nothing but vacuum energy ($w = -1$). The case $w < -1$ is called *phantom energy*. We shall always assume $w > -1$ (unless otherwise specified).

⁴Its square root turns out to be the speed of sound if our ρ and p describe ordinary fluid. Even if they do not, we nevertheless define this quantity, although the name may then be misleading.

⁵Well, some of the \mathcal{H} forms do not look very useful. I list them here for completeness.

3 The Perturbed Universe

We write the metric of the perturbed (around FRW(0)) universe as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu} \equiv a^2(\eta_{\mu\nu} + h_{\mu\nu}) \quad (\text{defines } h_{\mu\nu}), \quad (3.1)$$

where $h_{\mu\nu}$, as well as $h_{\mu\nu,\rho}$ and $h_{\mu\nu,\rho\sigma}$ are assumed small. Since we are doing first-order perturbation theory, we shall drop from all equations all terms with are of order $\mathcal{O}(h^2)$ or higher, and just write “=” to signify equality to first order in $h_{\mu\nu}$. Here the perturbation $h_{\mu\nu}$ is not a tensor in the perturbed universe, neither is $\eta_{\mu\nu}$, but we *define*

$$h_{\nu}^{\mu} \equiv \eta^{\mu\rho} h_{\rho\nu}, \quad h^{\mu\nu} \equiv \eta^{\mu\rho} \eta^{\nu\sigma} h_{\rho\sigma}. \quad (3.2)$$

One easily finds (**exercise**), that the inverse metric of the perturbed spacetime is

$$g^{\mu\nu} = a^{-2}(\eta^{\mu\nu} - h^{\mu\nu}) \quad (3.3)$$

(to first order).

We shall now give different names for the time and space components of the perturbed metric, defining⁶

$$[h_{\mu\nu}] \equiv \begin{bmatrix} -2A & -B_i \\ -B_i & -2D\delta_{ij} + 2E_{ij} \end{bmatrix} \quad (3.4)$$

where

$$D \equiv -\frac{1}{6}h_i^i \equiv -\frac{1}{6}h \quad (3.5)$$

carries the trace h of the spatial metric perturbation h_{ij} , and E_{ij} is traceless,

$$\delta^{ij}E_{ij} \equiv E_i^i \equiv E_{ii} = 0. \quad (3.6)$$

Since indices on $h_{\mu\nu}$ are raised and lowered with $\eta_{\mu\nu}$, we immediately have

$$[h^{\mu\nu}] \equiv \begin{bmatrix} -2A & +B_i \\ +B_i & -2D\delta_{ij} + 2E_{ij} \end{bmatrix} \quad (3.7)$$

On B_i and E_{ij} we do not raise/lower indices (or if we do, it is just the same thing, $B^i = \delta^{ij}B_j = B_i$).

The line element is thus

$$ds^2 = a^2(\eta) \left\{ -(1 + 2A)d\eta^2 - 2B_i d\eta dx^i + [(1 - 2D)\delta_{ij} + 2E_{ij}] dx^i dx^j \right\}. \quad (3.8)$$

The function $A(\eta, x^i)$ is called the *lapse function*, and $B_i(\eta, x^i)$ the *shift vector*.

Index gymnastics. We assume the reader is familiar with the standard use of index notation in GR. But in case this has been forgotten by some, we give a brief review here. We use greek indices ($\alpha, \beta, \gamma, \dots, \mu, \nu, \rho, \dots$) to denote spacetime components, so that they take values 0,1,2,3, where 0 denotes the time component, and 1,2,3 denote space components. We use latin indices (i, j, k, l) to denote just the space components, so they take only values 1,2,3. We use the *Einstein summation rule*: If an index appears twice in an equation term, once down and once up, this implies summation over its range of values (i.e., this is a device to avoid writing the summation sign \sum), e.g.,

$$A_{\nu}^{\mu} = B^{\mu\alpha} C_{\alpha\nu} \quad (3.9)$$

means that

$$A_{\nu}^{\mu} = B^{\mu 0} C_{0\nu} + B^{\mu 1} C_{1\nu} + B^{\mu 2} C_{2\nu} + B^{\mu 3} C_{3\nu} \quad (3.10)$$

⁶Liddle and Lyth ([3], Eq. (14.95); [7], Eq. (8.8)) have the opposite sign convention for D .

holds for all $\mu = 0, 1, 2, 3$ and $\nu = 0, 1, 2, 3$. For coordinates, x^μ refers to all spacetime coordinates x^0, x^1, x^2, x^3 , where x^0 is the time coordinate, and x^i just to the three space coordinates.

In full GR, the position of index (up or down) carries a specific meaning. Tensor and vector components with indices up ($T^{\mu\nu}, u^\mu$) are called *contravariant components* and those with indices down ($T_{\mu\nu}, u_\mu$) are called *covariant components*. These are related to each other via the metric tensor $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$ (raising and lowering indices):

$$u_\mu = g_{\mu\nu} u^\nu, \quad u^\mu = g^{\mu\nu} u_\nu, \quad T_{\mu\nu} = g_{\mu\alpha} g_{\nu\beta} T^{\alpha\beta}, \quad T^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} T_{\alpha\beta}, \quad T^\mu_\nu = g^{\mu\alpha} T_{\alpha\nu}, \quad T_\mu^\nu = g^{\nu\alpha} T_{\mu\alpha}. \quad (3.11)$$

The metric tensor is symmetric and $g^{\mu\alpha} g_{\alpha\nu} = \delta^\mu_\nu$. In case a tensor is symmetric, i.e., $T_{\mu\nu} = T_{\nu\mu} \Rightarrow T^{\mu\nu} = T^{\nu\mu}$ we get that $T^\mu_\nu = T_\nu^\mu$ and we write the indices on top of each other: T^μ_ν (but note that usually $T^\mu_\nu \neq T_\nu^\mu$ in this case).

A *free index*, i.e., one that appears only once in a term and is thus not summed over, has an unspecified value and thus “represents” its all possible values, and so, e.g. $A^{\mu\nu}$ represents its all 16 components (10, if it is symmetric), i.e., the entire tensor. We may switch between this component notation and matrix notation, when we want to show the values of all components:

$$A_{\mu\nu} = \begin{bmatrix} A_{00} & A_{01} & A_{02} & A_{03} \\ A_{10} & A_{11} & A_{12} & A_{13} \\ A_{20} & A_{21} & A_{22} & A_{23} \\ A_{30} & A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} A_{00} & A_{0i} \\ A_{i0} & A_{ij} \end{bmatrix}, \quad T^\mu_\nu = \begin{bmatrix} T^0_0 & T^0_1 & T^0_2 & T^0_3 \\ T^1_0 & T^1_1 & T^1_2 & T^1_3 \\ T^2_0 & T^2_1 & T^2_2 & T^2_3 \\ T^3_0 & T^3_1 & T^3_2 & T^3_3 \end{bmatrix} = \begin{bmatrix} T^0_0 & T^0_i \\ T^i_0 & T^i_j \end{bmatrix}. \quad (3.12)$$

The use of equals sign “=” in (3.12) is a somewhat relaxed practice, since on the left we have a single component of a tensor, and on the right the matrix of the full set of components. Up to now I had used the notation $[A_{\mu\nu}]$ to indicate the matrix of the components $A_{\mu\nu}$ but I drop the brackets from now on. Note that if indices are on top of each other, as in T^μ_ν , the upper index is taken to be the first index when switching to matrix notation, i.e., it indicates the row in the matrix. Note that $T^\mu_\mu = T^0_0 + T^1_1 + T^2_2 + T^3_3$, the *trace* of T^μ_ν , and $T^i_i = T^1_1 + T^2_2 + T^3_3$, the *trace* of T^i_j (space part of T^μ_ν).

We use the special symbols *Kronecker delta*

$$\delta_{ij} = \delta_j^i = \delta^{ij} \equiv (0 \text{ if } i \neq j, 1 \text{ if } i = j) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \quad (3.13)$$

and “Minkowski metric”

$$\eta^{\mu\nu} = \eta_{\mu\nu} \equiv (0 \text{ if } \mu \neq \nu, -1 \text{ if } \mu = \nu = 0, 1 \text{ if } \mu = \nu \neq 0) = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & & & \\ & \delta_{ij} & & \end{bmatrix}. \quad (3.14)$$

They are not tensors, they are just symbols whose “components” have the fixed values indicated above. The position of the index does not change their values (but we never write η^μ_ν , since in Minkowski space this was a tensor and raising one index in Minkowski space would change this to δ^μ_ν). Note that $\delta^i_i = \delta^1_1 + \delta^2_2 + \delta^3_3 = 3$ and $\delta^\mu_\mu = 4$. We can divide T^i_j into its trace part $\frac{1}{3}\delta^i_j T^k_k$ and traceless part $T^i_j - \frac{1}{3}\delta^i_j T^k_k$.

For partial derivatives with respect to spacetime coordinates we use the short-hand notation

$$T^\mu_{\nu,\alpha} \equiv \partial_\alpha T^\mu_\nu \equiv \frac{\partial T^\mu_\nu}{\partial x^\alpha} \quad (3.15)$$

(and $T^\mu_{\nu;\alpha}$ for covariant derivatives).

In the development of perturbation theory we switch from tensors in 4D spacetime to perturbation quantities like B_i and E_{ij} , which have only one kind of components, i.e., there is no concept of raising or lowering indices. They behave just like ordinary Euclidean/Cartesian 3D vector and tensor fields (except for gauge transformations). This is possible because the background space is flat. (In perturbation theory for open or closed FRW as background spacetime, we would encounter raising and lowering of indices with the 3D metric of the background space). The trace of E_{ij} is $\delta^{ij} E_{ij} = E_{11} + E_{22} + E_{33}$. (We could shorten this notation to E_{ii} , if we extend the Einstein summation rule to cover also cases where both indices are down, or to E^i_i if we allow ourselves to place the indices of Euclidean 3D vectors occasionally up. I will actually do this at some point.)

4 Gauge Transformations

The association between points in the background spacetime and the perturbed spacetime is via the coordinate system $\{x^\alpha\}$. As we noted earlier, for a given coordinate system in the background, there are many possible coordinate systems in the perturbed spacetime, all close to each other, that we could use. In GR perturbation theory, a *gauge transformation* means a coordinate transformation between such coordinate systems in the perturbed spacetime. (It may be helpful to temporarily forget at this point what you have learned about gauge transformations in other field theories, e.g. electrodynamics, so that you can learn the properties of this concept here with a fresh mind, without preconceptions.)

In this section, we denote the coordinates of the background by x^α , and two different coordinate systems in the perturbed spacetime (corresponding to two “gauges”) by \hat{x}^α and \tilde{x}^α .⁷ The coordinates \hat{x}^α and \tilde{x}^α are related by a coordinate transformation

$$\tilde{x}^\alpha = \hat{x}^\alpha + \xi^\alpha, \quad (4.1)$$

where ξ^α and the derivatives $\xi^\alpha_{,\beta}$ are first-order small. The difference between $\frac{\partial \xi^\alpha}{\partial \hat{x}^\beta}$ and $\frac{\partial \xi^\alpha}{\partial \tilde{x}^\beta}$ is second-order small, and thus ignored, so we can write just $\xi^\alpha_{,\beta}$. In fact, we shall think of ξ^α as living on the background spacetime.

The situation is illustrated in Fig. 2. The coordinate system $\{\hat{x}^\alpha\}$ associates point \bar{P} in the background with \hat{P} , whereas $\{\tilde{x}^\alpha\}$ associates the same background point \bar{P} with another point \tilde{P} . The association is by

$$\tilde{x}^\alpha(\tilde{P}) = \hat{x}^\alpha(\hat{P}) = x^\alpha(\bar{P}). \quad (4.2)$$

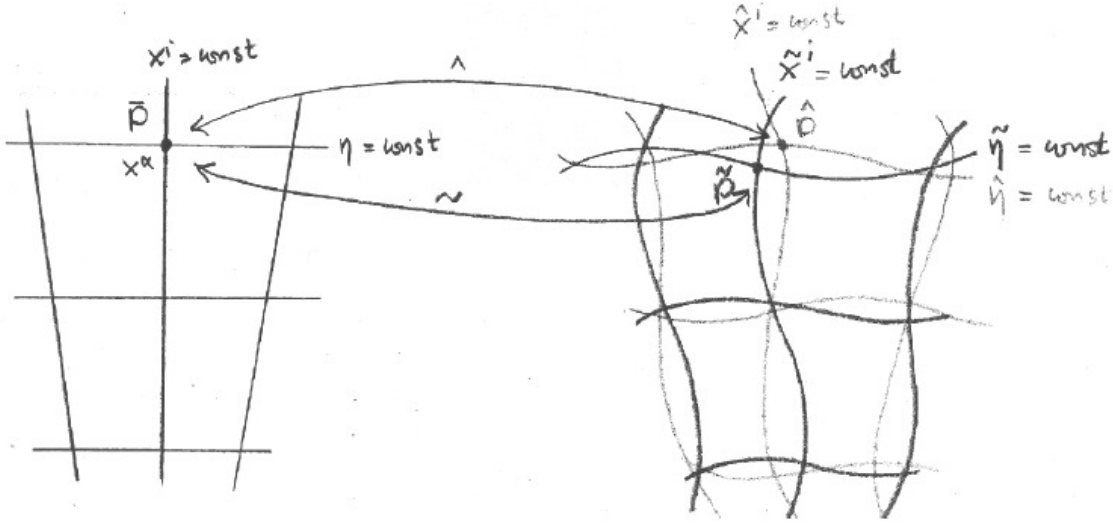


Figure 2: Gauge transformation. The background spacetime is on the left and the perturbed spacetime, with two different coordinate systems, is on the right. The “const” for the coordinate lines refer to the same constant for the two coordinate systems, i.e., $\tilde{\eta} = \hat{\eta} = \text{const} \equiv \eta(\bar{P})$ and $\tilde{x}^i = \hat{x}^i = \text{const} \equiv x^i(\bar{P})$.

The coordinate transformation relates the coordinates of the *same* point in the perturbed spacetime, i.e.,

$$\begin{aligned} \tilde{x}^\alpha(\tilde{P}) &= \hat{x}^\alpha(\tilde{P}) + \xi^\alpha \\ \tilde{x}^\alpha(\hat{P}) &= \hat{x}^\alpha(\hat{P}) + \xi^\alpha. \end{aligned} \quad (4.3)$$

⁷Ordinarily (when not doing gauge transformations) we write just x^α for both the background and perturbed spacetime coordinates.

Now the difference $\xi^\alpha(\tilde{P}) - \xi^\alpha(\hat{P})$ is second-order small. Thus we write just ξ^α and associate it with the background point:

$$\xi^\alpha = \xi^\alpha(\bar{P}) = \xi^\alpha(x^\beta).$$

Using Eqs. (4.2) and (4.3), we get the relation between the coordinates of the two different points in a given coordinate system,

$$\begin{aligned}\hat{x}^\alpha(\tilde{P}) &= \hat{x}^\alpha(\hat{P}) - \xi^\alpha \\ \tilde{x}^\alpha(\tilde{P}) &= \tilde{x}^\alpha(\hat{P}) - \xi^\alpha.\end{aligned}\tag{4.4}$$

Let us now consider how various tensors transform in the gauge transformation. We have, of course, the usual GR transformation rules for scalars (s), vectors (w^α), and other tensors,

$$\begin{aligned}s &= s \\ w^{\tilde{\alpha}} &= X^{\tilde{\alpha}}_{\hat{\beta}} w^{\hat{\beta}} \\ A^{\tilde{\alpha}}_{\tilde{\beta}} &= X^{\tilde{\alpha}}_{\hat{\gamma}} X^{\hat{\delta}}_{\tilde{\beta}} A^{\hat{\gamma}}_{\hat{\delta}} \\ B_{\tilde{\alpha}\tilde{\beta}} &= X^{\hat{\gamma}}_{\tilde{\alpha}} X^{\hat{\delta}}_{\tilde{\beta}} B_{\hat{\gamma}\hat{\delta}}\end{aligned}\tag{4.5}$$

where

$$\begin{aligned}X^{\tilde{\alpha}}_{\hat{\beta}} &\equiv \frac{\partial \tilde{x}^\alpha}{\partial \hat{x}^\beta} = \delta^\alpha_\beta + \xi^\alpha_{,\beta} \\ X^{\hat{\delta}}_{\tilde{\beta}} &\equiv \frac{\partial \hat{x}^\delta}{\partial \tilde{x}^\beta} = \delta^\delta_\beta - \xi^\delta_{,\beta}.\end{aligned}\tag{4.6}$$

These rules refer to the values of these quantities at a given point in the perturbed spacetime. However, this is not what we want now! Please, pay attention, since the following is central to understanding GR perturbation theory!

We shall be interested in perturbations of various quantities. In the background spacetime we may have various 4-scalar fields \bar{s} , 4-vector fields \bar{w}^α and tensor fields \bar{A}^α_β , $\bar{B}_{\alpha\beta}$. In the perturbed spacetime we have corresponding perturbed quantities,

$$\begin{aligned}s &= \bar{s} + \delta s \\ w^\alpha &= \bar{w}^\alpha + \delta w^\alpha \\ A^\alpha_\beta &= \bar{A}^\alpha_\beta + \delta A^\alpha_\beta \\ B_{\alpha\beta} &= \bar{B}_{\alpha\beta} + \delta B_{\alpha\beta}.\end{aligned}\tag{4.7}$$

Consider first the 4-scalar s . The full quantity $s = \bar{s} + \delta s$ lives on the perturbed spacetime. However, we cannot assign a unique background quantity \bar{s} to a point in the perturbed spacetime, because in different gauges this point is associated with different points in the background, with different values of \bar{s} . Therefore there is also no unique perturbation δs , but *the perturbation is gauge-dependent*. The perturbations in different gauges are defined as

$$\begin{aligned}\hat{\delta s}(x^\alpha) &\equiv s(\hat{P}) - \bar{s}(\bar{P}) \\ \tilde{\delta s}(x^\alpha) &\equiv s(\tilde{P}) - \bar{s}(\bar{P}).\end{aligned}\tag{4.8}$$

The perturbation δs is obtained from a subtraction between two spacetimes, and we consider it as *living on the background spacetime*. It changes in the gauge transformation. Relate now $\hat{\delta s}$ to $\tilde{\delta s}$:

$$s(\tilde{P}) = s(\hat{P}) + \frac{\partial s}{\partial \hat{x}^\alpha}(\hat{P}) \left[\hat{x}^\alpha(\tilde{P}) - \hat{x}^\alpha(\hat{P}) \right] = s(\hat{P}) - \frac{\partial s}{\partial \hat{x}^\alpha}(\hat{P}) \xi^\alpha = s(\hat{P}) - \frac{\partial \bar{s}}{\partial \hat{x}^\alpha}(\bar{P}) \xi^\alpha,$$

where we approximated $\frac{\partial s}{\partial \hat{x}^\alpha}(\hat{P}) \approx \frac{\partial \bar{s}}{\partial x^\alpha}(\bar{P})$, since the difference⁸ between them is a first order perturbation, and multiplication by ξ^α makes it second order.

Since our *background is homogeneous* (but time dependent!) , $\bar{s} = \bar{s}(\eta, x^i) = \bar{s}(\eta)$ only, and

$$\frac{\partial \bar{s}}{\partial x^\alpha}(\bar{P})\xi^\alpha = \frac{\partial \bar{s}}{\partial \eta}(\bar{P})\xi^0 = \bar{s}'\xi^0.$$

Thus we get

$$s(\tilde{P}) = s(\hat{P}) - \bar{s}'\xi^0, \quad (4.9)$$

and our final result for the gauge transformation of δs is

$$\widehat{\delta s}(x^\alpha) = s(\hat{P}) - \bar{s}'\xi^0 - \bar{s}(\bar{P}) = \widehat{\delta s}(x^\alpha) - \bar{s}'\xi^0. \quad (4.10)$$

In analogy with (4.8), the perturbations in vector and tensor fields in the two gauges are defined

$$\begin{aligned} \widehat{\delta w}^\alpha(x^\beta) &\equiv w^{\hat{\alpha}}(\hat{P}) - \bar{w}^\alpha(\bar{P}) \\ \widetilde{\delta w}^\alpha(x^\beta) &\equiv w^{\tilde{\alpha}}(\tilde{P}) - \bar{w}^\alpha(\bar{P}). \end{aligned} \quad (4.11)$$

and

$$\begin{aligned} \widehat{\delta A}^\alpha_\beta(x^\gamma) &\equiv A^{\hat{\alpha}}_{\hat{\beta}}(\hat{P}) - \bar{A}^\alpha_\beta(\bar{P}) \\ \widetilde{\delta A}^\alpha_\beta(x^\gamma) &\equiv A^{\tilde{\alpha}}_{\tilde{\beta}}(\tilde{P}) - \bar{A}^\alpha_\beta(\bar{P}) \\ \widehat{\delta B}_{\alpha\beta}(x^\gamma) &\equiv B_{\hat{\alpha}\hat{\beta}}(\hat{P}) - \bar{B}_{\alpha\beta}(\bar{P}) \\ \widetilde{\delta B}_{\alpha\beta}(x^\gamma) &\equiv B_{\tilde{\alpha}\tilde{\beta}}(\tilde{P}) - \bar{B}_{\alpha\beta}(\bar{P}). \end{aligned} \quad (4.12)$$

Consider the case of a type (0, 2) 4-tensor field. We have

$$B_{\hat{\mu}\hat{\nu}}(\tilde{P}) = B_{\hat{\mu}\hat{\nu}}(\hat{P}) + \frac{\partial B_{\hat{\mu}\hat{\nu}}}{\partial \hat{x}^\alpha} \left[\hat{x}^\alpha(\tilde{P}) - \hat{x}^\alpha(\hat{P}) \right] = B_{\hat{\mu}\hat{\nu}}(\hat{P}) - \frac{\partial \bar{B}_{\mu\nu}}{\partial x^\alpha}(\bar{P})\xi^\alpha \quad (4.13)$$

and

$$\begin{aligned} B_{\hat{\mu}\hat{\nu}}(\tilde{P}) &= X_{\hat{\mu}}^{\hat{\rho}} X_{\hat{\nu}}^{\hat{\sigma}} B_{\hat{\rho}\hat{\sigma}}(\tilde{P}) = (\delta_{\hat{\mu}}^{\hat{\rho}} - \xi_{\hat{\mu},\hat{\mu}}^{\hat{\rho}})(\delta_{\hat{\nu}}^{\hat{\sigma}} - \xi_{\hat{\nu},\hat{\nu}}^{\hat{\sigma}}) \left[B_{\hat{\rho}\hat{\sigma}}(\hat{P}) - \frac{\partial \bar{B}_{\rho\sigma}}{\partial x^\alpha}(\bar{P})\xi^\alpha \right] \\ &= B_{\hat{\mu}\hat{\nu}}(\hat{P}) - \xi_{\hat{\mu},\hat{\mu}}^{\hat{\rho}} B_{\hat{\rho}\hat{\nu}}(\hat{P}) - \xi_{\hat{\nu},\hat{\nu}}^{\hat{\sigma}} B_{\hat{\mu}\hat{\sigma}}(\hat{P}) - \frac{\partial \bar{B}_{\mu\nu}}{\partial x^\alpha}(\bar{P})\xi^\alpha \\ &= B_{\hat{\mu}\hat{\nu}}(\hat{P}) - \xi_{\hat{\mu},\hat{\mu}}^{\hat{\rho}} \bar{B}_{\rho\nu}(\bar{P}) - \xi_{\hat{\nu},\hat{\nu}}^{\hat{\sigma}} \bar{B}_{\mu\sigma}(\bar{P}) - \frac{\partial \bar{B}_{\mu\nu}}{\partial x^\alpha}(\bar{P})\xi^\alpha, \end{aligned} \quad (4.14)$$

where we can replace $B_{\hat{\mu}\hat{\sigma}}(\hat{P})$ with $\bar{B}_{\mu\sigma}(\bar{P})$ in the two middle terms, since it is multiplied by a first-order quantity $\xi_{\hat{\nu},\hat{\nu}}^{\hat{\sigma}}$ and we can thus ignore the perturbation part, which becomes second order.

Subtracting the background value at \bar{P} we get the gauge transformation rule for the tensor perturbation $\delta B_{\mu\nu}$,

$$\begin{aligned} \widetilde{\delta B}_{\mu\nu} &\equiv B_{\hat{\mu}\hat{\nu}}(\tilde{P}) - \bar{B}_{\mu\nu}(\bar{P}) \\ &= B_{\hat{\mu}\hat{\nu}}(\hat{P}) - \bar{B}_{\mu\nu}(\bar{P}) - \xi_{\hat{\mu},\hat{\mu}}^{\hat{\rho}} \bar{B}_{\rho\nu}(\bar{P}) - \xi_{\hat{\nu},\hat{\nu}}^{\hat{\sigma}} \bar{B}_{\mu\sigma}(\bar{P}) - \frac{\partial \bar{B}_{\mu\nu}}{\partial x^\alpha}(\bar{P})\xi^\alpha \\ &= \widehat{\delta B}_{\mu\nu} - \xi_{\hat{\mu},\hat{\mu}}^{\hat{\rho}} \bar{B}_{\rho\nu} - \xi_{\hat{\nu},\hat{\nu}}^{\hat{\sigma}} \bar{B}_{\mu\sigma} - \bar{B}_{\mu\nu,\alpha}\xi^\alpha. \end{aligned} \quad (4.15)$$

⁸This difference is the perturbation of the covariant vector $s_{,\alpha}$. We are assuming perturbations are first order small also in quantities derived by covariant derivation from the “primary” quantities.

In a similar manner we obtain the gauge transformation rules for 4-vector perturbations,

$$\widetilde{\delta w}^\alpha = \widehat{\delta w}^\alpha + \xi^\alpha_{,\beta} \bar{w}^\beta - \bar{w}^\alpha_{,\beta} \xi^\beta \quad (4.16)$$

and perturbations of type (1,1) 4-tensors (**exercise**),

$$\widetilde{\delta A}_\nu^\mu = \widehat{\delta A}_\nu^\mu + \xi^\mu_{,\rho} \bar{A}_\nu^\rho - \xi^\sigma_{,\nu} \bar{A}_\sigma^\mu - \bar{A}_{\nu,\alpha}^\mu \xi^\alpha. \quad (4.17)$$

Since the background is isotropic and homogeneous, and our background coordinate system fully respects these properties, the background 4-vectors and tensors must be of the form

$$\bar{w}^\alpha = (\bar{w}^0, \vec{0}) \quad \bar{A}_\beta^\alpha = \begin{bmatrix} \bar{A}_0^0 & 0 \\ 0 & \frac{1}{3} \delta_j^i \bar{A}_k^k \end{bmatrix}, \quad (4.18)$$

and they depend only on the (conformal) time coordinate η . Using these properties we can write (**exercise**) the gauge transformation rules for the individual components of 4-scalar, 4-vector and type (1,1) 4-tensor perturbations (we now drop the hats from the first gauge) as

$$\begin{aligned} \widetilde{\delta s} &= \delta s - \bar{s}' \xi^0 \\ \widetilde{\delta w}^0 &= \delta w^0 + \xi^0_{,0} \bar{w}^0 - \bar{w}^0_{,0} \xi^0 \\ \widetilde{\delta w}^i &= \delta w^i + \xi^i_{,0} \bar{w}^0 \\ \widetilde{\delta A}_0^0 &= \delta A_0^0 - \bar{A}_{0,0}^0 \xi^0 \\ \widetilde{\delta A}_i^0 &= \delta A_i^0 + \frac{1}{3} \xi^0_{,i} \bar{A}_k^k - \xi^0_{,i} \bar{A}_0^0 \\ \widetilde{\delta A}_0^i &= \delta A_0^i + \xi^i_{,0} \bar{A}_0^0 - \frac{1}{3} \xi^i_{,0} \bar{A}_k^k \\ \widetilde{\delta A}_j^i &= \delta A_j^i - \frac{1}{3} \delta_j^i \bar{A}_{k,0}^k \xi^0. \end{aligned} \quad (4.19)$$

$$\begin{aligned} \widetilde{\delta A}_k^k &= \delta A_k^k - \bar{A}_{k,0}^k \xi^0 \\ \widetilde{\delta A}_j^i - \frac{1}{3} \delta_j^i \widetilde{\delta A}_k^k &= \delta A_j^i - \frac{1}{3} \delta_j^i \delta A_k^k. \end{aligned} \quad (4.20)$$

The following combinations (the trace and the traceless part of $\widetilde{\delta A}_j^i$) are also useful:

$$\begin{aligned} \widetilde{\delta A}_k^k &= \delta A_k^k - \bar{A}_{k,0}^k \xi^0 \\ \widetilde{\delta A}_j^i - \frac{1}{3} \delta_j^i \widetilde{\delta A}_k^k &= \delta A_j^i - \frac{1}{3} \delta_j^i \delta A_k^k. \end{aligned} \quad (4.21)$$

Thus *the traceless part of δA_j^i is gauge-invariant!*

4.1 Gauge Transformation of the Metric Perturbations

Applying the gauge transformation equation (4.15) to the metric perturbation, we have

$$\widetilde{\delta g}_{\mu\nu} = \delta g_{\mu\nu} - \xi^\rho_{,\mu} \bar{g}_{\rho\nu} - \xi^\sigma_{,\nu} \bar{g}_{\mu\sigma} - \bar{g}_{\mu\nu,0} \xi^0, \quad (4.22)$$

where we have replaced the sum $\bar{g}_{\mu\nu,\alpha} \xi^\alpha$ with $\bar{g}_{\mu\nu,0} \xi^0$, since the background metric depends only on the time coordinate $x^0 = \eta$, and dropped the hats from the first gauge. Remembering $\bar{g}_{\mu\nu} = a^2(\eta) \eta_{\mu\nu}$ from Eq. (2.7), we have

$$\bar{g}_{\mu\nu,0} = 2a' a \eta_{\mu\nu} \quad (4.23)$$

and

$$\widetilde{\delta g}_{\mu\nu} = \delta g_{\mu\nu} + a^2 \left[-\xi^\rho_{,\mu} \eta_{\rho\nu} - \xi^\sigma_{,\nu} \eta_{\mu\sigma} - 2 \frac{a'}{a} \eta_{\mu\nu} \xi^0 \right]. \quad (4.24)$$

From Eqs. (3.1) and (3.4) we have

$$[\delta g_{\mu\nu}] = a^2 \begin{bmatrix} -2A & -B_i \\ -B_i & -2D\delta_{ij} + 2E_{ij} \end{bmatrix} \quad (4.25)$$

Applying the gauge transformation law (4.24) now separately to the different metric perturbation components, we get first

$$\begin{aligned} \tilde{\delta g}_{00} &\equiv -2a^2 \tilde{A} \\ &= \delta g_{00} + a^2 \left(-\xi^{\rho}_{,0} \eta_{\rho 0} - \xi^{\sigma}_{,0} \eta_{0\sigma} - 2 \frac{a'}{a} \eta_{00} \xi^0 \right) \\ &= -2a^2 A + a^2 \left(+\xi^0_{,0} + \xi^0_{,0} + 2 \frac{a'}{a} \xi^0 \right), \end{aligned} \quad (4.26)$$

from which we obtain the gauge transformation law

$$\tilde{A} = A - \xi^0_{,0} - \frac{a'}{a} \xi^0. \quad (4.27)$$

Similarly, from δg_{0i} we obtain

$$\tilde{B}_i = B_i + \xi^i_{,0} - \xi^0_{,i}, \quad (4.28)$$

and from δg_{ij} ,

$$-\tilde{D}\delta_{ij} + \tilde{E}_{ij} = -D\delta_{ij} + E_{ij} - \frac{1}{2}(\xi^i_{,j} + \xi^j_{,i}) - \frac{a'}{a} \xi^0 \delta_{ij}. \quad (4.29)$$

The trace of $\frac{1}{2}(\xi^i_{,j} + \xi^j_{,i})$ is $\xi^k_{,k}$, so we can write

$$\frac{1}{2}(\xi^i_{,j} + \xi^j_{,i}) = \frac{1}{3}\delta_{ij}\xi^k_{,k} + \frac{1}{2}(\xi^i_{,j} + \xi^j_{,i}) - \frac{1}{3}\delta_{ij}\xi^k_{,k}, \quad (4.30)$$

where the last two terms are the traceless part, and we can separate Eq. (4.29) into

$$\begin{aligned} \tilde{D} &= D + \frac{1}{3}\xi^k_{,k} + \frac{a'}{a} \xi^0 \\ \tilde{E}_{ij} &= E_{ij} - \frac{1}{2}(\xi^i_{,j} + \xi^j_{,i}) + \frac{1}{3}\delta_{ij}\xi^k_{,k}. \end{aligned} \quad (4.31)$$

Sometimes it may turn out that a gauge transformation causes all the perturbations to vanish. This means that the perturbations were not real (or physical)—the perturbed spacetime differed from the background spacetime only by having a perturbed coordinate system! We say that such perturbations are “pure gauge”.

4.2 Perturbation theory as gauge theory

The perturbations are defined as differences between the perturbed spacetime and the background spacetime. So in which spacetime do they “live”? What we are doing in perturbation theory, is that we are building a description of the perturbed spacetime as a set of functions, or fields, the perturbations, which live on the background spacetime⁹. *The background spacetime is the canvas on which we are painting the picture of the perturbed spacetime.* For the same perturbed spacetime there exists many equivalent descriptions of this kind and these are related by gauge transformations. The formalism of this theory of fields in the background spacetime is similar to gauge theories of particle physics.

⁹In practice, they are just functions of the four coordinates, η, x, y, z , of the background spacetime, and since in cosmological perturbation theory the time slicing of the background spacetime is fixed, and since we here just consider flat background spacetimes, the perturbations become just fields in Euclidean space that evolve in time.

5 Separation into Scalar, Vector, and Tensor Perturbations

In GR perturbation theory there are two kinds of coordinate transformations of interest. One is the gauge transformation just discussed, where the coordinates of the background are kept fixed, but the coordinates in the perturbed spacetime are changed, changing the correspondence between the points in the background and the perturbed spacetime.

The other kind is one where we keep the gauge, i.e., the correspondence between the background and perturbed spacetime points, fixed, but do a coordinate transformation in the background spacetime. This then induces a corresponding coordinate transformation in the perturbed spacetime. Our background coordinate system was chosen to respect the symmetries of the background, and we do not want to lose this property. In cosmological perturbation theory we have chosen the background coordinates to respect its homogeneity property, which gives us a unique slicing of the spacetime into homogeneous $t = \text{const.}$ spacelike slices. Thus we do not want to change this slicing. This leaves us:

1. homogeneous transformations of the time coordinate, i.e., reparameterizations of time, of which we already had an example, when we switched from cosmic time t to conformal time η ,
2. and transformations in the space coordinates¹⁰

$$x^{i'} = X^{i'}_k x^k, \quad (5.1)$$

where $X^{i'}_k$ is independent of time; which is the case we consider in this section.

We had chosen the coordinates for our background, FRW(0), so that the 3-metric was Euclidean,

$$g_{ij} = a^2 \delta_{ij}, \quad (5.2)$$

and we want to keep this property. This leaves us rotations. The full transformation matrices are then

$$X^{\mu'}_{\rho} = \begin{bmatrix} 1 & 0 \\ 0 & X^{i'}_k \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & R^{i'}_k \end{bmatrix} \quad \text{and} \quad X^{\mu}_{\rho'} = \begin{bmatrix} 1 & 0 \\ 0 & R^i_{k'} \end{bmatrix}, \quad (5.3)$$

where $R^{i'}_k$ is a rotation matrix¹¹, with the property $R^T R = I$, or $R^{i'}_k R^{i'}_l = (R^T R)_{kl} = \delta_{kl}$. Thus $R^T = R^{-1}$ so that $R^{i'}_k = R^k_{i'}$.

This coordinate transformation in the background induces the corresponding transformation,

$$x^{\mu'} = X^{\mu'}_{\rho} x^{\rho}, \quad (5.4)$$

into the perturbed spacetime. Here the metric is

$$g_{\mu\nu} = a^2 \begin{bmatrix} -1 - 2A & -B_i \\ -B_i & (1 - 2D)\delta_{ij} + 2E_{ij} \end{bmatrix} = a^2 \eta_{\mu\nu} + a^2 \begin{bmatrix} -2A & -B_i \\ -B_i & -2D\delta_{ij} + 2E_{ij} \end{bmatrix}. \quad (5.5)$$

Transforming the metric,

$$g_{\rho'\sigma'} = X^{\mu}_{\rho'} X^{\nu}_{\sigma'} g_{\mu\nu}, \quad (5.6)$$

¹⁰In this section we use $'$ to denote the other coordinate system. Do not confuse with $' \equiv d/d\eta$ in the other sections.

¹¹In this notation $R^{i'}_j$ and $R^i_{j'}$ are two different matrices, corresponding to opposite rotations; the position of the $'$ indicates which way we are rotating. We have put the first index upstairs to follow the Einstein summation convention—but we could have written $R_{i'j}$ and $R_{ij'}$ just as well.

we get for the different components

$$\begin{aligned}
g_{0'0'} &= X_{0'}^\mu X_{0'}^\nu g_{\mu\nu} = X_{0'}^0 X_{0'}^0 g_{00} = g_{00} = a^2(-1 - 2A) \\
g_{0'l'} &= X_{0'}^\mu X_{l'}^\nu g_{\mu\nu} = X_{0'}^0 X_{l'}^j g_{0j} = -a^2 R_{l'}^j B_j \\
g_{k'l'} &= X_{k'}^i X_{l'}^j g_{ij} = a^2 \left[(1 - 2D) \delta_{ij} R_{k'}^i R_{l'}^j + 2E_{ij} R_{k'}^i R_{l'}^j \right] \\
&= a^2 \left[(1 - 2D) \delta_{kl} + 2E_{ij} R_{k'}^i R_{l'}^j \right], \tag{5.7}
\end{aligned}$$

from which we identify the perturbations in the new coordinates,

$$\begin{aligned}
A' &= A \\
D' &= D \\
B_{l'} &= R_{l'}^j B_j \\
E_{k'l'} &= R_{k'}^i R_{l'}^j E_{ij}. \tag{5.8}
\end{aligned}$$

Thus A and D transform as scalars under rotations in the background spacetime coordinates, B_i transforms as a 3-vector, and E_{ij} as a 3D tensor. While staying in a fixed gauge, we can thus think of them as scalar, vector, and tensor fields on the 3D Euclidean background space. We are, however, not yet satisfied. We can extract two more scalar quantities and one more vector quantity from B_i and E_{ij} .

We know from Euclidean 3D vector calculus, that a vector field can be divided into two parts, the first one with zero curl, the second one with zero divergence,

$$\vec{B} = \vec{B}^S + \vec{B}^V, \quad \text{with} \quad \nabla \times \vec{B}^S = 0 \quad \text{and} \quad \nabla \cdot \vec{B}^V = 0, \tag{5.9}$$

and that the first one can be expressed as (minus) a gradient of some scalar field¹²

$$\vec{B}^S = -\nabla B. \tag{5.10}$$

In component notation,

$$B_i = -B_{,i} + B_i^V, \quad \text{where} \quad \delta^{ij} B_{i,j}^V = 0. \tag{5.11}$$

In like manner, the symmetric traceless tensor field E_{ij} can be divided into three parts,

$$E_{ij} = E_{ij}^S + E_{ij}^V + E_{ij}^T, \tag{5.12}$$

where E_{ij}^S and E_{ij}^V can be expressed in terms of a scalar field E and a vector field E_i ,¹³

$$E_{ij}^S = \left(\partial_i \partial_j - \frac{1}{3} \delta_{ij} \nabla^2 \right) E = E_{,ij} - \frac{1}{3} \delta_{ij} \delta^{kl} E_{,kl} \tag{5.13}$$

$$E_{ij}^V = -\frac{1}{2} (E_{i,j} + E_{j,i}), \quad \text{where} \quad \delta^{ij} E_{i,j} = \nabla \cdot \vec{E} = 0 \tag{5.14}$$

$$\text{and} \quad \delta^{ik} E_{ij,k}^T = 0, \quad \delta^{ij} E_{ij}^T = 0. \tag{5.15}$$

We see that E_{ij}^S is symmetric and traceless by construction. E_{ij}^V is symmetric by construction, and the condition on E_i makes it traceless. The tensor E_{ij}^T is assumed symmetric, and the two

¹²This sign convention corresponds to thinking of the scalar function B as a “potential”, where the vector field \vec{B}^S “flows downhill”. We use the same letter B here for both the original vector field B_i and this scalar potential B . There should be no confusion since the vector field always has an index (or “”). Same goes for the E .

¹³I have a tendency to write things like $\delta^{kl} E_{,kl}$ and (later) $\delta^{ik} k_k E_{ij}^T$, to respect the traditional Einstein summation convention, where the indices to be summed over are one up, one down; but since here the location of the space index has no effect, one could as well write these in the shorter forms $E_{,kk}$ and $k_i E_{ij}^T$.

conditions on it make it *transverse* and traceless. The meaning of “transverse” and the nature of the above construction becomes clearer in the next section when we do this in Fourier space.

Under rotations in background space,

$$\begin{aligned} A' &= A, & B' &= B, & D' &= D, & E' &= E, \\ B_{l'}^V &= R_{l'}^j B_j^V, & E_{l'} &= R_{l'}^j E_j, \\ E_{k'l'}^T &= R_{k'}^i R_{l'}^j E_{ij}^T. \end{aligned} \quad (5.16)$$

The metric perturbation can thus be divided into

1. a *scalar* part, consisting of A , B , D , and E ,
2. a *vector* part, consisting of B_i^V and E_i ,
3. and a *tensor* part E_{ij}^T .

The names “scalar”, “vector”, and “tensor” refer to their *transformation properties under rotations in the background space*.¹⁴

The Einstein tensor perturbation δG_ν^μ and the energy tensor perturbation δT_ν^μ can likewise be divided into scalar+vector+tensor; the scalar part of δG_ν^μ coming only from the scalar part of $\delta g_{\mu\nu}$ and so on.

The important thing about this division is that *the scalar, vector, and tensor parts do not couple to each other* (in first-order perturbation theory), but they evolve independently. This allows us to treat them separately: We can study, e.g., scalar perturbations as if the vector and tensor perturbations were absent. The total evolution of the full perturbation is just a linear superposition of the independent evolution of the scalar, vector, and tensor part of the perturbation.

We imposed one constraint on each of the 3-vectors B_i^V and E_i , and $3 + 1 = 4$ constraints on the symmetric 3-d tensor E_{ij}^T leaving each of them 2 independent components. Thus the 10 degrees of freedom corresponding to the 10 components of the metric perturbation $h_{\mu\nu}$ are divided into

$$\begin{aligned} 1 + 1 + 1 + 1 &= 4 & \text{scalar} & & (2 \text{ physical, } 2 \text{ gauge}) \\ 2 + 2 &= 4 & \text{vector} & & (2 \text{ physical, } 2 \text{ gauge}) \\ 2 &= 2 & \text{tensor} & & \end{aligned} \quad (5.17)$$

degrees of freedom.

The scalar perturbations are the most important. They couple to density and pressure perturbations and exhibit gravitational instability: overdense regions grow more overdense. They are responsible for the formation of structure in the universe from small initial perturbations.

Vector perturbations couple to rotational velocity perturbations in the cosmic fluid. They tend to decay in an expanding universe, and are therefore probably not important in cosmology.

We have done all of the above in a fixed gauge. It turns out that gauge transformations affect scalar and vector perturbations, but tensor perturbations are gauge-invariant (shown in Sec. 6.1). Tensor perturbations are gravitational waves, this time¹⁵ in an expanding universe. When they are extracted from the total perturbation by the above separation procedure, they are automatically in the “transverse traceless gauge”. (This expression is clarified in the next section.) Tensor perturbations have cosmological importance since, if strong enough, they have an observable effect on the anisotropy of the cosmic microwave background. Gravitational waves have also been detected, although so far (2024) only with astrophysical, not cosmological, origin.

¹⁴Thus “scalar” does *not* mean, e.g., that the perturbation would be invariant under gauge transformations—scalar perturbations are *not gauge-invariant*, as we have already seen, e.g. in Eqs. (4.27) and (4.31).

¹⁵Contrasted to perturbation theory around Minkowski space, which is the way gravitational waves are usually introduced in GR.

6 Perturbations in Fourier Space

Because our background space is flat we can Fourier expand the perturbations. For an arbitrary perturbation $f = f(\eta, x^i) = f(\eta, \vec{x})$, we write

$$f(\eta, \vec{x}) = \sum_{\vec{k}} f_{\vec{k}}(\eta) e^{i\vec{k} \cdot \vec{x}}. \quad (6.1)$$

(Using a Fourier sum implies using a fiducial box with some volume V . At the end of the day we can let $V \rightarrow \infty$, and replace remaining Fourier sums with integrals.) In first-order perturbation theory each Fourier component evolves independently. We can thus just study the evolution of a single Fourier component, with some arbitrary wave vector \vec{k} , and we drop the subscript \vec{k} from the Fourier amplitudes. (This is analogous to not writing explicitly the \vec{x} -dependence in coordinate space: in coordinate space we write just f for $f(\eta, \vec{x})$ and in Fourier space we write just f for $f_{\vec{k}}(\eta)$).

Since $\vec{x} = (x^1, x^2, x^3)$ is a comoving coordinate, \vec{k} is a *comoving wave vector*. The comoving (or coordinate) wave number $k \equiv |\vec{k}|$ and wavelength $\lambda = 2\pi/k$ are related to the physical wavelength and wave number of the Fourier mode by

$$k_{\text{phys}} = \frac{2\pi}{\lambda_{\text{phys}}} = \frac{2\pi}{a\lambda} = a^{-1}k. \quad (6.2)$$

Thus the wavelength λ_{phys} of the Fourier mode \vec{k} grows in time as the universe expands.

In the separation into scalar+vector+tensor, we follow Liddle&Lyth and **include an additional factor** $k \equiv |\vec{k}|$ in the Fourier components of B and E_i , and a factor k^2 in E , so that we have, e.g.,

$$\begin{aligned} B(\eta, \vec{x}) &= \sum_{\vec{k}} \frac{B_{\vec{k}}(\eta)}{k} e^{i\vec{k} \cdot \vec{x}} \\ E(\eta, \vec{x}) &= \sum_{\vec{k}} \frac{E_{\vec{k}}(\eta)}{k^2} e^{i\vec{k} \cdot \vec{x}}. \end{aligned} \quad (6.3)$$

The purpose of this is to make them have the same dimension and magnitude as B_i^S , E_{ij}^S and E_{ij}^V .¹⁶ That is,

$$B_i = B_i^S + B_i^V, \quad \text{and} \quad E_{ij} = E_{ij}^S + E_{ij}^V + E_{ij}^T, \quad (6.4)$$

where

$$\begin{aligned} B_i^S &= -B_{,i} & \text{becomes} & & B_i^S &= -i \frac{k_i}{k} B \\ E_{ij}^S &= (\partial_i \partial_j - \frac{1}{3} \delta_{ij} \nabla^2) E & \text{becomes} & & E_{ij}^S &= \left(-\frac{k_i k_j}{k^2} + \frac{1}{3} \delta_{ij} \right) E, \\ E_{ij}^V &= -\frac{1}{2} (E_{i,j} + E_{j,i}) & \text{becomes} & & E_{ij}^V &= -\frac{i}{2k} (k_i E_j + k_j E_i), \end{aligned} \quad (6.5)$$

and the conditions

$$\delta^{ij} B_{i,j}^V = 0, \quad \delta^{ij} E_{i,j} = 0, \quad \text{and} \quad \delta^{ik} E_{ij,k}^T = \delta^{ij} E_{ik}^T = 0 \quad (6.6)$$

¹⁶Powers of k cancel in Eqs. (6.5). The metric perturbations, A , B , D , E , B_i^V , E_i , and E_{ij}^T will then all have the same dimension in Fourier space, which facilitates comparison of their magnitudes. This unorthodox convention is actually very confusing and you have to learn to watch for it. Additional reasons for this convention are given at the end of this section.

become

$$\delta^{ij}k_j B_i^V = \vec{k} \cdot \vec{B}^V = 0, \quad \delta^{ij}k_j E_i = \vec{k} \cdot \vec{E} = 0, \quad \text{and} \quad \delta^{ik}k_k E_{ij}^T = \delta^{ij}E_{ij}^T = 0. \quad (6.7)$$

To make the separation into scalar+vector+tensor parts as clear as possible, rotate the background coordinates so that the z axis becomes parallel to \vec{k} ,

$$\vec{k} = k\hat{z} = (0, 0, k) \quad (6.8)$$

(\hat{z} denoting the unit vector in z direction.) Then

$$B_i^S = (0, 0, -iB) \quad (6.9)$$

and

$$E_{ij}^S = \begin{bmatrix} 0 & & \\ & 0 & \\ & & -E \end{bmatrix} + \begin{bmatrix} \frac{1}{3}E & & \\ & \frac{1}{3}E & \\ & & \frac{1}{3}E \end{bmatrix} = \begin{bmatrix} \frac{1}{3}E & & \\ & \frac{1}{3}E & \\ & & -\frac{2}{3}E \end{bmatrix} \quad (6.10)$$

and we can write the scalar part of $\delta g_{\mu\nu}$ as

$$\delta g_{\mu\nu}^S = a^2 \begin{bmatrix} -2A & & & +iB \\ & 2(-D + \frac{1}{3}E) & & \\ & & 2(-D + \frac{1}{3}E) & \\ +iB & & & 2(-D - \frac{2}{3}E) \end{bmatrix} \quad (6.11)$$

For the vector part we have then

$$\vec{k} \cdot \vec{B}^V = 0 \Rightarrow \vec{B}^V = (B_1, B_2, 0) \quad (6.12)$$

$$\vec{k} \cdot \vec{E} = 0 \Rightarrow \vec{E} = (E_1, E_2, 0) \quad (6.13)$$

and

$$E_{ij}^V = \frac{-i}{2k} (k_i E_j + k_j E_i) = -\frac{i}{2} \begin{bmatrix} & & E_1 \\ & & E_2 \\ E_1 & E_2 & \end{bmatrix}, \quad (6.14)$$

so that the vector part of $\delta g_{\mu\nu}$ is

$$\delta g_{\mu\nu}^V = a^2 \begin{bmatrix} & -B_1 & -B_2 & \\ -B_1 & & & -iE_1 \\ -B_2 & & & -iE_2 \\ & -iE_1 & -iE_2 & \end{bmatrix}. \quad (6.15)$$

For the tensor part,

$$\delta^{ik}k_k E_{ij}^T \equiv \sum_i k_i E_{ij}^T = 0 \Rightarrow E_{3j}^T \equiv E_{j3}^T = 0 \quad (6.16)$$

so that

$$E_{ij}^T = \begin{bmatrix} E_{11}^T & E_{12}^T \\ E_{12}^T & -E_{11}^T \end{bmatrix}. \quad (6.17)$$

The tensor part of $\delta g_{\mu\nu}$ becomes

$$\delta g_{\mu\nu}^T = a^2 \begin{bmatrix} & 2E_{11}^T & 2E_{12}^T \\ 2E_{12}^T & -2E_{11}^T & \end{bmatrix} = a^2 \begin{bmatrix} & h_+ & h_\times \\ h_\times & -h_+ & \end{bmatrix}, \quad (6.18)$$

where we have denoted the two gravitational wave polarization amplitudes by $E_{11}^T = \frac{1}{2}h_+$ and $E_{12}^T = \frac{1}{2}h_\times$.

We see how the scalar part of the perturbation is associated with the time direction, the wave direction \vec{k} and the trace. The vector part is associated with the two remaining space directions, those *transverse* to the wave vector. Thus the vectors have only two independent components. The tensor part is also associated with these two transverse directions; being also symmetric and traceless, it thus has only two independent components.

Putting all together, the full metric perturbation (for a Fourier mode in the z direction) is

$$\delta g_{\mu\nu} = a^2 \begin{bmatrix} -2A & -B_1 & -B_2 & +iB \\ -B_1 & 2(-D + \frac{1}{3}E) + h_+ & h_\times & -iE_1 \\ -B_2 & h_\times & 2(-D + \frac{1}{3}E) - h_+ & -iE_2 \\ +iB & -iE_1 & -iE_2 & 2(-D - \frac{2}{3}E) \end{bmatrix}. \quad (6.19)$$

6.1 Gauge Transformation in Fourier Space

Consider then how the gauge transformation appears in Fourier space. We need now the Fourier transform of the gauge transformation vector

$$\xi^\mu(\eta, \vec{x}) = \sum_{\vec{k}} \xi_{\vec{k}}^\mu(\eta) e^{i\vec{k} \cdot \vec{x}}. \quad (6.20)$$

For a single Fourier mode, the gauge transformation Eqs. (4.27,4.28,4.31) become

$$\tilde{A} = A - (\xi^0)' - \frac{a'}{a} \xi^0 \quad (6.21)$$

$$\tilde{B}_i = B_i + (\xi^i)' - i k_i \xi^0 \quad (6.22)$$

$$\tilde{D} = D + \frac{1}{3} i k_i \xi^i + \frac{a'}{a} \xi^0 \quad (6.23)$$

$$\tilde{E}_{ij} = E_{ij} - \frac{1}{2} i (k_i \xi^j + k_j \xi^i) + \frac{1}{3} i \delta_{ij} k_k \xi^k. \quad (6.24)$$

For illustration, consider again a mode in the z direction, $\vec{k} = (0, 0, k)$. Now the new part added into the matrix of Eq. (6.19) is

$$a^2 \begin{bmatrix} 2(\xi^0)' + 2\frac{a'}{a} \xi^0 & -(\xi^1)' & -(\xi^2)' & -(\xi^3)' + i k \xi^0 \\ -(\xi^1)' & -2\frac{a'}{a} \xi^0 & & -i k \xi^1 \\ -(\xi^2)' & & -2\frac{a'}{a} \xi^0 & -i k \xi^2 \\ -(\xi^3)' + i k \xi^0 & -i k \xi^1 & -i k \xi^2 & -2\frac{a'}{a} \xi^0 - 2i k \xi^3 \end{bmatrix} \quad (6.25)$$

(note the cancelations on the diagonal from the D and E_{ij} parts). We see that no new tensor part is introduced. Thus *the tensor part of the metric perturbation is gauge-invariant*¹⁷. But we see that the components ξ^0 and ξ^3 are responsible for a new scalar part and the components ξ^1 and ξ^2 are responsible for a new vector part.

For Fourier modes in an arbitrary direction, the above means that the time component ξ^0 and the component of the space part $\vec{\xi}$ parallel to \vec{k} are responsible for a change in the scalar perturbation and the *transverse* part of $\vec{\xi}$ is responsible for a change in the vector perturbation.

¹⁷To make the meaning of this statement clear: Consider a perturbation that is initially purely tensor, and do an arbitrary gauge transformation. The parts that get added to the perturbation are of scalar and/or vector nature. Thus this perturbation is not gauge-invariant; but its *tensor part*—because of the way we have defined the tensor part of a perturbation—is. The scalar and vector parts that appeared in the gauge transformation are pure gauge.

This freedom of doing gauge transformations can be used, e.g., to set 2 of the 4 scalar quantities of scalar perturbations and 2 of the 4 independent components of vector perturbations to zero. Thus only two of the degrees of freedom are real physical degrees of freedom in each case. Thus there are in total 6 physical degrees of freedom, 2 scalar, 2 vector, and 2 tensor. The other 4 (of the total 10) are just gauge degrees of freedom, representing perturbing just the coordinates, not the spacetime.

If the perturbation can be completely eliminated by a gauge transformation, we say the perturbation is “pure gauge”, i.e., it is not a real perturbation of spacetime, just a perturbation in the coordinates.

About the Fourier Convention. Let us call the preceding method of introducing the Fourier components of scalar, vector, and tensor parts method A. Consider a vector field

$$\vec{v}(\vec{x}) = \sum_{\vec{k}} \vec{v}_{\vec{k}} e^{i\vec{k}\cdot\vec{x}}. \quad (6.26)$$

Its scalar part is $\vec{v}^S(\vec{x}) = -\nabla v(\vec{x})$, where the potential $v(\vec{x})$ is Fourier expanded according to Eq. (6.3), so that

$$\begin{aligned} v(\vec{x}) &= \sum_{\vec{k}} \frac{v_{\vec{k}}^{(A)}}{k} e^{i\vec{k}\cdot\vec{x}} \\ \vec{v}^S(\vec{x}) &= \sum_{\vec{k}} \frac{-i\vec{k}}{k} v_{\vec{k}}^{(A)} e^{i\vec{k}\cdot\vec{x}} \end{aligned} \quad (6.27)$$

Another way (method B) to introduce the separation into scalar, vector, and tensor parts is to do it in Fourier space. We divide $\vec{v}_{\vec{k}} = \vec{v}_{\vec{k}}^S + \vec{v}_{\vec{k}}^V$ into scalar and vector parts so that $\vec{v}_{\vec{k}}^S$ is the component parallel to \vec{k} , with

$$v_{\vec{k}}^{(B)} \equiv |\vec{v}_{\vec{k}}^S| \quad (6.28)$$

and $\vec{v}_{\vec{k}}^V$ the component orthogonal to \vec{k} . Thus $\vec{v}_{\vec{k}}^S = (\vec{k}/k) v_{\vec{k}}^{(B)}$ and

$$\vec{v}^S(\vec{x}) = \sum_{\vec{k}} \frac{\vec{k}}{k} v_{\vec{k}}^{(B)} e^{i\vec{k}\cdot\vec{x}}. \quad (6.29)$$

In this method there is no “unconventional” division of Fourier components with k (or k^2 for scalar parts of tensor fields). In both methods the final quantity to represent the scalar perturbation in Fourier space is $v_{\vec{k}}$. In both methods, or conventions, $v_{\vec{k}}$ has the same magnitude, corresponding to the magnitude of $\vec{v}_{\vec{k}}^S$, which is important later when we compare terms in equations based on powers of (k/\mathcal{H}) ; but they have different phase:

$$v_{\vec{k}}^{(B)} = -i v_{\vec{k}}^{(A)}. \quad (6.30)$$

Convention A has the advantage that the coefficients in all perturbation equations stay real, whereas in convention B the imaginary unit i appears here and there. (Note that in Fourier space the perturbations themselves have complex values: their relative phases are related to how the maxima and minima of different quantities are located in different places in the plane waves. It could also be considered an advantage of convention B that the i in the equations reminds us of this.)

Liddle and Lyth[3] use convention A. Dodelson[5] uses convention B. We will keep using convention A. We will introduce a number of scalar quantities for which this applies. These include the scalar parts of vector fields: B , v , ξ ; and the scalar parts of tensor fields: E , Π .

7 Scalar Perturbations

From here on (except for the beginning of Sec. 9, where we discuss perturbations in the energy tensor) we shall consider scalar perturbations only, until Sec. 29, where we return to tensor perturbations. Scalar perturbations are the ones responsible for the structure of the universe (i.e., the formation and clustering of galaxies).

The metric is now

$$ds^2 = a(\eta)^2 \{ -(1 + 2A)d\eta^2 + 2B_{,i}d\eta dx^i + [(1 - 2\psi)\delta_{ij} + 2E_{,ij}] dx^i dx^j \} , \quad (7.1)$$

where we have defined¹⁸¹⁹ the *curvature perturbation*

$$\psi \equiv D + \frac{1}{3}\nabla^2 E . \quad (7.2)$$

In Fourier space this reads

$$\psi_{\vec{k}} = D_{\vec{k}} - \frac{1}{3}E_{\vec{k}} . \quad (7.3)$$

The components of $h_{\mu\nu}$ are

$$h_{\mu\nu} = \begin{bmatrix} -2A & B_{,i} \\ B_{,i} & -2\psi\delta_{ij} + 2E_{,ij} \end{bmatrix} . \quad (7.4)$$

Exercise: Curvature of the spatial hypersurface. The hypersurface $\eta = \text{const.}$ is a 3-dimensional curved manifold with the metric

$$g_{ij} = a(\eta)^2 [(1 - 2\psi)\delta_{ij} + 2E_{,ij}] \quad (7.5)$$

and inverse metric

$$g^{ij} = a(\eta)^{-2} [(1 + 2\psi)\delta_{ij} - 2E_{,ij}] . \quad (7.6)$$

Calculate the connection coefficients

$$\Gamma_{jk}^i \equiv \frac{1}{2}g^{im}(\partial_j g_{km} + \partial_k g_{mj} - \partial_m g_{jk}) \quad (7.7)$$

and the scalar curvature ${}^{(3)}R \equiv g^{ij}{}^{(3)}R_{ij}$, where the Ricci curvature tensor is

$${}^{(3)}R_{ij} \equiv \Gamma_{ji,m}^m - \Gamma_{mi,j}^m + \Gamma_{km}^k \Gamma_{ji}^m - \Gamma_{jm}^k \Gamma_{ki}^m , \quad (7.8)$$

of this 3-space for a scalar perturbation in terms of ψ and E to first order in perturbations.

Now if we start from a pure scalar perturbation and do an arbitrary gauge transformation, represented by the field $\xi^\mu = (\xi^0, \xi^i)$, we may introduce also a vector perturbation. This vector perturbation is however, pure gauge, and thus of no interest. Just like we did for the shift vector B_i earlier, we can divide ξ^i into a part with zero divergence (a transverse part) and a part with zero curl, expressible as a gradient of some function ξ ,

$$\xi^i = \xi_{\text{tr}}^i - \delta^{ij}\xi_{,j} = \vec{\xi}_{\text{tr}} - \nabla\xi \quad \text{where} \quad \xi_{\text{tr},i}^i = \nabla \cdot \vec{\xi}_{\text{tr}} = 0 . \quad (7.9)$$

The part ξ_{tr}^i is responsible for the spurious vector perturbation, whereas ξ^0 and $\xi_{,j}$ change the scalar perturbation. For our discussion of scalar perturbations we thus lose nothing, if we decide that we only consider gauge transformations, where the ξ_{tr}^i part is absent. These “scalar gauge transformations” are fully specified by two functions, ξ^0 and ξ ,

$$\begin{aligned} \tilde{\eta} &= \eta + \xi^0(\eta, \vec{x}) \\ \tilde{x}^i &= x^i - \delta^{ij}\xi_{,j}(\eta, \vec{x}) \end{aligned} \quad (7.10)$$

¹⁸In my spring 2003 lecture notes (CMB Physics / Cosmological Perturbation Theory) I was using the symbol ψ for what I am now denoting D . The present notation is better in line with common usage.

¹⁹My sign convention for ψ is that of MFB[1]. The opposite sign convention is common.

Fourier conventions for scalar perturbations						
Coordinate space	B	E	ξ	v	Π	μ
Fourier space	$\frac{B}{k}$	$\frac{E}{k^2}$	$\frac{\xi}{k}$	$\frac{v}{k}$	$\frac{\Pi}{k^2}$	$\frac{\mu}{k^2}$

Table 1: The Fourier convention of including an extra factor of k or k^2 in Fourier coefficients of scalar “potentials” may be quite confusing. We provide here a table for aid in converting equations between coordinate and Fourier space. Some of the quantities appearing here will be introduced later.

and they preserve the scalar nature of the perturbation. Also for ξ we use the Fourier convention (6.3a).

Applied to scalar perturbations and gauge transformations, our transformation equations (4.27, 4.28, 4.31) become

$$\begin{aligned}
\tilde{A} &= A - \xi^{0'} - \frac{a'}{a} \xi^0 \\
\tilde{B} &= B + \xi' + \xi^0 \\
\tilde{D} &= D - \frac{1}{3} \nabla^2 \xi + \frac{a'}{a} \xi^0 \\
\tilde{E} &= E + \xi,
\end{aligned} \tag{7.11}$$

where we use the notation $' \equiv \partial/\partial\eta$ for quantities which depend on both η and \vec{x} . The quantity ψ defined in Eq. (7.2) is often used as the fourth scalar variable instead of D . For it, we get

$$\tilde{\psi} = \psi + \frac{a'}{a} \xi^0 = \psi + \mathcal{H} \xi^0. \tag{7.12}$$

In Fourier space the last three equations in (7.11) become

$$\begin{aligned}
\tilde{B} &= B + \xi' + k \xi^0 \\
\tilde{D} &= D + \frac{1}{3} k \xi + \mathcal{H} \xi^0 \\
\tilde{E} &= E + k \xi.
\end{aligned} \tag{7.13}$$

7.1 Bardeen Potentials

We now define the following two quantities, called the *Bardeen potentials*,^{20,21}

$$\begin{aligned}
\Phi &\equiv A + \mathcal{H}(B - E') + (B - E')' \\
\Psi &\equiv D + \frac{1}{3} \nabla^2 E - \mathcal{H}(B - E') = \psi - \mathcal{H}(B - E').
\end{aligned} \tag{7.14}$$

These quantities are *invariant* under gauge transformations (**exercise**).

²⁰These may not appear well-motivated definitions just now, but wait until you see (8.3) in Sec. 8.

²¹For the second Bardeen potential, the opposite sign convention is common. Often Ψ is used to denote my Φ , and Φ to denote my Ψ or $-\Psi$. My sign and naming convention here is that of MFB[1]. Bardeen[2] originally called them Φ_A and $-\Phi_H$.

8 Conformal-Newtonian Gauge

We can use the gauge freedom to set the scalar perturbations B and E equal to zero. From Eq. (7.11) we see that this is accomplished by choosing

$$\begin{aligned}\xi &= -E \\ \xi^0 &= -B + E'.\end{aligned}\tag{8.1}$$

Doing this gauge transformation we arrive at a commonly used gauge, which has many names: the *conformal-Newtonian* gauge (or sometimes, for short, just the *Newtonian* gauge), the *longitudinal* gauge, Poisson gauge, and the *zero-shear* gauge. We shall denote quantities in this gauge with the sub- or superscript N . Thus $B^N = E^N = 0$, whereas you immediately see that

$$\begin{aligned}A^N &= \Phi \\ D^N &= \psi^N = \Psi.\end{aligned}\tag{8.2}$$

Thus the Bardeen potentials are equal to the two nonzero metric perturbations in the conformal-Newtonian gauge. In the Newtonian limit both Bardeen potentials become equal to the Newtonian gravitational potential perturbation.

From here on, until otherwise noted, we shall calculate in the conformal-Newtonian gauge. The metric is thus just²²

$$ds^2 = a(\eta)^2 \left[-(1 + 2\Phi)d\eta^2 + (1 - 2\Psi)\delta_{ij}dx^i dx^j \right],\tag{8.3}$$

or

$$g_{\mu\nu} = a^2 \begin{bmatrix} -1 - 2\Phi & \\ & (1 - 2\Psi)\delta_{ij} \end{bmatrix} \quad \text{and} \quad g^{\mu\nu} = a^{-2} \begin{bmatrix} -1 + 2\Phi & \\ & (1 + 2\Psi)\delta_{ij} \end{bmatrix},\tag{8.4}$$

or

$$h_{\mu\nu} = \begin{bmatrix} -2\Phi & \\ & -2\Psi\delta_{ij} \end{bmatrix} \quad \text{and} \quad h^{\mu\nu} = \begin{bmatrix} -2\Phi & \\ & -2\Psi\delta_{ij} \end{bmatrix}.\tag{8.5}$$

8.1 Perturbation in the Curvature Tensors

From the conformal-Newtonian metric (8.3) we get the connection coefficients (Christoffel symbols)

$$\Gamma_{\mu\nu}^\rho = \frac{1}{2}g^{\rho\sigma} (g_{\sigma\mu,\nu} + g_{\sigma\nu,\mu} - g_{\mu\nu,\sigma})\tag{8.6}$$

as (**exercise**)

$$\begin{aligned}\Gamma_{00}^0 &= \frac{a'}{a} + \Phi' & \Gamma_{0k}^0 &= \Phi_{,k} & \Gamma_{ij}^0 &= \frac{a'}{a}\delta_{ij} - \left[2\frac{a'}{a}(\Phi + \Psi) + \Psi' \right] \delta_{ij} \\ \Gamma_{00}^i &= \Phi_{,i} & \Gamma_{0j}^i &= \frac{a'}{a}\delta_j^i - \Psi'\delta_j^i & \Gamma_{kl}^i &= -(\Psi_{,l}\delta_k^i + \Psi_{,k}\delta_l^i) + \Psi_{,i}\delta_{kl}\end{aligned}\tag{8.7}$$

and the sums

$$\begin{aligned}\Gamma_{0\alpha}^\alpha &= 4\frac{a'}{a} + \Phi' - 3\Psi' \\ \Gamma_{i\alpha}^\alpha &= \Phi_{,i} - 3\Psi_{,i}\end{aligned}\tag{8.8}$$

²²Dodelson ([5], Eq. (4.9)) has $ds^2 = a(\eta)^2 [-(1 + 2\Psi)d\eta^2 + (1 + 2\Phi)\delta_{ij}dx^i dx^j]$, so he has both the opposite sign and naming convention. Lyth&Liddle ([7], Eq. (8.32)) has $ds^2 = a(\eta)^2 [-(1 + 2\Psi)d\eta^2 + (1 - 2\Phi)\delta_{ij}dx^i dx^j]$, so they have my sign convention but opposite naming convention.

where we have dropped all terms higher than first order in the small quantities Φ and Ψ . Thus these expressions contain only 0th and 1st order terms, and separate into the background and perturbation, accordingly:

$$\Gamma_{\beta\gamma}^\alpha = \bar{\Gamma}_{\beta\gamma}^\alpha + \delta\Gamma_{\beta\gamma}^\alpha, \quad (8.9)$$

where

$$\begin{aligned} \bar{\Gamma}_{00}^0 &= \mathcal{H} & \bar{\Gamma}_{0k}^0 &= 0 & \bar{\Gamma}_{ij}^0 &= \mathcal{H}\delta_{ij} \\ \bar{\Gamma}_{00}^i &= 0 & \bar{\Gamma}_{0j}^i &= \mathcal{H}\delta_j^i & \bar{\Gamma}_{kl}^i &= 0 \end{aligned} \quad (8.10)$$

and

$$\begin{aligned} \delta\Gamma_{00}^0 &= \Phi' & \delta\Gamma_{0k}^0 &= \Phi_{,k} & \delta\Gamma_{ij}^0 &= -[2\mathcal{H}(\Phi + \Psi) + \Psi']\delta_{ij} \\ \delta\Gamma_{00}^i &= \Phi_{,i} & \delta\Gamma_{0j}^i &= -\Psi'\delta_j^i & \delta\Gamma_{kl}^i &= -(\Psi_{,l}\delta_k^i + \Psi_{,k}\delta_l^i) + \Psi_{,i}\delta_{kl}. \end{aligned} \quad (8.11)$$

The Ricci tensor is

$$\begin{aligned} R_{\mu\nu} &= \Gamma_{\nu\mu,\alpha}^\alpha - \Gamma_{\alpha\mu,\nu}^\alpha + \Gamma_{\alpha\beta}^\alpha \Gamma_{\nu\mu}^\beta - \Gamma_{\nu\beta}^\alpha \Gamma_{\alpha\mu}^\beta \\ &= \bar{R}_{\mu\nu} + \delta\Gamma_{\nu\mu,\alpha}^\alpha - \delta\Gamma_{\alpha\mu,\nu}^\alpha + \bar{\Gamma}_{\alpha\beta}^\alpha \delta\Gamma_{\nu\mu}^\beta + \bar{\Gamma}_{\nu\mu}^\beta \delta\Gamma_{\alpha\beta}^\alpha - \bar{\Gamma}_{\nu\beta}^\alpha \delta\Gamma_{\alpha\mu}^\beta - \bar{\Gamma}_{\alpha\mu}^\beta \delta\Gamma_{\nu\beta}^\alpha. \end{aligned} \quad (8.12)$$

Calculation gives (**exercise**)

$$\begin{aligned} R_{00} &= -3\mathcal{H}' + 3\Psi'' + \nabla^2\Phi + 3\mathcal{H}(\Phi' + \Psi') \\ R_{0i} &= 2(\Psi' + \mathcal{H}\Phi)_{,i} \\ R_{ij} &= (\mathcal{H}' + 2\mathcal{H}^2)\delta_{ij} \\ &\quad + [-\Psi'' + \nabla^2\Psi - \mathcal{H}(\Phi' + 5\Psi') - (2\mathcal{H}' + 4\mathcal{H}^2)(\Phi + \Psi)]\delta_{ij} \\ &\quad + (\Psi - \Phi)_{,ij} \end{aligned} \quad (8.13)$$

Next we raise an index to get R_ν^μ . Note that we can not just raise the index of the background and perturbation parts separately, since

$$R_\nu^\mu = g^{\mu\alpha} R_{\alpha\nu} = (\bar{g}^{\mu\alpha} + \delta g^{\mu\alpha})(\bar{R}_{\alpha\nu} + \delta R_{\alpha\nu}) = \bar{R}_\nu^\mu + \delta g^{\mu\alpha} \bar{R}_{\alpha\nu} + \bar{g}^{\mu\alpha} \delta R_{\alpha\nu}. \quad (8.14)$$

We get

$$\begin{aligned} R_0^0 &= 3a^{-2}\mathcal{H}' + a^{-2}[-3\Psi'' - \nabla^2\Phi - 3\mathcal{H}(\Phi' + \Psi') - 6\mathcal{H}'\Phi] \\ R_i^0 &= -2a^{-2}(\Psi' + \mathcal{H}\Phi)_{,i} \\ R_0^i &= -R_i^0 = 2a^{-2}(\Psi' + \mathcal{H}\Phi)_{,i} \\ R_j^i &= a^{-2}(\mathcal{H}' + 2\mathcal{H}^2)\delta_j^i \\ &\quad + a^{-2}[-\Psi'' + \nabla^2\Psi - \mathcal{H}(\Phi' + 5\Psi') - (2\mathcal{H}' + 4\mathcal{H}^2)\Phi]\delta_{ij} \\ &\quad + a^{-2}(\Psi - \Phi)_{,ij}. \end{aligned} \quad (8.15)$$

and summing for the curvature scalar

$$\begin{aligned} R &= R_0^0 + R_i^i \\ &= 6a^{-2}(\mathcal{H}' + \mathcal{H}^2) \\ &\quad + a^{-2}[-6\Psi'' + 2\nabla^2(2\Psi - \Phi) - 6\mathcal{H}(\Phi' + 3\Psi') - 12(\mathcal{H}' + \mathcal{H}^2)\Phi]. \end{aligned} \quad (8.16)$$

And, finally, the Einstein tensor

$$\begin{aligned}
G_0^0 &= R_0^0 - \frac{1}{2}R \\
&= -3a^{-2}\mathcal{H}^2 + a^{-2}[-2\nabla^2\Psi + 6\mathcal{H}\Psi' + 6\mathcal{H}^2\Phi] \\
G_i^0 &= R_i^0 \\
G_0^i &= R_0^i = -R_i^0 = -G_i^0 \\
G_j^i &= R_j^i - \frac{1}{2}\delta_j^i R \\
&= a^{-2}(-2\mathcal{H}' - \mathcal{H}^2)\delta_j^i \\
&\quad + a^{-2}[2\Psi'' + \nabla^2(\Phi - \Psi) + \mathcal{H}(2\Phi' + 4\Psi') + (4\mathcal{H}' + 2\mathcal{H}^2)\Phi] \delta_j^i \\
&\quad + a^{-2}(\Psi - \Phi)_{,ij} .
\end{aligned} \tag{8.17}$$

Note the background (written first) and perturbation parts in all these quantities. Since the background \bar{R}_ν^μ and \bar{G}_ν^μ are diagonal, the off-diagonals contain just the perturbation, and we have

$$R_i^0 = G_i^0 = \delta R_i^0 = \delta G_i^0 . \tag{8.18}$$

9 Perturbation in the Energy Tensor

Consider then the energy tensor²³.

The background energy tensor is necessarily of the perfect fluid form²⁴

$$\begin{aligned}\bar{T}^{\mu\nu} &= (\bar{\rho} + \bar{p})\bar{u}^\mu\bar{u}^\nu + \bar{p}\bar{g}^{\mu\nu} \\ \bar{T}_\nu^\mu &= (\bar{\rho} + \bar{p})\bar{u}^\mu\bar{u}_\nu + \bar{p}\delta_\nu^\mu.\end{aligned}\tag{9.1}$$

Because of homogeneity, $\bar{\rho} = \bar{\rho}(\eta)$ and $\bar{p} = \bar{p}(\eta)$. Because of isotropy, the fluid is at rest, $\bar{u}^i = 0 \Rightarrow \bar{u}^\mu = (\bar{u}^0, 0, 0, 0)$ in the background universe. Since

$$\bar{u}_\mu\bar{u}^\mu = \bar{g}_{\mu\nu}\bar{u}^\mu\bar{u}^\nu = a^2\eta_{\mu\nu}\bar{u}^\mu\bar{u}^\nu = -a^2(\bar{u}^0)^2 = -1,\tag{9.2}$$

we have

$$\bar{u}^\mu = \frac{1}{a}(1, \vec{0}) \quad \text{and} \quad \bar{u}_\mu = a(-1, \vec{0}).\tag{9.3}$$

The energy tensor of the perturbed universe is

$$T_\nu^\mu = \bar{T}_\nu^\mu + \delta T_\nu^\mu.\tag{9.4}$$

Just like the metric perturbation, the energy tensor perturbation has 10 degrees of freedom, of which 6 are physical and 4 are gauge. It can likewise be divided into scalar+vector+tensor, with 4+4+2 degrees of freedom, of which 2+2+2 are physical. The perturbation can also be divided into perfect fluid + non-perfect, with 5+5 degrees of freedom.

The perfect fluid degrees of freedom in δT_ν^μ are those which keep T_ν^μ in the perfect fluid form

$$T_\nu^\mu = (\rho + p)u^\mu u_\nu + p\delta_\nu^\mu.\tag{9.5}$$

Thus they can be taken as the density perturbation, pressure perturbation, and velocity perturbation

$$\rho = \bar{\rho} + \delta\rho, \quad p = \bar{p} + \delta p, \quad \text{and} \quad u^i = \bar{u}^i + \delta u^i = \delta u^i \equiv \frac{1}{a}v_i.\tag{9.6}$$

The δu^0 is not an independent degree of freedom, because of the constraint $u_\mu u^\mu = -1$. We shall call

$$v_i \equiv au^i\tag{9.7}$$

the velocity perturbation. It is equal to the coordinate velocity, since (to first order)

$$\frac{dx^i}{d\eta} = \frac{u^i}{u^0} = \frac{u^i}{\bar{u}^0} = au^i = v_i.\tag{9.8}$$

It is also equal to the fluid velocity observed by a comoving (i.e., one whose $x^i = \text{const.}$) observer, since the ratio of change in comoving coordinate dx^i to change in conformal time $d\eta$ equals the ratio of the corresponding physical distance adx^i to the change in cosmic time $dt = ad\eta$.

We also define the *relative energy density perturbation*

$$\delta \equiv \frac{\delta\rho}{\bar{\rho}},\tag{9.9}$$

which is a dimensionless quantity in coordinate space (but not in Fourier space).

²³This section could actually have been earlier. We do not specify a gauge here, and the restriction to scalar perturbations is done only in the end.

²⁴The “imperfections” can only show up in the energy tensor if there is inhomogeneity or anisotropy. Whether an observer would “feel” the \bar{p} as pressure is another matter, which depends on the interactions of the fluid particles. But gravity only cares about the energy tensor.

To express u^μ and u_ν in terms of v_i , write them as

$$\begin{aligned} u^\mu = \bar{u}^\mu + \delta u^\mu &\equiv (a^{-1} + \delta u^0, a^{-1}v_1, a^{-1}v_2, a^{-1}v_3) \\ u_\nu = \bar{u}_\nu + \delta u_\nu &\equiv (-a + \delta u_0, \delta u_1, \delta u_2, \delta u_3) . \end{aligned} \quad (9.10)$$

These are related by $u_\nu = g_{\mu\nu}u^\mu$ and $u_\mu u^\mu = -1$. Using

$$g_{\mu\nu} = a^2 \begin{bmatrix} -1 - 2A & -B_i \\ -B_i & (1 - 2D)\delta_{ij} + 2E_{ij} \end{bmatrix} , \quad (9.11)$$

we get

$$\begin{aligned} u_0 = g_{0\mu}u^\mu &= a^2(-1 - 2A)(a^{-1} + \delta u^0) - \delta^{ij}a^2B_ia^{-1}v_j \\ &= -a - a^2\delta u^0 - 2aA \end{aligned} \quad (9.12)$$

(where we dropped higher than 1st order quantities, like B_iv_j), from which follows

$$\delta u_0 = -a^2\delta u^0 - 2aA . \quad (9.13)$$

Likewise

$$\delta u_i = u_i = g_{i\mu}u^\mu = -aB_i + av_i . \quad (9.14)$$

We solve the remaining unknown, δu^0 from

$$u_\mu u^\mu = \dots = -1 - 2a\delta u^0 - 2A = -1 \quad \Rightarrow \quad \delta u^0 = -\frac{1}{a}A \quad (9.15)$$

Thus we have for the 4-velocity

$$u^\mu = \frac{1}{a}(1 - A, v_i) \quad \text{and} \quad u_\mu = a(-1 - A, v_i - B_i) . \quad (9.16)$$

Inserting this into Eq. (9.5) we get

$$\begin{aligned} T_\nu^\mu &= \bar{T}_\nu^\mu + \delta T_\nu^\mu \\ &= \begin{bmatrix} -\bar{\rho} & 0 \\ 0 & \bar{p}\delta_j^i \end{bmatrix} + \begin{bmatrix} -\delta\rho & (\bar{\rho} + \bar{p})(v_i - B_i) \\ -(\bar{\rho} + \bar{p})v_i & \delta p\delta_j^i \end{bmatrix} . \end{aligned} \quad (9.17)$$

There are 5 remaining degrees of freedom in the space part, δT_j^i , corresponding to perturbations away from the perfect fluid form. We write them as

$$\delta T_j^i = \delta p\delta_j^i + \Sigma_{ij} \equiv \bar{p} \left(\frac{\delta p}{\bar{p}} + \Pi_{ij} \right) . \quad (9.18)$$

Here Σ_{ij} and $\Pi_{ij} \equiv \Sigma_{ij}/\bar{p}$ are symmetric and traceless, which makes the separation into

$$\delta p \equiv \frac{1}{3}\delta T_k^k \quad (9.19)$$

and

$$\Sigma_{ij} \equiv \delta T_j^i - \frac{1}{3}\delta_j^i\delta T_k^k \quad (9.20)$$

unique (the trace and the traceless part of δT_j^i). Σ_{ij} is called anisotropic stress or anisotropic pressure. Π_{ij} is its dimensionless version. For a perfect fluid $\Sigma_{ij} = \Pi_{ij} = 0$.

9.1 Separation into Scalar, Vector, and Tensor Parts

The energy tensor perturbation δT_ν^μ is built out of the scalar perturbations $\delta\rho$, δp , the 3-vector $\vec{v} = v_i$ and the traceless 3-tensor Π_{ij} . Just like for the metric perturbations, we can extract a scalar perturbation out of \vec{v} :

$$\begin{aligned} v_i &= v_i^S + v_i^V, & \text{where } v_i^S &= -v_{,i} \\ & & \text{and } \nabla \cdot \vec{v}^V &= 0. \end{aligned} \quad (9.21)$$

and a scalar + a vector perturbation out of Π_{ij} :

$$\Pi_{ij} = \Pi_{ij}^S + \Pi_{ij}^V + \Pi_{ij}^T, \quad (9.22)$$

where

$$\begin{aligned} \Pi_{ij}^S &= (\partial_i \partial_j - \tfrac{1}{3} \delta_{ij} \nabla^2) \Pi \\ \Pi_{ij}^V &= -\tfrac{1}{2} (\Pi_{i,j} + \Pi_{j,i}) \quad \text{and} \end{aligned} \quad (9.23)$$

$$\delta^{ik} \Pi_{ij,k}^T = 0. \quad (9.24)$$

We see that perfect fluid perturbations ($\Pi_{ij} = 0$) do not have a tensor perturbation component.

For Fourier components of v_i and Π_{ij} we use the same (Liddle&Lyth) convention as for B_i and E_{ij} (see Sec. (6)), so that

$$\begin{aligned} v_i^S &= -i \frac{k_i}{k} v \\ \Pi_{ij}^S &= \left(-\frac{k_i k_j}{k^2} + \tfrac{1}{3} \delta_{ij} \right) \Pi. \end{aligned} \quad (9.25)$$

In the early universe, we have anisotropic pressure from the cosmic neutrino background during and after neutrino decoupling, and from the cosmic microwave background during and after photon decoupling. Perturbations in the metric will make the momentum distribution of noninteracting particles anisotropic (this is anisotropic pressure). If there are sufficient interactions among the particles, these will isotropize the momentum distribution. Decoupling means that the interactions become too weak for this. If we aim for high precision in our calculations, we need to take this anisotropic pressure into account. For a more approximate treatment, the perfect fluid approximation can be made, which simplifies the calculations significantly.

9.2 Gauge Transformation of the Energy Tensor Perturbations

9.2.1 General Rule

Using the gauge transformation rules from Sect. 4 we have

$$\begin{aligned} \widetilde{\delta T}_0^0 &= -\widetilde{\delta\rho} = \delta T_0^0 - \bar{T}_{0,0}^0 \xi^0 = -\delta\rho + \bar{\rho}' \xi^0 \\ \widetilde{\delta T}_0^i &= -(\bar{\rho} + \bar{p}) \tilde{v}_i = \delta T_0^i + \xi^i_{,0} (\bar{T}_0^0 - \tfrac{1}{3} \bar{T}_k^k) \\ &= -(\bar{\rho} + \bar{p}) v_i - \xi^i_{,0} (\bar{\rho} + \bar{p}) \\ \tfrac{1}{3} \widetilde{\delta T}_k^k &= \widetilde{\delta p} = \tfrac{1}{3} (\delta T_k^k - \bar{T}_{k,0}^k \xi^0) = \delta p - \bar{p}' \xi^0 \\ \widetilde{\delta T}_j^i - \tfrac{1}{3} \delta_j^i \widetilde{\delta T}_k^k &= \bar{p} \tilde{\Pi}_{ij} = \delta T_j^i - \tfrac{1}{3} \delta_j^i \delta T_k^k = \bar{p} \Pi_{ij} \end{aligned} \quad (9.26)$$

and we get the gauge transformation laws for the different parts of the energy tensor perturbation:

$$\tilde{\delta\rho} = \delta\rho - \bar{\rho}'\xi^0 \quad (9.27)$$

$$\tilde{\delta p} = \delta p - \bar{p}'\xi^0 \quad (9.28)$$

$$\tilde{v}_i = v_i + \xi_{,0}^i \quad (9.29)$$

$$\tilde{\Pi}_{ij} = \Pi_{ij} \quad (9.30)$$

$$\tilde{\delta} = \delta - \frac{\bar{\rho}'}{\bar{\rho}}\xi^0 = \delta + 3\mathcal{H}(1+w)\xi^0. \quad (9.31)$$

Thus the anisotropic stress is gauge-invariant (being the traceless part of δT_j^i). Note that the $\delta\rho$ and δp equations are those of a perturbation of a 4-scalar, as they should be, as ρ and p are, indeed, 4-scalars.

9.2.2 Scalar Perturbations

For scalar perturbations, $v_i = -v_{,i}$ and $\xi^i = -\xi_{,i}$, so that we have

$$\begin{aligned} \tilde{v} &= v + \xi' \\ \tilde{\Pi} &= \Pi. \end{aligned} \quad (9.32)$$

These hold both in coordinate space and Fourier space (we use the same Fourier convention for ξ as for v and B).

9.2.3 Conformal-Newtonian Gauge

We get to the conformal-Newtonian gauge by $\xi^0 = -B + E'$ and $\xi = -E$. Thus

$$\begin{aligned} \delta\rho^N &= \delta\rho + \bar{\rho}'(B - E') = \delta\rho - 3\mathcal{H}(1+w)\bar{\rho}(B - E') \\ \delta p^N &= \delta p + \bar{p}'(B - E') = \delta p - 3\mathcal{H}(1+w)c_s^2\bar{\rho}(B - E') \\ v^N &= v - E' \\ \Pi^N &= \Pi. \end{aligned} \quad (9.33)$$

9.3 Scalar Perturbations in the Conformal-Newtonian Gauge

From here on we shall (unless otherwise noted)

1. consider scalar perturbations only, so that $v_i = -v_{,i}$ and $B_i = -B_{,i}$
2. use the conformal-Newtonian gauge (until Sec. 16), so that $B = 0$.

Thus the energy tensor perturbation has the form

$$\delta T_\nu^\mu = \begin{bmatrix} -\delta\rho^N & -(\bar{\rho} + \bar{p})v_{,i}^N \\ (\bar{\rho} + \bar{p})v_{,i}^N & \delta p^N \delta_j^i + \bar{p}(\Pi_{,ij} - \frac{1}{3}\delta_{ij}\nabla^2\Pi) \end{bmatrix}. \quad (9.34)$$

Note how the signs appear in δT_i^0 and δT_0^i and compare to (9.17) – we got an extra minus sign from $v_i = -v_{,i}$.

10 Field Equations for Scalar Perturbations in the Newtonian Gauge

We can now write the Einstein equations

$$\delta G_\nu^\mu = 8\pi G \delta T_\nu^\mu \quad (10.1)$$

for scalar perturbations in the conformal-Newtonian gauge. We have the left-hand side δG_ν^μ from Sect. 8.1 and the right-hand side δT_ν^μ from Sect. 9.3:

$$\begin{aligned} \delta G_0^0 &= a^{-2} [-2\nabla^2 \Psi + 6\mathcal{H}(\Psi' + \mathcal{H}\Phi)] = -8\pi G \delta \rho^N \\ \delta G_i^0 &= -2a^{-2} (\Psi' + \mathcal{H}\Phi)_{,i} = -8\pi G (\bar{\rho} + \bar{p}) v_{,i}^N \\ \delta G_0^i &= 2a^{-2} (\Psi' + \mathcal{H}\Phi)_{,i} = 8\pi G (\bar{\rho} + \bar{p}) v_{,i}^N \\ \delta G_j^i &= a^{-2} [2\Psi'' + \nabla^2(\Phi - \Psi) + \mathcal{H}(2\Phi' + 4\Psi') + (4\mathcal{H}' + 2\mathcal{H}^2)\Phi] \delta_j^i \\ &\quad + a^{-2} (\Psi - \Phi)_{,ij} = 8\pi G [\delta p^N \delta_j^i + \bar{p}(\Pi_{,ij} - \frac{1}{3}\delta_{ij}\nabla^2\Pi)] . \end{aligned} \quad (10.2)$$

Separating the δG_j^i equation into its trace and traceless part (the trace of δ_j^i is 3, and the trace of $(\Psi - \Phi)_{,ij}$ is $\nabla^2(\Psi - \Phi)$) the full set of Einstein equations is

$$3\mathcal{H}(\Psi' + \mathcal{H}\Phi) - \nabla^2\Psi = -4\pi G a^2 \delta \rho^N \quad (10.3)$$

$$(\Psi' + \mathcal{H}\Phi)_{,i} = 4\pi G a^2 (\bar{\rho} + \bar{p}) v_{,i}^N \quad (10.4)$$

$$\Psi'' + \mathcal{H}(\Phi' + 2\Psi') + (2\mathcal{H}' + \mathcal{H}^2)\Phi + \frac{1}{3}\nabla^2(\Phi - \Psi) = 4\pi G a^2 \delta p^N \quad (10.5)$$

$$(\partial_i \partial_j - \frac{1}{3}\delta_j^i \nabla^2)(\Psi - \Phi) = 8\pi G a^2 \bar{p}(\partial_i \partial_j - \frac{1}{3}\delta_j^i \nabla^2)\Pi . \quad (10.6)$$

The off-diagonal part of the last equation gives

$$(\Psi - \Phi)_{,ij} = 8\pi G a^2 \bar{p} \Pi_{,ij} \quad \text{for} \quad i \neq j . \quad (10.7)$$

In Fourier space this reads

$$-k_i k_j (\Psi_{\vec{k}} - \Phi_{\vec{k}}) = -\frac{k_i k_j}{k^2} 8\pi G a^2 \bar{p} \Pi_{\vec{k}} \quad \text{for} \quad i \neq j . \quad (10.8)$$

(with the Liddle&Lyth Fourier convention for Π). Since we can always rotate the background coordinate system so that more than one of the components of \vec{k} are non-zero, this means that

$$k^2(\Psi_{\vec{k}} - \Phi_{\vec{k}}) = 8\pi G a^2 \bar{p} \Pi_{\vec{k}} \quad \text{for} \quad \vec{k} \neq \vec{0} . \quad (10.9)$$

The 0th Fourier component represents a constant offset. But the split into a background and a perturbation is always chosen so that the spatial average of the perturbation vanishes (and the background value thus represents the spatial average of the full perturbed quantity).

Thus we have (going back to \vec{x} -space)

$$\Psi - \Phi = 8\pi G a^2 \bar{p} \Pi . \quad (10.10)$$

Likewise, since the spatial average of a perturbation is always zero, the equality of gradients of two perturbations means the equality of those perturbations themselves. Thus Eq. (10.4) says that

$$\Psi' + \mathcal{H}\Phi = 4\pi G a^2 (\bar{\rho} + \bar{p}) v^N . \quad (10.11)$$

Inserting this into Eq. (10.3) gives

$$\nabla^2\Psi = 4\pi G a^2 \bar{p} [\delta^N + 3\mathcal{H}(1+w)v^N] . \quad (10.12)$$

The final form of the Einstein equations can be divided into two *constraint equations*

$$\nabla^2 \Psi = \frac{3}{2} \mathcal{H}^2 [\delta^N + 3\mathcal{H}(1+w)v^N] \quad (10.13)$$

$$\Psi - \Phi = 3\mathcal{H}^2 w \Pi \quad (10.14)$$

that apply to any given time slice; and to two *evolution equations*

$$\Psi' + \mathcal{H}\Phi = \frac{3}{2} \mathcal{H}^2 (1+w)v^N \quad (10.15)$$

$$\Psi'' + \mathcal{H}(\Phi' + 2\Psi') + (2\mathcal{H}' + \mathcal{H}^2)\Phi + \frac{1}{3}\nabla^2(\Phi - \Psi) = \frac{3}{2} \mathcal{H}^2 \delta p^N / \bar{\rho} \quad (10.16)$$

that determine how the metric perturbation evolves in time. In the above we used the background relation

$$4\pi G a^2 \bar{\rho} = \frac{3}{2} \mathcal{H}^2, \quad (10.17)$$

which follows directly from the Friedmann equation (2.2).

In Fourier space the Einstein equations can be written as

$$\left(\frac{k}{\mathcal{H}}\right)^2 \Psi = -\frac{3}{2} \left[\delta^N + 3(1+w) \frac{\mathcal{H}}{k} v^N \right] \quad (10.18)$$

$$\left(\frac{k}{\mathcal{H}}\right)^2 (\Psi - \Phi) = 3w\Pi \quad (10.19)$$

$$\mathcal{H}^{-1} \Psi' + \Phi = \frac{3}{2} (1+w) \frac{\mathcal{H}}{k} v^N \quad (10.20)$$

$$\mathcal{H}^{-2} \Psi'' + \mathcal{H}^{-1} (\Phi' + 2\Psi') + \left(1 + \frac{2\mathcal{H}'}{\mathcal{H}^2}\right) \Phi - \frac{1}{3} \left(\frac{k}{\mathcal{H}}\right)^2 (\Phi - \Psi) = \frac{3}{2} \frac{\delta p^N}{\bar{\rho}}, \quad (10.21)$$

where the powers of \mathcal{H} are arranged so that the distance scales k^{-1} and time scales $d\eta$ are always related to the conformal Hubble scale \mathcal{H}^{-1} .

11 Energy-Momentum Continuity Equations

We know that from the Einstein equation, $G^\mu_\nu = 8\pi GT^\mu_\nu$, the energy-momentum continuity equation,

$$T^\mu_{\nu;\mu} = 0, \quad (11.1)$$

follows. Just like for the background universe, we may use the energy-momentum continuity equations instead of some of the Einstein equations.

Calculating

$$T^\mu_{\nu;\mu} = T^\mu_{\nu,\mu} + \Gamma^\mu_{\alpha\mu} T^\alpha_\nu - \Gamma^\alpha_{\nu\mu} T^\mu_\alpha = 0 \quad (11.2)$$

to first order in perturbations (note that, e.g., $\Gamma^\mu_{\alpha\mu} T^\alpha_\nu = (\bar{\Gamma}^\mu_{\alpha\mu} + \delta\Gamma^\mu_{\alpha\mu})(\bar{T}^\alpha_\nu + \delta T^\alpha_\nu) = \bar{\Gamma}^\mu_{\alpha\mu} \bar{T}^\alpha_\nu + \bar{\Gamma}^\mu_{\alpha\mu} \delta T^\alpha_\nu + \delta\Gamma^\mu_{\alpha\mu} \bar{T}^\alpha_\nu$), one obtains the 0th order (background) equation

$$\bar{\rho}' = -3\mathcal{H}(\bar{\rho} + \bar{p}) \quad (11.3)$$

and the 1st order (perturbation) equations, which for scalar perturbations in the conformal-Newtonian gauge are (**exercise**)

$$(\delta\rho^N)' = -3\mathcal{H}(\delta\rho^N + \delta p^N) + (\bar{\rho} + \bar{p})(\nabla^2 v^N + 3\Psi') \quad (11.4)$$

$$(\bar{\rho} + \bar{p})(v^N)' = -(\bar{\rho} + \bar{p})'v^N - 4\mathcal{H}(\bar{\rho} + \bar{p})v^N + \delta p^N + \frac{2}{3}\bar{p}\nabla^2\Pi + (\bar{\rho} + \bar{p})\Phi \quad (11.5)$$

Note that v^N is the velocity potential, $\vec{v}^N = -\nabla v^N$. It is easy to interpret the various terms in these equations.

In the energy perturbation equation (11.4), we have first the effect of the background expansion, then the effect of velocity divergence (local fluid expansion) and then the effect of the expansion/contraction in the metric perturbation.

In the momentum perturbation equation (11.5), the lhs and the first term on the right represent the change in inertia \times velocity. The second on the right is the effect of background expansion. The third and last terms represent forces due to gradients in pressure and gravitational potential.

With manipulations involving background relations, these can be worked (**exercise**) into the form

$$(\delta^N)' = (1+w)(\nabla^2 v^N + 3\Psi') + 3\mathcal{H}\left(w\delta^N - \frac{\delta p^N}{\bar{\rho}}\right) \quad (11.6)$$

$$(v^N)' = -\mathcal{H}(1-3w)v^N - \frac{w'}{1+w}v^N + \frac{\delta p^N}{\bar{\rho} + \bar{p}} + \frac{2}{3}\frac{w}{1+w}\nabla^2\Pi + \Phi. \quad (11.7)$$

In Fourier space,

$$(\delta^N)' = -(1+w)(kv^N - 3\Psi') + 3\mathcal{H}\left(w\delta^N - \frac{\delta p^N}{\bar{\rho}}\right) \quad (11.8)$$

$$(v^N)' = -\mathcal{H}(1-3w)v^N - \frac{w'}{1+w}v^N + k\frac{\delta p^N}{\bar{\rho} + \bar{p}} - \frac{2}{3}k\frac{w}{1+w}\Pi + k\Phi. \quad (11.9)$$

Equations (11.4), (11.6), (11.8), and related equations may be referred to as the *energy continuity equation*; and (11.5), (11.7), (11.9), and related equations as the *Euler equation*. These two *fluid evolution equations* are not independent of the Einstein equations, but they can be used instead of the two Einstein evolution equations (10.15) and (10.16).

12 Perfect Fluid Scalar Perturbations in the Newtonian Gauge

12.1 Field Equations

For a perfect fluid, things simplify a lot, since now $\Pi = 0$ and thus *for a perfect fluid*

$$\Psi = \Phi, \quad (12.1)$$

and we have *only one degree of freedom in the scalar metric perturbation*. We can now replace Ψ with Φ in the field equations. The original set becomes

$$\nabla^2 \Phi - 3\mathcal{H}(\Phi' + \mathcal{H}\Phi) = 4\pi G a^2 \delta \rho^N \quad (12.2)$$

$$(\Phi' + \mathcal{H}\Phi)_{,i} = 4\pi G a^2 (\bar{\rho} + \bar{p}) v_{,i}^N \quad (12.3)$$

$$\Phi'' + 3\mathcal{H}\Phi' + (2\mathcal{H}' + \mathcal{H}^2)\Phi = 4\pi G a^2 \delta p^N, \quad (12.4)$$

and the reworked set becomes²⁵

$$\begin{aligned} \nabla^2 \Phi &= 4\pi G a^2 \bar{\rho} [\delta^N + 3\mathcal{H}(1+w)v^N] \\ &= \frac{3}{2}\mathcal{H}^2 [\delta^N + 3\mathcal{H}(1+w)v^N] \end{aligned} \quad (12.8)$$

$$\Phi' + \mathcal{H}\Phi = 4\pi G a^2 (\bar{\rho} + \bar{p}) v^N = \frac{3}{2}\mathcal{H}^2 (1+w)v^N \quad (12.9)$$

$$\Phi'' + 3\mathcal{H}\Phi' + (2\mathcal{H}' + \mathcal{H}^2)\Phi = 4\pi G a^2 \delta p^N = \frac{3}{2}\mathcal{H}^2 \delta p^N / \bar{\rho}, \quad (12.10)$$

where we have used Eq. (2.9). Note how (12.8) resembles the Newtonian perturbation theory result $\nabla^2 \Phi = 4\pi G a^2 \bar{\rho} \delta$ – indeed, we expect to recover the Newtonian perturbation theory results in the appropriate limit!

We define the *total entropy perturbation* as

$$\mathcal{S} \equiv \mathcal{H} \left(\frac{\delta p}{\bar{p}'} - \frac{\delta \rho}{\bar{\rho}'} \right) \equiv H \left(\frac{\delta p}{\dot{\bar{p}}} - \frac{\delta \rho}{\dot{\bar{\rho}}} \right). \quad (12.11)$$

From the gauge transformation equations (9.27,9.28) we see that it is *gauge invariant*.

Using the background relations $\bar{\rho}' = -3\mathcal{H}(1+w)\bar{\rho}$ and $\bar{p}' = c_s^2 \bar{\rho}'$ we can also write

$$\mathcal{S} = \frac{1}{3(1+w)} \left(\frac{\delta \rho}{\bar{\rho}} - \frac{1}{c_s^2} \frac{\delta p}{\bar{\rho}} \right), \quad (12.12)$$

from which we get

$$\delta p = c_s^2 [\delta \rho - 3(\bar{\rho} + \bar{p})\mathcal{S}], \quad (12.13)$$

which holds in any gauge.

Using the entropy perturbation, we can now write in the rhs of Eq. (12.10)

$$\delta p^N / \bar{\rho} = c_s^2 [\delta^N - 3(1+w)\mathcal{S}], \quad (12.14)$$

where, from (12.8) and (12.9),

$$\delta^N = -3\mathcal{H}(1+w)v^N + \frac{2}{3\mathcal{H}^2} \nabla^2 \Phi = -\frac{2}{\mathcal{H}}(\Phi' + \mathcal{H}\Phi) + \frac{2}{3\mathcal{H}^2} \nabla^2 \Phi. \quad (12.15)$$

Using the background relation (2.22a), the evolution equation (12.10) becomes

$$\mathcal{H}^{-2} \Phi'' + 3(1+c_s^2)\mathcal{H}^{-1} \Phi' + 3(c_s^2 - w)\Phi = c_s^2 \mathcal{H}^{-2} \nabla^2 \Phi - \frac{9}{2} c_s^2 (1+w)\mathcal{S}. \quad (12.16)$$

²⁵If we change the time variable from conformal time η to cosmic time t , they read

$$\nabla^2 \Phi = 4\pi G a^2 \bar{\rho} [\delta^N + 3aH(1+w)v^N] \quad (12.5)$$

$$\dot{\Phi} + H\Phi = 4\pi G a (\bar{\rho} + \bar{p}) v^N \quad (12.6)$$

$$\ddot{\Phi} + 4H\dot{\Phi} + (2\dot{H} + 3H^2)\Phi = 4\pi G \delta p^N. \quad (12.7)$$

12.2 Adiabatic perturbations

Perturbations where the total entropy perturbation vanishes,²⁶

$$\mathcal{S} = 0 \quad \Leftrightarrow \quad \delta p = c_s^2 \delta \rho \quad (12.17)$$

are called *adiabatic perturbations*.²⁷ For adiabatic perturbations, Eq. (12.16) becomes (going to Fourier space)

$$\mathcal{H}^{-2} \Phi_k'' + 3(1 + c_s^2) \mathcal{H}^{-1} \Phi_k' + 3(c_s^2 - w) \Phi_k = - \left(\frac{c_s k}{\mathcal{H}} \right)^2 \Phi_k, \quad (12.18)$$

from which $\Phi_k(\eta)$ can be solved, given the initial conditions. From Eq. (12.9) we then get $v_k^N(\eta)$, and after that, from Eq. (12.8), $\delta_k^N(\eta)$.

In terms of ordinary cosmic time, Eq. (12.18) becomes

$$H^{-2} \ddot{\Phi}_k + (4 + 3c_s^2) H^{-1} \dot{\Phi}_k + 3(c_s^2 - w) \Phi_k = - \left(\frac{c_s k}{H} \right)^2 \Phi_k, \quad (12.19)$$

12.3 Fluid Equations

For a perfect fluid, Eqs. (11.6) and (11.7) become

$$(\delta^N)' = (1 + w) (\nabla^2 v^N + 3\Phi') + 3\mathcal{H} \left(w \delta^N - \frac{\delta p^N}{\bar{\rho}} \right) \quad (12.20)$$

$$(v^N)' = -\mathcal{H}(1 - 3w)v^N - \frac{w'}{1 + w} v^N + \frac{\delta p^N}{\bar{\rho} + \bar{p}} + \Phi. \quad (12.21)$$

12.4 Adiabatic perturbations at superhorizon scales

For superhorizon scales, $k \ll \mathcal{H}$, we can drop the rhs of Eq. (12.19). Using background relations to rewrite the equation-of-state parameters in terms of the Hubble parameter, it becomes

$$\ddot{\Phi} - \frac{1}{\dot{H}} (\ddot{H} - H\dot{H}) \dot{\Phi} - \frac{1}{\dot{H}} (H\ddot{H} - 2\dot{H}^2) \Phi = 0. \quad (12.22)$$

The general solution is (**exercise**)

$$\Phi_k(t) = A_k \left(1 - \frac{H}{a} \int_0^t a dt \right) + B_k \frac{H}{a}. \quad (12.23)$$

The lower integration limit (written as $t = 0$) is arbitrary, since the effect of changing it can be absorbed in the constant B_k . The second term is a decaying mode, it decays at least as fast as $1/a$ (unless $w < -1$).

²⁶For pressureless matter, where $\delta p = \bar{p} = w = c_s^2 = 0$, Eq. (12.11) is not defined, but $\delta p = c_s^2 \delta \rho$ holds always ($0 = 0$). We use then this latter definition for adiabaticity; and the perturbations are necessarily adiabatic.

²⁷Warning: Sometimes people may say “adiabatic perturbations”, when they mean perturbations which were *initially* adiabatic. Such perturbations do not usually stay adiabatic as the universe evolves. The proper expression for these is “adiabatic mode”.

13 Scalar Perturbations for a Barotropic Perfect Fluid

An equation of state of the form

$$p = p(\rho), \quad (13.1)$$

i.e., there are no other thermodynamic variables than ρ that the pressure would depend on, is called *barotropic*.²⁸ In this case the perturbations are guaranteed to be adiabatic, since now

$$\frac{\delta p}{\delta \rho} = \frac{\bar{p}'}{\bar{\rho}'} = \frac{dp}{d\rho} = c_s^2. \quad (13.2)$$

(Also, we keep assuming the fluid is perfect.)

This discussion will also apply to adiabatic perturbations of general perfect fluids.²⁹ That is, when the fluid in principle may have more state variables, but these other degrees of freedom are not “used”. Let us show that the adiabaticity of perturbations,

$$\delta p = c_s^2 \delta \rho \equiv \frac{\bar{p}'}{\bar{\rho}'} \delta \rho, \quad (13.3)$$

implies that a unique relation (13.1) holds everywhere and -when:

Note first, that in the background solution, where $\bar{p} = \bar{p}(t)$ and $\bar{\rho} = \bar{\rho}(t)$, we can (assuming $\bar{\rho}$ decreases monotonously with time) invert for $t(\bar{\rho})$, and thus $\bar{p} = \bar{p}(t(\bar{\rho}))$, defining a function $\bar{p}(\bar{\rho})$, whose derivative is c_s^2 . Now Eq. (13.3) guarantees that also $p = \bar{p} + \delta p$ and $\rho = \bar{\rho} + \delta \rho$ satisfy this same relation.

We note a property, which illuminates the nature of adiabatic perturbations: A small region of the perturbed universe is just like the background universe at a slightly earlier or later time. We can thus think of adiabatic perturbations as a perturbation in the “timing” of the different parts of the universe. (In adiabatic oscillations, this “corresponding background solution time” may oscillate back and forth.)

Once we have solved $\Phi_{\vec{k}}(\eta)$ from (12.18),

$$\Phi'' + 3(1 + c_s^2)\mathcal{H}\Phi' + 3(c_s^2 - w)\mathcal{H}^2\Phi + (c_s k)^2\Phi = 0, \quad (13.4)$$

we get $v_{\vec{k}}^N(\eta)$ and $\delta_{\vec{k}}^N(\eta)$ from (12.9) and (12.8), which read as

$$\begin{aligned} v^N &= \frac{2k}{3(1+w)} (\mathcal{H}^{-2}\Phi' + \mathcal{H}^{-1}\Phi) \\ \delta^N &= -\frac{2}{3} \left(\frac{k}{\mathcal{H}} \right)^2 \Phi - 3(1+w) \left(\frac{\mathcal{H}}{k} \right) v^N = -\frac{2}{3} \left(\frac{k}{\mathcal{H}} \right)^2 \Phi - 2(\mathcal{H}^{-1}\Phi' + \Phi) \end{aligned} \quad (13.5)$$

in Fourier space.

²⁸This *barotropic* equation of state is not the same concept as the *baryotropic* equation of state $p = (\gamma - 1)\rho$ one encounters in some other branches of physics.

²⁹Thus we could have called this section “Adiabatic perfect fluid scalar perturbations”. The reason we did not, is that mentioned in the earlier footnote. In this section we require the perturbations stay adiabatic the whole time. Perturbations of general fluids, which are initially (when they are at superhorizon scales) adiabatic, acquire entropy perturbations when they approach and enter the horizon.

14 Scalar Perturbations in the Matter-Dominated Universe

Let us now consider density perturbations in the simplest case, the *matter-dominated universe*. By “matter” we mean here non-relativistic matter, whose pressure is so small compared to energy density, that we can ignore it here.^{30,31} In general relativity this is often called “dust”.

According to our present understanding, the universe was radiation-dominated for the first few ten thousand years, after which it became matter-dominated. For our present discussion, we take “matter-dominated” to mean that matter dominates energy density to the extent that we can ignore the other components. This approximation becomes valid after the first few million years.

Until late 1990’s it was believed that this matter-dominated state persists until (and beyond) the present time. But new observational data points towards another component in the energy density of the universe, with a large negative pressure, resembling vacuum energy, or a cosmological constant. This component is called “dark energy”. The dark energy seems to have become dominant a few billion (10^9) years ago. Thus the validity of the matter-dominated approximation is not as extensive as was thought before; but anyway there was a significant period in the history of the universe, when it holds good.

We now make the matter-dominated approximation, i.e., we ignore pressure,

$$\bar{p} = w = c_s^2 = 0 \quad \text{and} \quad \delta p = \Pi = 0. \quad (14.1)$$

This is our first example of solving a perturbation theory problem. The order of work is always:

1. Solve the background problem.
2. Using the background quantities as known functions of time, solve the perturbation problem.

In the present case, the background equations are

$$\mathcal{H}^2 = \left(\frac{a'}{a}\right)^2 = \frac{8\pi G}{3}\bar{\rho}a^2 \quad (14.2)$$

$$\mathcal{H}' = -\frac{4\pi G}{3}\bar{\rho}a^2, \quad (14.3)$$

from which we have

$$2\mathcal{H}' + \mathcal{H}^2 = 0. \quad (14.4)$$

The background solution is the familiar $k = 0$ matter-dominated Friedmann model, $a \propto t^{2/3}$. But let us review the solution in terms of conformal time. Since $\bar{\rho} \propto a^{-3(1+w)} \propto a^{-3}$, (14.2) says that $a' \propto a^{1/2}$, which gives

$$a(\eta) \propto \eta^2. \quad (14.5)$$

Since $dt = a d\eta$, or $dt/d\eta = a$, we get $t(\eta) \propto \eta^3 \propto a^{3/2}$ or $a \propto t^{2/3}$.

From $a \propto \eta^2$ we get

$$\mathcal{H} \equiv \frac{a'}{a} = \frac{2}{\eta} \quad \text{and} \quad \mathcal{H}' = -\frac{2}{\eta^2} \quad (14.6)$$

Thus, from Eq. (14.2),

$$4\pi G a^2 \bar{\rho} = \frac{3}{2} \mathcal{H}^2 = \frac{6}{\eta^2}. \quad (14.7)$$

³⁰Likewise, we can ignore its anisotropic stress. Thus nonrelativistic matter is a perfect fluid for our purposes.

³¹Note that there are situations where we can make the matter-dominated approximation at the background level, but for the perturbations pressure gradients are still important at small distance scales (large k). Here we, however, make the approximation that also pressure perturbations can be ignored.

The perturbation equations (12.8), (12.9), and (12.10) with $\bar{p} = w = \delta p = 0$, are

$$\nabla^2 \Phi = 4\pi G a^2 \bar{\rho} [\delta^N + 3\mathcal{H} v^N] \quad (14.8)$$

$$\Phi' + \mathcal{H}\Phi = 4\pi G a^2 \bar{\rho} v^N \quad (14.9)$$

$$\Phi'' + 3\mathcal{H}\Phi' + (2\mathcal{H}' + \mathcal{H}^2)\Phi = 0. \quad (14.10)$$

Using Eq. (14.4), Eq. (14.10) becomes (or directly from (13.4))

$$\Phi'' + 3\mathcal{H}\Phi' = \Phi'' + \frac{6}{\eta}\Phi' = 0, \quad (14.11)$$

whose solution is

$$\Phi(\eta, \vec{x}) = C_1(\vec{x}) + C_2(\vec{x})\eta^{-5}. \quad (14.12)$$

The second term, $\propto \eta^{-5}$ is the *decaying part*. We get $C_1(\vec{x})$ and $C_2(\vec{x})$ from the initial values $\Phi_{\text{in}}(\vec{x})$, $\Phi'_{\text{in}}(\vec{x})$ at some initial time $\eta = \eta_{\text{in}}$,

$$\Phi_{\text{in}}(\vec{x}) = C_1(\vec{x}) + C_2(\vec{x})\eta_{\text{in}}^{-5} \quad (14.13)$$

$$\Phi'_{\text{in}}(\vec{x}) = -5C_2(\vec{x})\eta_{\text{in}}^{-6} \quad (14.14)$$

as

$$C_1(\vec{x}) = \Phi_{\text{in}}(\vec{x}) + \frac{1}{5}\eta_{\text{in}}\Phi'_{\text{in}}(\vec{x}) \quad (14.15)$$

$$C_2(\vec{x}) = -\frac{1}{5}\eta_{\text{in}}^6\Phi'_{\text{in}}(\vec{x}) \quad (14.16)$$

Unless we have very special initial conditions, conspiring to make $C_1(\vec{x})$ vanishingly small, the decaying part soon becomes $\ll C_1(\vec{x})$ and can be ignored. Thus we have the important result

$$\Phi(\eta, \vec{x}) = \Phi(\vec{x}), \quad (14.17)$$

i.e., the Bardeen potential Φ is *constant in time* for perturbations in the flat matter-dominated universe.

Ignoring the decaying part, we have $\Phi' = 0$ and we get for the velocity perturbation from Eq. (14.9)

$$v^N = \frac{\mathcal{H}\Phi}{4\pi G a^2 \bar{\rho}} = \frac{2\Phi}{3\mathcal{H}} = \frac{1}{3}\Phi\eta \propto \eta \propto a^{1/2} \propto t^{1/3}. \quad (14.18)$$

and Eq. (14.8) becomes

$$\nabla^2 \Phi = 4\pi G a^2 \bar{\rho} [\delta^N + 2\Phi] = \frac{3}{2}\mathcal{H}^2 [\delta^N + 2\Phi] \quad (14.19)$$

or

$$\delta^N = -2\Phi + \frac{2}{3\mathcal{H}^2}\nabla^2 \Phi. \quad (14.20)$$

In Fourier space this reads

$$\delta_k^N(\eta) = -\left[2 + \frac{2}{3}\left(\frac{k}{\mathcal{H}}\right)^2\right]\Phi_{\vec{k}}. \quad (14.21)$$

Thus we see that for *superhorizon* scales, $k \ll \mathcal{H}$, or $k_{\text{phys}} \ll H$, the density perturbation stays constant,

$$\delta_k^N = -2\Phi_{\vec{k}} = \text{const.} \quad (14.22)$$

whereas for *subhorizon* scales, $k \gg \mathcal{H}$, or $k_{\text{phys}} \gg H$, they grow proportional to the scale factor,

$$\delta_{\vec{k}} = -\frac{2}{3} \left(\frac{k}{\mathcal{H}} \right)^2 \Phi_{\vec{k}} \propto \eta^2 \propto a \propto t^{2/3}. \quad (14.23)$$

Since the comoving Hubble scale \mathcal{H}^{-1} grows with time, various scales k are superhorizon to begin with, but later become subhorizon as \mathcal{H}^{-1} grows past k^{-1} . (In physical terms, in the expanding universe $\lambda_{\text{phys}}/2\pi = k_{\text{phys}}^{-1} \propto a$ grows slower than the Hubble scale $H^{-1} \propto a^{3/2}$.) We say that the scale in question “enters the horizon”. (The word “horizon” in this context refers just to the Hubble scale $1/\mathcal{H}$, and not to other definitions of “horizon”.) We see that *density perturbations begin to grow when they enter the horizon*, and after that they grow proportional to the scale factor. Thus the present magnitude of the density perturbation at comoving scale k should be a_0/a_k times its primordial value³²

$$\delta_{\vec{k}}(t_0) \sim \frac{a_0}{a_k} \delta_{\vec{k},\text{pr}}^N, \quad (14.25)$$

where a_0 is the present value of the scale factor, and a_k is its value at the time the scale k “entered horizon”. The “primordial” density perturbation $\delta_{\vec{k},\text{pr}}^N$ refers to the constant value it had at superhorizon scales, after the decaying part of Φ had died out. Of course Eq. (14.25) is valid only for those (large) scales where the perturbation is still small today. Once the perturbation becomes large, $\delta \sim 1$, perturbation theory is no more valid. We say the scale in question “goes nonlinear”³³.

One has to remember that these results refer to the density and velocity perturbations in the conformal-Newtonian gauge only. In some other gauge these perturbations, and their growth laws would be different. However, for subhorizon scales general relativistic effects become unimportant and a Newtonian description becomes valid. In this limit, the issue of gauge choice becomes irrelevant as all “sensible gauges” approach each other, and the conformal-Newtonian density and velocity perturbations become those of a Newtonian description. The Bardeen potential can then be understood as a Newtonian gravitational potential due to density perturbations. (Eq. (14.8) acquires the form of the Newtonian law of gravity as the second term on the right becomes small compared to the first term, δ^N . The factor a^2 appears on the right since ∇^2 on the left refers to comoving coordinates.)

³²If we take into account the presence of “dark energy”, the above result is modified to

$$\delta_{\vec{k}}(t_0) \sim \frac{a_{DE}}{a_k} \delta_{\vec{k},\text{pr}}^N, \quad (14.24)$$

where $a_{DE} \gtrsim a_0/2$ is the value of a when dark energy became dominant, since that stops the growth of density perturbations. For Λ CDM, this modification was done accurately in Cosmology II, Sec. 8.3.5.

³³Since in our universe this happens only at subhorizon scales, the nonlinear growth of perturbations can be treated with Newtonian physics. First the growth of the density perturbation (for overdensities) becomes much faster than in linear perturbation theory, but then the system “virializes”, settling into a relatively stable structure, a galaxy or a galaxy cluster, where further collapse is prevented by the conservation of angular momentum, as the different parts of the system begin to orbit the center of mass of the system. For underdensities, we have of course $\rho \geq 0 \Rightarrow \delta \geq -1$ always, so the underdensity cannot “grow” beyond that.

15 Scalar Perturbations in the Radiation-Dominated Universe

For radiation,

$$p = \frac{1}{3}\rho \quad \Rightarrow \quad w = c_s^2 = \frac{1}{3}, \quad \bar{\rho} \propto a^{-4} \quad \text{and} \quad \delta p = \frac{1}{3}\delta\rho. \quad (15.1)$$

From the Friedmann equation,

$$\left(\frac{a'}{a}\right)^2 = \frac{8\pi G}{3}\bar{\rho}a^2 \propto a^{-2} \quad \Rightarrow \quad a' = \text{const}, \quad (15.2)$$

we get the background solution

$$a \propto \eta \quad \Rightarrow \quad \mathcal{H} \equiv \frac{a'}{a} = \frac{1}{\eta} \quad \text{and} \quad \mathcal{H}' = -\frac{1}{\eta^2}. \quad (15.3)$$

From (13.4),

$$\boxed{\Phi'' + \frac{4}{\eta}\Phi' + \frac{1}{3}k^2\Phi = 0} \quad (15.4)$$

or

$$\eta^2\Phi'' + 4\eta\Phi' + \frac{1}{3}(k\eta)^2\Phi = 0. \quad (15.5)$$

The obvious differences from the corresponding matter-dominated equation, Eq. (14.11), is that now we have k -dependence and that $\Phi = \text{const.} \neq 0$ is not a solution.

Equation (15.4) can be converted to a Bessel equation for $u \equiv \eta\Phi$:

$$u'' + \frac{2}{\eta}u' + \left(\frac{k^2}{3} - \frac{2}{\eta^2}\right)u = 0. \quad (15.6)$$

The equation for spherical Bessel functions is

$$\frac{d^2j_l}{dx^2} + \frac{2}{x}\frac{dj_l}{dx} + \left[1 - \frac{l(l+1)}{x^2}\right]j_l = 0. \quad (15.7)$$

Writing

$$x \equiv \frac{k\eta}{\sqrt{3}} = c_s k\eta, \quad (15.8)$$

where c_s is the speed of sound, Eq. (15.6) becomes Eq. (15.7) with $l = 1$. Thus the solutions are

$$u(\eta) = Aj_1\left(\frac{k\eta}{\sqrt{3}}\right) + Bn_1\left(\frac{k\eta}{\sqrt{3}}\right). \quad (15.9)$$

Spherical Bessel functions can be written in terms of trigonometric functions. In particular,

$$\begin{aligned} j_1(x) &= \frac{\sin x - x \cos x}{x^2} \\ n_1(x) &= \frac{-\cos x - x \sin x}{x^2} \end{aligned} \quad (15.10)$$

Consider the limiting behavior at superhorizon scales $x \rightarrow 0$. Since

$$\sin x \sim x - \frac{1}{6}x^3 \quad \text{and} \quad \cos x \sim 1 - \frac{1}{2}x^2, \quad (15.11)$$

we have

$$j_1(x) \sim \frac{x}{3} \quad \text{and} \quad n_1(x) \sim -\frac{1}{x^2} \rightarrow \infty \quad (15.12)$$

Thus the n_1 solution diverges at early times, i.e., it is a decaying solution, so we discard it. The final solution is

$$\Phi_{\vec{k}}(\eta) = \frac{1}{\eta} u = A_{\vec{k}} \frac{\sin\left(\frac{k\eta}{\sqrt{3}}\right) - \frac{k\eta}{\sqrt{3}} \cos\left(\frac{k\eta}{\sqrt{3}}\right)}{\left(\frac{k\eta}{\sqrt{3}}\right)^3} = A_{\vec{k}} \frac{\sqrt{3}}{k\eta} j_1\left(\frac{k\eta}{\sqrt{3}}\right) \quad (15.13)$$

(where $A_{\vec{k}}$ is $k/\sqrt{3}$ times the earlier A). From (13.5)

$$\begin{aligned} v^N &= \frac{1}{2}(k\eta^2\Phi' + k\eta\Phi) \\ \delta^N &= -\frac{2}{3}(k\eta)^2\Phi - \frac{4}{k\eta}v^N = -\frac{2}{3}(k\eta)^2\Phi - 2(\eta\Phi' + \Phi). \end{aligned} \quad (15.14)$$

For superhorizon scales ($k\eta \ll 1$, i.e., at early times $\eta \ll k^{-1}$), we have $\Phi(\eta) \approx \frac{1}{3}A_{\vec{k}} = \text{const.}$ Likewise $\delta^N \approx -2\Phi \approx \text{const.}$, but $v^N \approx \frac{1}{2}k\eta\Phi$ grows $\propto \eta$.

For subhorizon scales, i.e., at later times, after horizon entry ($\eta \gg k^{-1}$ so that $k\eta \gg 1$), the cosine part dominates, so that for the gravitational potential we have

$$\Phi_{\vec{k}}(\eta) \approx -3A_{\vec{k}} \frac{\cos(k\eta/\sqrt{3})}{(k\eta)^2}, \quad (15.15)$$

which *oscillates* with frequency $\omega = 2\pi f = k/\sqrt{3} = c_s k$ and *decaying* amplitude

$$\frac{3A_{\vec{k}}}{(k\eta)^2}. \quad (15.16)$$

The fluid quantities δ^N and v^N

$$\begin{aligned} v_{\vec{k}}^N &= \frac{1}{2}(k\eta^2\Phi'_{\vec{k}} + k\eta\Phi_{\vec{k}}) \approx \frac{1}{2}k\eta^2\Phi'_{\vec{k}} \approx \frac{3}{2}A_{\vec{k}}c_s \sin(c_s k\eta) \\ \delta_{\vec{k}}^N &\approx -\frac{2}{3}(k\eta)^2\Phi_{\vec{k}} \approx 2A_{\vec{k}} \cos(c_s k\eta), \end{aligned} \quad (15.17)$$

oscillate with constant amplitude.

16 Other Gauges

Different gauges are good for different purposes, and therefore it is useful to be able to work in different gauges, and to switch from one gauge to another in the course of a calculation. When one uses more than one gauge it is important to be clear about which gauge each quantity refers to. One useful gauge is the *comoving gauge*. Particularly useful quantities which refer to the comoving gauge, are the comoving density perturbation δ^C and the *comoving curvature perturbation*

$$\mathcal{R} \equiv -\psi^C. \quad (16.1)$$

This is often just called the *curvature perturbation*, however that term is also used to refer to ψ in any other gauge, so beware! (There are different sign conventions for \mathcal{R} and ψ . In my sign conventions, positive $\mathcal{R}_{\vec{k}}$, but negative $\psi_{\vec{k}}$, correspond to positive curvature of the 3D $\eta = \text{const}$ slice.

Gauges are usually specified by giving gauge conditions. These may involve the metric perturbation variables A , D , B , E (e.g., the Newtonian gauge condition $E = B = 0$), the energy-momentum perturbation variables $\delta\rho$, δp , v , or both kinds.

16.1 Slicing and Threading

The gauge corresponds to the coordinate system $\{x^\mu\} = \{\eta, x^i\}$ in the perturbed spacetime. The conformal time η gives the *slicing* of the perturbed spacetime into $\eta = \text{const}$ time slices (3D spacelike hypersurfaces). The spatial coordinates x^i give the *threading* of the perturbed spacetime into $x^i = \text{const}$ threads (1D timelike curves). See Fig. 3. Slicing and threading are orthogonal to each other if and only if the shift vector vanishes, $B^i = 0$.

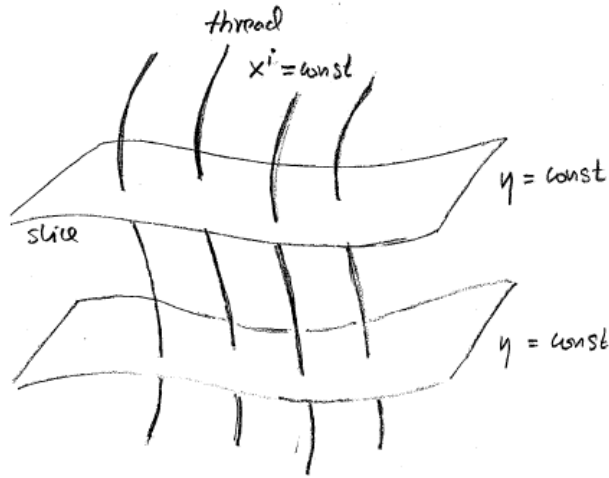


Figure 3: Slicing and threading the perturbed spacetime.

In the gauge transformation,

$$\tilde{x}^\alpha = x^\alpha + \xi^\alpha, \quad (16.2)$$

or

$$\begin{aligned}\tilde{\eta} &= \eta + \xi^0 \\ \tilde{x}^i &= x^i + \xi^i,\end{aligned}\tag{16.3}$$

ξ^0 changes the slicing, and ξ^i changes the threading. From the 4-scalar transformation law, $\delta\tilde{s} = \delta s - \xi'^0$, we see that perturbations in 4-scalars, e.g., $\delta\rho$ and δp depend only on the slicing. Thus slicing is a more important property of the gauge than its threading. In some cases, a gauge is specified by defining the slicing only, leaving the threading unspecified.

16.2 Comoving Gauge

We say that the *slicing is comoving*, if the time slices are orthogonal to the fluid 4-velocity. For scalar perturbations such a slicing always exists. This condition turns out to be equivalent to the condition that the fluid velocity perturbation v^i equals the shift vector B^i . For scalar perturbations,

$$\text{Comoving slicing} \quad \Leftrightarrow \quad v = B.\tag{16.4}$$

From the gauge transformation equations (7.11) and (9.32),

$$\begin{aligned}\tilde{v} &= v + \xi' \\ \tilde{B} &= B + \xi' + \xi^0\end{aligned}\tag{16.5}$$

we see that we get to comoving slicing by $\xi^0 = v - B$.

We say that the threading is comoving, if the threads are world lines of fluid elements, i.e., the velocity perturbation vanishes, $v^i = 0$.³⁴ For scalar perturbations,

$$\text{Comoving threading} \quad \Leftrightarrow \quad v = 0.\tag{16.6}$$

We get to comoving threading by the gauge transformation $\xi' = -v$. Note that comoving threads are usually not geodesics, since pressure gradients cause the fluid flow to deviate from free fall.

The *comoving gauge* is defined by requiring both comoving slicing and comoving threading. Thus

$$\text{Comoving gauge} \quad \Leftrightarrow \quad v = B = 0.\tag{16.7}$$

(We assume that we are working with scalar perturbations.) The threading is now orthogonal to the slicing. We denote the comoving gauge by the sub- or superscript C . Thus the statement $v^C = B^C = 0$ is generally true, whereas the statement $v = B = 0$ holds only in the comoving gauge.

We get to the comoving gauge from an arbitrary gauge by the gauge transformation

$$\begin{aligned}\xi' &= -v \\ \xi^0 &= v - B.\end{aligned}\tag{16.8}$$

This does not fully specify the coordinate system in the perturbed spacetime, since only ξ' is specified, not ξ . Thus we remain free to do time-independent transformations

$$\tilde{x}^i = x^i - \xi(\vec{x})_{,i},\tag{16.9}$$

while staying in the comoving gauge. This, however, does not change the way the spacetime is sliced and threaded by the coordinate system, it just relabels the threads with different coordinate values x^i .

³⁴Thus here “comoving” means with respect to the fluid: the coordinate system is comoving with the fluid flow.

Applying Eq. (16.8) to the general scalar gauge transformation Eqs. (7.11,7.12,9.27,9.28,9.32), we get the rules to relate the comoving gauge perturbations to perturbations in an arbitrary gauge. For the metric:

$$\begin{aligned} A^C &= A - (v - B)' - \mathcal{H}(v - B) \\ B^C &= B - v + (v - B) = 0 \\ D^C &= D - \frac{1}{3}\nabla^2\xi + \mathcal{H}(v - B) \\ E^C &= E + \xi \\ \psi^C &\equiv -\mathcal{R} = \psi + \mathcal{H}(v - B). \end{aligned} \tag{16.10}$$

For the energy tensor:

$$\begin{aligned} \delta\rho^C &= \delta\rho - \bar{\rho}'(v - B) = \delta\rho + 3\mathcal{H}(1 + w)\bar{\rho}(v - B) \\ \delta p^C &= \delta p - \bar{p}'(v - B) = \delta p + 3\mathcal{H}(1 + w)c_s^2\bar{\rho}(v - B) \\ \delta^C &= \delta + 3\mathcal{H}(1 + w)(v - B) \\ v^C &= v - v = 0 \\ \Pi^C &= \Pi. \end{aligned} \tag{16.11}$$

Because of the remaining gauge freedom (relabeling the threading) left by the comoving gauge condition (only ξ' is fixed, not ξ), D^C and E^C are not fully fixed. However, ψ^C is, and likewise

$$E^{C'} = E' - v. \tag{16.12}$$

In particular, we get the transformation rule from the Newtonian gauge (where $A = \Phi$, $B = 0$, $\psi = D = \Psi$, $E = 0$) to the comoving gauge. For the metric:

$$\begin{aligned} A^C &= \Phi - v^{N'} - \mathcal{H}v^N \\ \mathcal{R} &= -\Psi - \mathcal{H}v^N \\ E^{C'} &= -v^N. \end{aligned} \tag{16.13}$$

For the energy tensor:

$$\begin{aligned} \delta\rho^C &= \delta\rho^N + 3\mathcal{H}(1 + w)\bar{\rho}v^N \\ \delta p^C &= \delta p^N + 3\mathcal{H}(1 + w)c_s^2\bar{\rho}v^N \\ \delta^C &= \delta^N + 3\mathcal{H}(1 + w)v^N. \end{aligned} \tag{16.14}$$

Thus, for example, we see that in the matter-dominated universe discussed in Sect. 14, Eqs. (14.18,14.20) lead to

$$\delta^C = \delta^N + 3\mathcal{H}v^N = -2\Phi + \frac{2}{3\mathcal{H}^2}\nabla^2\Phi + 2\Phi = \frac{2}{3\mathcal{H}^2}\nabla^2\Phi \propto \mathcal{H}^{-2} \propto a \tag{16.15}$$

both inside and outside (and through) the horizon.

Note that in Fourier space we have included an extra factor of k in v , so that, e.g., Eq. (16.14) reads in Fourier space as

$$\delta^C = \delta^N + 3\mathcal{H}(1 + w)k^{-1}v^N. \tag{16.16}$$

16.3 Mixing Gauges

We see that Eq. (16.14c) is the rhs of Eq. (10.13), which we can thus write in the shorter form:

$$\boxed{\nabla^2 \Psi = 4\pi G a^2 \bar{\rho} \delta^C} . \quad (16.17)$$

This is an equation which has a Newtonian gauge metric perturbation on the lhs, but a comoving gauge density perturbation on the rhs. Is it dangerous to mix gauges like this? No, if we know what we are doing, i.e., in which gauge each quantity is, and the equations were derived correctly for this combination of quantities.

We could also say that, instead of working in any particular gauge, we work with *gauge-invariant quantities*, that the quantity that we denote by δ^C is the gauge-invariant quantity defined by

$$\delta^C = \delta + 3\mathcal{H}(1+w)(v-B)$$

which just happens to coincide with the density perturbation in the comoving gauge. Just like Ψ happens equal the metric perturbation ψ in the Newtonian gauge.

Switching to the comoving gauge for δ and δp , but keeping the velocity perturbation in the Newtonian gauge, the Einstein and continuity equations can be rewritten as (**exercise**):

$$\nabla^2 \Psi = \frac{3}{2} \mathcal{H}^2 \delta^C \quad (16.18)$$

$$\Psi - \Phi = 3\mathcal{H}^2 w \Pi \quad (16.19)$$

$$\Psi' + \mathcal{H}\Phi = \frac{3}{2} \mathcal{H}^2 (1+w) v^N \quad (16.20)$$

$$\Psi'' + (2 + 3c_s^2) \mathcal{H} \Psi' + \mathcal{H} \Phi' + 3(c_s^2 - w) \mathcal{H}^2 \Phi + \frac{1}{3} \nabla^2 (\Phi - \Psi) = \frac{3}{2} \mathcal{H}^2 \frac{\delta p^C}{\bar{\rho}} \quad (16.21)$$

$$\delta'_C - 3\mathcal{H} w \delta_C = (1+w) \nabla^2 v^N + 2\mathcal{H} w \nabla^2 \Pi \quad (16.22)$$

$$v'_N + \mathcal{H} v_N = \frac{\delta p^C}{\bar{\rho} + \bar{p}} + \frac{2}{3} \frac{w}{1+w} \nabla^2 \Pi + \Phi . \quad (16.23)$$

16.4 Comoving Curvature Perturbation

The comoving curvature perturbation

$$\mathcal{R} \equiv -\psi^C = -\psi - \mathcal{H}(v-B) \quad (16.24)$$

turns out to be a useful quantity for discussing superhorizon perturbations.

From (16.20) we have that

$$v^N = \frac{2}{3\mathcal{H}^2(1+w)} (\Psi' + \mathcal{H}\Phi) , \quad (16.25)$$

so that the relation of the comoving curvature perturbation and the Bardeen potentials is³⁵

$$\boxed{\mathcal{R} = -\Psi - \frac{2}{3(1+w)} (\mathcal{H}^{-1} \Psi' + \Phi) .} \quad (16.27)$$

³⁵In terms of ordinary cosmic time t ,

$$\mathcal{R} = -\Psi - \frac{2}{3(1+w)H} (\dot{\Psi} + H\Phi) = -\Psi + \frac{H}{\dot{H}} (\dot{\Psi} + H\Phi) , \quad (16.26)$$

where we used the background equation (2.22a).

Derivating Eq. (16.27),

$$\begin{aligned}\mathcal{R}' &= -\Psi' + \frac{2w'}{3(1+w)^2} (\mathcal{H}^{-1}\Psi' + \Phi) - \frac{2}{3(1+w)} \left(-\frac{\mathcal{H}'}{\mathcal{H}^2}\Psi' + \mathcal{H}^{-1}\Psi'' + \Phi' \right) \\ &= -\frac{2\mathcal{H}^{-1}}{3(1+w)}\Psi'' - \frac{4+6c_s^2}{3(1+w)}\Psi' - \frac{2}{3(1+w)}\Phi' + 2\mathcal{H}\frac{w-c_s^2}{1+w}\Phi,\end{aligned}$$

(where we used some background relations), and using the Einstein equations (10.21) and (10.18), we get an evolution equation for \mathcal{R} ,

$$\begin{aligned}-\frac{3}{2}(1+w)\mathcal{H}^{-1}\mathcal{R}' &= \mathcal{H}^{-2}\Psi'' + \mathcal{H}^{-1}(\Phi' + 2\Psi') + 3c_s^2(\mathcal{H}^{-1}\Psi' + \Phi) - 3w\Phi \\ &= \mathcal{H}^{-2}\Psi'' + (2+3c_s^2)\mathcal{H}^{-1}\Psi' + \mathcal{H}^{-1}\Phi' + 3(c_s^2-w)\Phi \\ &= 3c_s^2(\mathcal{H}^{-1}\Psi' + \Phi) + \frac{1}{3}\left(\frac{k}{\mathcal{H}}\right)^2(\Phi - \Psi) + \frac{3}{2}\frac{\delta p^N}{\bar{\rho}} \\ &= -c_s^2\left(\frac{k}{\mathcal{H}}\right)^2\Psi + \frac{1}{3}\left(\frac{k}{\mathcal{H}}\right)^2(\Phi - \Psi) + \frac{3}{2}\left(\frac{\delta p^N}{\bar{\rho}} - c_s^2\delta^N\right).\end{aligned}\quad (16.28)$$

From Sec. 12 we have

$$\mathcal{S} = \frac{1}{3(1+w)}\left(\delta - \frac{1}{c_s^2}\frac{\delta p}{\bar{\rho}}\right) \Rightarrow \delta p = c_s^2[\delta\rho - 3(\bar{\rho} + \bar{p})\mathcal{S}] \quad (16.29)$$

(valid in any gauge), so that Eq. (16.28) becomes the important result (**exercise**)

$$\frac{3}{2}(1+w)\mathcal{H}^{-1}\mathcal{R}' = \left(\frac{k}{\mathcal{H}}\right)^2 [c_s^2\Psi + \frac{1}{3}(\Psi - \Phi)] + \frac{9}{2}c_s^2(1+w)\mathcal{S}. \quad (16.30)$$

or

$$\mathcal{H}^{-1}\mathcal{R}' = \frac{2}{3(1+w)}\left(\frac{k}{\mathcal{H}}\right)^2 [c_s^2\Psi + \frac{1}{3}(\Psi - \Phi)] + 3c_s^2\mathcal{S}. \quad (16.31)$$

As an evolution equation, this does not appear very useful, since it mixes metric perturbations from two different gauges. However, the importance of this equation comes from the two observations we can now make immediately: 1) For adiabatic perturbations, $\mathcal{S} = 0$, the second term on the rhs vanishes. 2) For superhorizon perturbations, i.e., for Fourier modes whose wavelength is much larger than the Hubble distance, $k \ll \mathcal{H}$, we can ignore the first term on the rhs. Thus:

For adiabatic perturbations, the comoving curvature perturbation stays constant outside the horizon.

Adiabatic scalar perturbations have only one degree of freedom, and thus their full evolution is captured in the evolution of \mathcal{R} .

A general perturbation at a given time can be decomposed into an adiabatic component, which has $\mathcal{S} = 0$, and an isocurvature component, which has $\mathcal{R} = 0$. Because of the linearity of first order perturbation theory, these components evolve independently, and the evolution of the general perturbation is just the superposition of the evolution of these two components, and thus they can be studied separately. Beware, however, that the "adiabatic" component does not necessarily remain adiabatic in its evolution, and the "isocurvature" component does not necessarily maintain zero comoving curvature. We later show (in Sec. 18.3) that adiabatic perturbations stay adiabatic while they are well outside the horizon.

Adiabatic perturbations are important, since the simplest theory for the origin of structure of the universe, single-field inflation, produces adiabatic perturbations. Using the constancy of \mathcal{R} , it is easy to follow the evolution of these perturbations while they are well outside the horizon.

Using Eq. (16.21) we can write the second line of Eq. (16.28) as

$$\begin{aligned} -\frac{3}{2}(1+w)\mathcal{H}\mathcal{R}' &= \Psi'' + (2+3c_s^2)\mathcal{H}\Psi' + \mathcal{H}\Phi' + 3(c_s^2-w)\mathcal{H}^2\Phi \\ &= \frac{3}{2}\mathcal{H}^2\frac{\delta p_C}{\bar{\rho}} - \frac{1}{3}\nabla^2(\Phi - \Psi) \end{aligned} \quad (16.32)$$

or (using Eq. 16.19)

$$\mathcal{H}^{-1}\mathcal{R}' = -\frac{\delta p^C}{\bar{\rho} + \bar{p}} - \frac{2}{3}\frac{w}{1+w}\nabla^2\Pi = -c_s^2\left(\frac{\delta^C}{1+w} - 3\mathcal{S}\right) - \frac{2}{3}\frac{w}{1+w}\nabla^2\Pi \quad (16.33)$$

which is completely in the comoving gauge.

16.5 Perfect Fluid Scalar Perturbations, Again

With $\Pi = 0 \Rightarrow \Psi = \Phi$, Eqs. (16.18,16.19,16.20,16.21,16.22,16.23) become

$$\nabla^2\Phi = \frac{3}{2}\mathcal{H}^2\delta_C \quad (16.34)$$

$$\Phi' + \mathcal{H}\Phi = \frac{3}{2}\mathcal{H}^2(1+w)v_N \quad (16.35)$$

$$\Phi'' + 3(1+c_s^2)\mathcal{H}\Phi' + 3(c_s^2-w)\mathcal{H}^2\Phi = \frac{3}{2}\mathcal{H}^2\frac{\delta p_C}{\bar{\rho}} \quad (16.36)$$

$$\delta'_C - 3\mathcal{H}w\delta_C = (1+w)\nabla^2v_N \quad (16.37)$$

$$v'_N + \mathcal{H}v_N = \frac{\delta p_C}{\bar{\rho} + \bar{p}} + \Phi. \quad (16.38)$$

and Eqs. (16.27,16.33) become

$$\mathcal{R} = -\Phi - \frac{2}{3(1+w)\mathcal{H}}(\Phi' + \mathcal{H}\Phi) \quad (16.39)$$

$$\mathcal{H}^{-1}\mathcal{R}' = -\frac{\delta p^C}{\bar{\rho} + \bar{p}} = -c_s^2\left(\frac{\delta^C}{1+w} - 3\mathcal{S}\right). \quad (16.40)$$

We can write Eq. (16.36) as

$$\mathcal{H}^{-2}\Phi'' + 3(1+c_s^2)\mathcal{H}^{-1}\Phi' + 3(c_s^2-w)\Phi = \frac{3}{2}c_s^2[\delta^C - 3(1+w)\mathcal{S}]. \quad (16.41)$$

which is Eq. (12.16), with just $\nabla^2\Phi$ replaced by δ^C using Eq. (16.34). Our aim is to get a differential equation with preferably just one perturbation quantity to solve from, so this might seem a step backwards, replacing one of the Φ with δ^C , but we can now go ahead and replace also all the other Φ with δ^C :

Taking the Laplacian of this (**exercise**), we get the *Bardeen equation*

$$\mathcal{H}^{-2}\delta_C'' + (1-6w+3c_s^2)\mathcal{H}^{-1}\delta_C' - \frac{3}{2}(1+8w-6c_s^2-3w^2)\delta_C = c_s^2\mathcal{H}^{-2}\nabla^2[\delta_C - 3(1+w)\mathcal{S}], \quad (16.42)$$

a differential equation from which we can solve the evolution of the comoving density perturbation for superhorizon scales (when one can ignore the rhs) and for adiabatic perturbations at all scales (when $\mathcal{S} = 0$).

For the general case we need also an equation for \mathcal{S} . To be able to do this, we need to take a closer look at the fluid, see Sec. 18 and beyond.

16.5.1 Adiabatic Perfect Fluid Perturbations at Superhorizon Scales

Rewrite Eq. (16.39) as

$$\frac{2}{3}\mathcal{H}^{-1}\Phi' + \frac{5+3w}{3}\Phi = -(1+w)\mathcal{R}. \quad (16.43)$$

If we have a period in the history of the universe, where we can approximate $w = \text{const}$, then, for adiabatic perturbations at superhorizon scales, Eq. (16.43) is a differential equation for Φ with $w = \mathcal{R} = \text{const}$ for that period. This equation has a special solution

$$\Phi = -\frac{3+3w}{5+3w}\mathcal{R}.$$

The corresponding homogeneous equation is

$$\begin{aligned} \mathcal{H}^{-1}\Phi' + \frac{5+3w}{2}\Phi &= 0 \\ \Rightarrow a \frac{d\Phi}{da} &= -\frac{5+3w}{2}\Phi \\ \Rightarrow \Phi &= Ca^{-\frac{5+3w}{2}} \end{aligned}$$

so that the general solution to Eq. (16.43) is

$$\Phi = -\frac{3+3w}{5+3w}\mathcal{R} + Ca^{-\frac{5+3w}{2}}. \quad (16.44)$$

If $w \approx \text{const}$ for a long enough time, the second part becomes negligible, and we have

$$\Phi = \Psi = -\frac{3+3w}{5+3w}\mathcal{R} = \text{const} \quad (16.45)$$

In particular, we have the relations

$$\Phi_{\mathbf{k}} = -\frac{2}{3}\mathcal{R}_{\mathbf{k}} \quad (\text{adiab.}, \text{rad.dom}, w = \frac{1}{3}, k \ll \mathcal{H}) \quad (16.46)$$

$$\Phi_{\mathbf{k}} = -\frac{3}{5}\mathcal{R}_{\mathbf{k}} \quad (\text{adiab.}, \text{mat.dom}, w = 0, k \ll \mathcal{H}). \quad (16.47)$$

While the universe goes from radiation domination to matter domination, w is not a constant, so Eq. (16.45) does not apply, but we know from Eq. (16.47) that, $\Phi_{\mathbf{k}}$ changes from $-\frac{2}{3}\mathcal{R}_{\mathbf{k}}$ to $-\frac{3}{5}\mathcal{R}_{\mathbf{k}}$, i.e., changes by a factor 9/10 (assuming $k \ll \mathcal{H}$ the whole time, so that $\mathcal{R}_{\mathbf{k}}$ stays constant).

16.6 Uniform Energy Density Gauge

The uniform energy density gauge (time slicing)³⁶ U is defined by the condition

$$\delta\rho^U = 0. \quad (16.48)$$

Since

$$\tilde{\delta\rho} = \delta\rho - \bar{\rho}'\xi^0, \quad (16.49)$$

we get to this gauge by

$$\xi^0 = \frac{\delta\rho}{\bar{\rho}'}. \quad (16.50)$$

³⁶Sometimes we refer to a gauge just by the condition on time slicing, i.e., ξ^0 , since that is more important in CPT, leaving ξ unspecified.

Thus

$$\psi^U = \psi + \mathcal{H}\xi^0 = \psi + \mathcal{H}\frac{\delta\rho}{\bar{\rho}'} . \quad (16.51)$$

The *uniform energy density curvature perturbation* ζ , defined as

$$\zeta \equiv -\psi^U = -\psi - \mathcal{H}\frac{\delta\rho}{\bar{\rho}'} = -\psi + \frac{\delta}{3(1+w)} \quad (16.52)$$

is a widely used quantity in inflation literature. Going to U from C ,

$$\zeta = -\psi^C + \frac{\delta^C}{3(1+w)} = \mathcal{R} + \frac{2}{9(1+w)}\mathcal{H}^{-2}\nabla^2\Psi . \quad (16.53)$$

In Fourier space,

$$\zeta_{\mathbf{k}} = \mathcal{R}_{\mathbf{k}} - \frac{2}{9(1+w)}\left(\frac{k}{\mathcal{H}}\right)^2\Psi_{\mathbf{k}} , \quad (16.54)$$

so that at superhorizon scales

$$\zeta \approx \mathcal{R} \quad (k \ll \mathcal{H}) . \quad (16.55)$$

Therefore, ζ has the same useful property as \mathcal{R} : *For adiabatic perturbations, ζ remains constant outside the horizon.*

16.7 Spatially Flat Gauge

The spatially flat gauge, denoted by the sub/superscript Q , is defined by the condition that the curvature perturbation ψ vanishes, i.e.,

$$\psi^Q = 0 . \quad (16.56)$$

Since the metric perturbation ψ transforms as

$$\tilde{\psi} = \psi + \mathcal{H}\xi^0 . \quad (16.57)$$

we get to the spatially flat gauge by the gauge transformation

$$\xi^0 = -\mathcal{H}^{-1}\psi . \quad (16.58)$$

From the gauge transformation rule of the relative density perturbation (9.31),

$$\tilde{\delta} = \delta + 3\mathcal{H}(1+w)\xi^0 , \quad (16.59)$$

we get that

$$\delta^Q = \delta - 3(1+w)\psi . \quad (16.60)$$

In particular,

$$\delta^Q = \delta^C + 3(1+w)\mathcal{R} = 3(1+w)\zeta , \quad (16.61)$$

so that

$$\zeta = \frac{1}{3(1+w)}\delta^Q \quad (16.62)$$

and the comoving curvature perturbation \mathcal{R} is proportional to the difference between the relative density perturbations in the spatially flat and comoving gauges,

$$\mathcal{R} = \frac{1}{3(1+w)}(\delta^Q - \delta^C) = \frac{1}{3(1+w)}\left[\delta^Q + \frac{2}{3}\left(\frac{k}{\mathcal{H}}\right)^2\Psi\right] , \quad (16.63)$$

where we used Eq. (16.18) for the latter equality. Thus for superhorizon scales we have the correspondence

$$\mathcal{R} \approx \frac{1}{3(1+w)}\delta^Q \quad (16.64)$$

between the comoving gauge curvature perturbation and the flat gauge density perturbation.

We shall later use the spatially flat gauge to solve perturbation equations for scalar fields.

17 Synchronous Gauge

The synchronous gauge was the first one to be used in cosmological perturbation theory (by Lifshitz in 1946) and it is often used in numerical work. The synchronous gauge is defined by the requirement $A = B_i = 0$. Synchronous gauge can be used both for scalar and vector perturbations. In this section we consider only scalar perturbations, where it means that

$$A^Z = B^Z = 0. \quad (17.1)$$

(We use Z instead of S to denote synchronous gauge, so as not to confuse it with the scalar part of a perturbation.)

One gets to synchronous gauge from an arbitrary gauge with a gauge transformation ξ^μ that satisfies

$$\xi^{0'} + \mathcal{H}\xi^0 = A \quad (17.2)$$

$$\xi' = -\xi^0 - B. \quad (17.3)$$

This is a differential equation from which to solve ξ^0 . Thus it is not easy to switch from another gauge to synchronous gauge. We see that, like for comoving gauge, only the time derivative of ξ is determined. In addition, ξ^0 is determined only up to solutions of the homogeneous equation $\xi^{0'} + \mathcal{H}\xi^0 = 0$. Thus the gauge is not fully specified by the synchronous condition, which historically led to confusion until theorists learned to separate gauge modes within the synchronous gauge from physical ones.

In the synchronous gauge the threads are orthogonal to the slices, since $B = 0$, and the metric perturbation is only in the space part of the metric,

$$\begin{aligned} h_{ij} &= -2D^Z\delta_{ij} + 2(E_{,ij}^Z - \frac{1}{3}\nabla^2 E^Z\delta_{ij}) \\ &= -2\psi^Z\delta_{ij} + 2E_{,ij}^Z. \end{aligned} \quad (17.4)$$

The line element is

$$ds^2 = a^2 \{ -d\eta^2 + [(1 - 2\psi)\delta_{ij} + 2E_{,ij}^Z] dx^i dx^j \}. \quad (17.5)$$

For a comoving observer, i.e., one moving along a thread (a space coordinate line $x^i = \text{const.}$), the proper time $d\tau$ is given by $d\tau^2 = -ds^2 = a^2 d\eta^2$, so that $d\tau = a d\eta = dt$. Thus her clock shows the coordinate time (t , not η). Moreover, *the threads are geodesics*:

From Eq. (A.2), setting $A = B = 0$, the Christoffel symbols are

$$\begin{aligned} \Gamma_{00}^0 &= \mathcal{H} \\ \Gamma_{0i}^0 &= \Gamma_{00}^i = 0 \\ \Gamma_{ij}^0 &= \mathcal{H}[(1 - 2\psi)\delta_{ij} + 2E_{,ij}^Z] - \delta_{ij}\psi' + E_{,ij}' \\ \Gamma_{0j}^i &= \mathcal{H}\delta_{ij} - \delta_{ij}\psi' + E_{,ij}' \\ \Gamma_{jk}^i &= -\delta_{ij}\psi_{,k} - \delta_{ik}\psi_{,j} + \delta_{jk}\psi_{,i} + E_{,ijk}. \end{aligned} \quad (17.6)$$

From $\Gamma_{0i}^0 = \Gamma_{00}^i = 0$ follows that *the space coordinate lines are geodesics*. Namely, the geodesic equations

$$\ddot{x}^0 + \Gamma_{\alpha\beta}^0 \dot{x}^\alpha \dot{x}^\beta = \ddot{x}^0 + \mathcal{H}\dot{x}^0\dot{x}^0 + \Gamma_{ij}^0 \dot{x}^i \dot{x}^j = 0 \quad (17.7)$$

$$\ddot{x}^i + \Gamma_{\alpha\beta}^i \dot{x}^\alpha \dot{x}^\beta = \ddot{x}^i + 2\Gamma_{0j}^i \dot{x}^0 \dot{x}^j + \Gamma_{jk}^i \dot{x}^j \dot{x}^k = 0, \quad (17.8)$$

where now $\cdot \equiv d/d\tau$, where τ is proper time along the geodesic, are satisfied by $x^i = \text{const.} \Rightarrow \dot{x}^i = 0$ and $\dot{x}^0 \equiv d\eta/d\tau = a^{-1}$. Thus one can construct the synchronous gauge coordinate system

by choosing an initial spacelike hypersurface, distributing observers carrying space coordinate values on that hypersurface, with their initial 4-velocities orthogonal to the hypersurface (i.e., they are at rest with respect to that surface), synchronizing the clocks of the observers, and letting the observers then fall freely. The world lines of these observers are then the space coordinate lines and their clocks show the time coordinate.

Note that the synchronous gauge leaves the choice of this initial hypersurface free (this is the remaining gauge freedom in ξ^0 already mentioned above). In the case of dust ($p = 0$), the fluid flow lines are geodesics, and we can choose them to be the threads (and the initial hypersurface orthogonal to them), in which case our synchronous gauge becomes equal to the comoving gauge. Pressure makes the comoving gauge different from the synchronous gauge, since pressure gradients accelerate the fluid away from geodesics.

We use Ma & Bertschinger[4] (hereafter MB) as our main reference for synchronous gauge and adopt from their notation

$$\begin{aligned} h &\equiv -6D^Z \equiv h_i^{iZ} \\ \eta &\equiv \psi^Z = D^Z + \frac{1}{3}\nabla^2 E^Z \\ \mu &\equiv 2E^Z \end{aligned} \tag{17.9}$$

(only two of these are independent), so that

$$\begin{aligned} h_{ij} &= \frac{1}{3}h\delta_{ij} + (\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2)\mu \\ &= -2\eta\delta_{ij} + \mu_{,ij}. \end{aligned} \tag{17.10}$$

Unfortunately, the MB notation for the synchronous curvature perturbation is the same as our notation for conformal time (MB use τ for the latter). Let's hope this does not lead to confusion. (In this Section, from here on, η stands always for the synchronous metric perturbation (17.9b); we avoid writing a symbol for conformal time.) MB discuss the synchronous metric perturbation in terms of the variables h and η , instead of the pair h, μ or η, μ . In coordinate space this appears impractical, since to solve μ from h and η requires integration ($\eta = \frac{1}{6}(-h + \nabla^2\mu)$). However, MB work entirely in Fourier space, where $\eta = -\frac{1}{6}(h + \mu)$ or $\mu = -h - 6\eta$.

From this point on, in this Section, we work in **Fourier space**. The metric perturbation μ follows the Fourier convention for E , so, from (17.9),

$$\eta = -\frac{1}{6}(h + \mu) \quad \text{or} \quad \mu = -h - 6\eta. \tag{17.11}$$

The metric is

$$\begin{aligned} h_{ij} &= -2D^Z\delta_{ij} + 2\left(-\frac{k_i k_j}{k^2} + \frac{1}{3}\delta_{ij}\right)E^Z \\ &= -2D^Z\delta_{ij} - 2\hat{k}_i\hat{k}_jE^Z + \frac{2}{3}E^Z\delta_{ij} \\ &= \frac{1}{3}h\delta_{ij} - \hat{k}_i\hat{k}_j\mu + \frac{1}{3}\mu\delta_{ij} \\ &= \hat{k}_i\hat{k}_jh + (\hat{k}_i\hat{k}_j - \frac{1}{3}\delta_{ij})6\eta, \end{aligned} \tag{17.12}$$

which is MB Eq. (4), where $\hat{k}_i \equiv k_i/k$.

From Eq. (8.1), we get from the synchronous gauge to the Newtonian gauge by

$$\begin{aligned} \xi_{Z \rightarrow N} &= -\frac{1}{k}E^Z = -\frac{1}{2k}\mu = \frac{1}{2k}(h + 6\eta) \\ \xi_{Z \rightarrow N}^0 &= +\frac{1}{k^2}E^{Z'} = -\frac{1}{k}\xi'_{Z \rightarrow N} = +\frac{1}{2k^2}\mu' \end{aligned} \tag{17.13}$$

From Eq. (7.14), the Bardeen potentials are

$$\begin{aligned}\Phi &= -\frac{1}{k^2} \left(\mathcal{H}E^{Z'} + E^{Z''} \right) = \frac{1}{2k^2} (-\mathcal{H}\mu' - \mu'') = \frac{1}{2k^2} [h'' + 6\eta'' + \mathcal{H}(h' + 6\eta')] \\ \Psi &= \psi^Z + \frac{1}{k^2} \mathcal{H}E^{Z'} = \eta - \frac{1}{2k^2} \mathcal{H}(h' + 6\eta'),\end{aligned}\quad (17.14)$$

which is MB Eq. (18).

The opposite transformation, from Newtonian to synchronous gauge is, of course, just

$$\xi_{N \rightarrow Z}^0 = -\xi_{Z \rightarrow N}^0 \quad \xi_{N \rightarrow Z} = -\xi_{Z \rightarrow N}, \quad (17.15)$$

so we can easily express it in synchronous gauge quantities, which may actually be what we want.

We get the Einstein tensor perturbations from Eq. (A.10), setting $A = B = 0$,

$$\begin{aligned}\delta G_0^0 &= a^{-2} [2k^2\psi + 6\mathcal{H}D'] \\ \delta G_i^0 &= a^{-2} [-2ik_i\psi'] \\ \delta G_0^i &= a^{-2} [2ik_i\psi'] \\ \delta G_j^i &= a^{-2} \left[2D'' + k^2D + 4\mathcal{H}D' - \frac{1}{3}k^2E + \frac{1}{3}E'' + \frac{2}{3}\mathcal{H}E' \right] \delta_j^i \\ &\quad - k_ik_j a^{-2} \left[D - \frac{1}{3}E + \frac{1}{k^2}(E'' + 2\mathcal{H}E') \right] \\ \delta G_i^i &= a^{-2} [6D'' + 2k^2\psi + 12\mathcal{H}D'] .\end{aligned}\quad (17.16)$$

In the MB η, h notation,

$$\begin{aligned}\delta G_0^0 &= a^{-2} [2k^2\eta - \mathcal{H}h'] \\ \delta G_i^0 &= a^{-2} [-2ik_i\eta'] \\ \delta G_0^i &= a^{-2} [2ik_i\eta'] \\ \delta G_j^i &= a^{-2} \left[-\frac{1}{2}h'' - \eta'' - \mathcal{H}(h' + 2\eta') + k^2\eta \right] \delta_j^i \\ &\quad - k_ik_j a^{-2} \left[\eta - \frac{1}{k^2} \left(\frac{1}{2}h'' + 3\eta'' + \mathcal{H}h' + 6\mathcal{H}\eta' \right) \right] \\ \delta G_i^i &= a^{-2} [-h'' + 2k^2\eta - 2\mathcal{H}h'] .\end{aligned}\quad (17.17)$$

The Einstein equations are thus

$$\begin{aligned}k^2\eta - \frac{1}{2}\mathcal{H}h' &= -4\pi G a^2 \delta\rho^Z = -\frac{3}{2}\mathcal{H}^2\delta^Z \\ k^2\eta' &= 4\pi G a^2 (\bar{\rho} + \bar{p})kv^Z = \frac{3}{2}\mathcal{H}^2(1+w)kv^Z \\ h'' + 2\mathcal{H}h' - 2k^2\eta &= -24\pi G a^2 \delta p^Z = -9\mathcal{H}^2 \frac{\delta p^Z}{\bar{\rho}} \\ h'' + 6\eta'' + 2\mathcal{H}h' + 12\mathcal{H}\eta' - 2k^2\eta &= -16\pi G a^2 \bar{p}\Pi = -6\mathcal{H}^2 w\Pi,\end{aligned}\quad (17.18)$$

which is MB Eq. (21). Note that MB uses the notation

$$\theta \equiv \nabla \cdot \vec{v} = -\nabla^2 v = kv \quad \text{and} \quad \sigma \equiv \frac{2}{3} \frac{\bar{p}}{(\bar{\rho} + \bar{p})} \Pi = \frac{2}{3} \frac{w}{1+w} \Pi. \quad (17.19)$$

We get the continuity equations from Eq. (A.17), setting $A = B = 0$ and $D = -\frac{1}{6}h$,

$$\begin{aligned}
 \delta\rho^{Z'} &= -3\mathcal{H}(\delta\rho^Z + \delta p^Z) - (\bar{\rho} + \bar{p})(\tfrac{1}{2}h' + kv^Z) \\
 (\bar{\rho} + \bar{p})v^{Z'} &= -(\bar{\rho} + \bar{p})'v^Z - 4\mathcal{H}(\bar{\rho} + \bar{p})v^Z + k\delta p^Z - \tfrac{2}{3}k\bar{p}\Pi \\
 \delta^{Z'} &= -(1+w)(kv^Z + \tfrac{1}{2}h') + 3\mathcal{H}\left(w\delta^Z - \frac{\delta p^Z}{\bar{\rho}}\right) \\
 v^{Z'} &= -\mathcal{H}(1-3w)v^Z - \frac{w'}{1+w}v^Z + \frac{k\delta p^Z}{\bar{\rho} + \bar{p}} - \tfrac{2}{3}\frac{w}{1+w}k\Pi.
 \end{aligned} \tag{17.20}$$

The two last equations are MB Eq. (29).

Exercise: Derive the synchronous gauge Einstein equations and continuity equations from the corresponding Newtonian gauge equations by gauge transformation.

18 Fluid Components

18.1 Division into Components

In practice, the cosmological fluid consists of many components (e.g., particle species: photons, baryons, CDM, neutrinos, ...). It is useful to divide the energy tensor into such components:

$$T_\nu^\mu = \sum_i T_{\nu(i)}^\mu, \quad (18.1)$$

which have their corresponding background and perturbation parts

$$\bar{T}_\nu^\mu = \sum_i \bar{T}_{\nu(i)}^\mu \quad \text{and} \quad \delta T_\nu^\mu = \sum_i \delta T_{\nu(i)}^\mu. \quad (18.2)$$

Here i labels the different components (so I'll use l and m for space indices).

Often the component pressure is a unique function of the component energy density, $p_i = p_i(\rho_i)$, or can be so approximated (common cases are $p_i = 0$ and $p_i = \rho_i/3$), although this is not true for the total pressure and energy density.

The total fluid and component quantities for the background are related as

$$\begin{aligned} \bar{\rho} &= \sum \bar{\rho}_i \\ \bar{p} &= \sum \bar{p}_i = \sum w_i \bar{\rho}_i \\ w &\equiv \frac{\bar{p}}{\bar{\rho}} = \sum \frac{\bar{\rho}_i}{\bar{\rho}} w_i \\ c_s^2 &\equiv \frac{\bar{p}'}{\bar{\rho}'} = \frac{\sum \bar{p}_i'}{\bar{\rho}'} = \sum \frac{\bar{\rho}_i'}{\bar{\rho}'} c_i^2, \end{aligned} \quad (18.3)$$

where

$$w_i \equiv \frac{\bar{p}_i}{\bar{\rho}_i} \quad \text{and} \quad c_i^2 \equiv \frac{\bar{p}_i'}{\bar{\rho}_i'}. \quad (18.4)$$

The total fluid and component quantities for the perturbations are related as

$$\begin{aligned} \delta\rho &= \sum \delta\rho_i = \sum \bar{\rho}_i \delta_i \\ \delta p &= \sum \delta p_i \\ \delta &\equiv \frac{\delta\rho}{\bar{\rho}} = \sum \frac{\bar{\rho}_i}{\bar{\rho}} \delta_i, \end{aligned} \quad (18.5)$$

where $\delta_i \equiv \delta\rho_i/\bar{\rho}_i$. From δT_0^l ,

$$(\bar{\rho} + \bar{p})v_l = \sum (\bar{\rho}_i + \bar{p}_i)v_{l(i)} \quad (18.6)$$

we get that

$$v_l = \sum \frac{\bar{\rho}_i + \bar{p}_i}{\bar{\rho} + \bar{p}} v_{l(i)} = \sum \frac{1 + w_i}{1 + w} \frac{\bar{\rho}_i}{\bar{\rho}} v_{l(i)} \quad (18.7)$$

and from

$$\Sigma_{lm} = \sum \Sigma_{lm(i)} = \sum \bar{p}_i \Pi_{lm(i)},$$

where $\Pi_{lm(i)} \equiv \Sigma_{lm(i)}/\bar{p}_i$, we get that

$$\Pi_{lm} \equiv \frac{\Sigma_{lm}}{\bar{p}} = \sum \frac{\bar{p}_i}{\bar{p}} \Pi_{lm(i)} = \sum \frac{w_i \bar{\rho}_i}{w \bar{\rho}} \Pi_{lm(i)}. \quad (18.8)$$

From here on, we consider scalar perturbations only.

18.2 Gauge Transformations and Entropy Perturbations

The gauge transformations for the fluid component perturbations are

$$\tilde{\delta\rho}_i = \delta\rho_i - \bar{\rho}'_i \xi^0 \quad (18.9)$$

$$\tilde{\delta p}_i = \delta p_i - \bar{p}'_i \xi^0 \quad (18.10)$$

$$\begin{aligned} \tilde{v}_i &= v_i + \xi' \\ \tilde{\Pi}_i &= \Pi_i \end{aligned} \quad (18.11)$$

$$\tilde{\delta}_i = \delta_i - \frac{\bar{\rho}'_i}{\bar{\rho}_i} \xi^0 \quad (18.12)$$

Note that the ξ^0 and ξ are the same for all fluid components.

Those gauge conditions on ξ^0 and ξ that refer to fluid perturbations that we have discussed so far refer to the total fluid. Thus, e.g., in the comoving gauge the total velocity perturbation v vanishes, but the component velocities v_i do not vanish (unless they happen to be all equal). Thus, the velocity v that appears in the gauge transformation equations to comoving gauge refers to the *total velocity perturbation*.³⁷ For example the component gauge transformation equations that correspond to Eq. (16.14) read

$$\begin{aligned} \delta\rho_i^C &= \delta\rho_i^N - \bar{\rho}'_i v^N \\ \delta p_i^C &= \delta p_i^N - \bar{p}'_i v^N \\ \delta_i^C &= \delta_i^N - \frac{\bar{\rho}'_i}{\bar{\rho}_i} v^N. \end{aligned} \quad (18.13)$$

Since the gauge transformations are the same for all components, we find some gauge invariances. The *relative velocity perturbation* between two components i and j ,

$$v_i - v_j \quad \text{is gauge invariant.} \quad (18.14)$$

Like the total anisotropic stress, also the component anisotropic stresses

$$\Pi_i \quad \text{are gauge invariant.} \quad (18.15)$$

We can also define a kind of *entropy perturbation* (different from the total entropy perturbation \mathcal{S} defined earlier!)

$$S_{ij} \equiv -3\mathcal{H} \left(\frac{\delta\rho_i}{\bar{\rho}'_i} - \frac{\delta\rho_j}{\bar{\rho}'_j} \right) \quad (18.16)$$

between two fluid components i and j which turns out to be gauge invariant due to the way the density perturbations $\delta\rho_i$ transform. This is actually the most common type of quantity called “entropy perturbation” in the literature. It is a special case of a *generalized entropy perturbation*

$$S_{xy} \equiv \mathcal{H} \left(\frac{\delta x}{\bar{x}'} - \frac{\delta y}{\bar{y}'} \right) \quad (18.17)$$

between any two 4-scalar quantities x and y , which is gauge invariant due to the way 4-scalar perturbations transform. Beware of the many different quantities called “entropy perturbation”! Some of them can be interpreted as perturbations in some entropy/particle ratio. What is common to all of them, is that they all vanish in the case of adiabatic perturbations. (Entropy is a useful quantity in cosmology, since for most of the evolution of the universe, entropy is conserved to high accuracy. However, in these notes we are not using it, so we do not need the possible connection between the concepts “entropy” and “entropy perturbation”.)

³⁷In some situations we may want to assign special status to one fluid component, and define a gauge which is comoving with that fluid component, so then the v_i^N of that component would appear in the gauge transformation equations for all fluid components.

18.3 Equations

The Einstein equations (both background and perturbation) involve the total fluid quantities. The metric perturbations are not divided into components³⁸ due to different fluid components! There is a single gravity, due to the total fluid, which each fluid component obeys.

If there is no *energy transfer between the fluid components in the background universe*, the background energy continuity equation is satisfied separately by such independent components,

$$\bar{\rho}'_i = -3\mathcal{H}(\bar{\rho}_i + \bar{p}_i), \quad (18.18)$$

and in that case we can write Eq. (18.3) as

$$c_s^2 = \sum \frac{\bar{\rho}_i + \bar{p}_i}{\bar{\rho} + \bar{p}} c_i^2, \quad (18.19)$$

and Eq. (18.16) as

$$S_{ij} = \frac{\delta\rho_i}{(1+w_i)\bar{\rho}_i} - \frac{\delta\rho_j}{(1+w_j)\bar{\rho}_j} = \frac{\delta_i}{1+w_i} - \frac{\delta_j}{1+w_j}. \quad (18.20)$$

We also get (2.20) componentwise,

$$\frac{w'_i}{1+w_i} = 3\mathcal{H}(w_i - c_i^2). \quad (18.21)$$

Likewise, the gauge transformation equations (18.13) become

$$\begin{aligned} \delta\rho_i^C &= \delta\rho_i^N + 3\mathcal{H}(1+w_i)\bar{\rho}_i v^N \\ \delta p_i^C &= \delta p_i^N + 3\mathcal{H}(1+w_i)c_i^2 \bar{\rho}_i v^N \\ \delta_i^C &= \delta_i^N + 3\mathcal{H}(1+w_i)v^N. \end{aligned} \quad (18.22)$$

Note that this transformation involves the total v^N , not the component v_i^N , which makes the comoving gauge less practical for a many-component fluid.

But if there is energy transfer between two fluid components, then their component energy continuity equations acquire an interaction term.

Even if there is no energy transfer between fluid components at the background level, the perturbations often introduce energy and momentum transfer between the components. It may also be the case that the energy transfer can be neglected in practice, but the momentum transfer remains important.

In the case of noninteracting fluid components (no energy or momentum transfer), we have the perturbation energy and momentum continuity equations separately for each such fluid component,

$$T_{\nu(i);\mu}^\mu = 0. \quad (18.23)$$

For the case of scalar perturbations in the conformal-Newtonian gauge, they read

$$(\delta_i^N)' = (1+w_i)(\nabla^2 v_i^N + 3\Psi') + 3\mathcal{H}\left(w_i \delta_i^N - \frac{\delta p_i^N}{\bar{\rho}_i}\right) \quad (18.24)$$

$$(v_i^N)' = -\mathcal{H}(1-3w_i)v_i^N - \frac{w'_i}{1+w_i}v_i^N + \frac{\delta p_i^N}{\bar{\rho}_i + \bar{p}_i} + \frac{2}{3}\frac{w_i}{1+w_i}\nabla^2\Pi_i + \Phi. \quad (18.25)$$

³⁸There are exceptions to this in the literature, where contributions to some metric perturbation quantities due to different fluid components are defined.

In Fourier space they become

$$(\delta_i^N)' = (1 + w_i) (-k v_i^N + 3\Psi') + 3\mathcal{H} \left(w_i \delta_i^N - \frac{\delta p_i^N}{\bar{\rho}_i} \right) \quad (18.26)$$

$$(v_i^N)' = -\mathcal{H}(1 - 3w_i)v_i^N - \frac{w_i'}{1 + w_i}v_i^N + \frac{k\delta p_i^N}{\bar{\rho}_i + \bar{p}_i} - \frac{2}{3} \frac{w_i}{1 + w_i} k\Pi_i + k\Phi. \quad (18.27)$$

If there are interactions between the fluid components, there will be interaction terms (“collision terms”) in these equations.

Note that one cannot write the mixed gauge equations for fluid components by just replacing the fluid quantities in equations (16.22, 16.23) with component quantities, since these equations were derived by making a gauge transformation between the comoving and Newtonian gauges, which involved the total fluid velocity.

For the case where energy transfer between components can be neglected both at the background and perturbation level, we can use Eqs. (18.20), (18.21), and (18.25) to find the time derivative of the entropy perturbation S_{ij} ,

$$\begin{aligned} S'_{ij} &= \nabla^2 (v_i^N - v_j^N) + 3\mathcal{H} \left(\frac{c_i^2 \delta \rho_i^N - \delta p_i^N}{\bar{\rho}_i + \bar{p}_i} - \frac{c_j^2 \delta \rho_j^N - \delta p_j^N}{\bar{\rho}_j + \bar{p}_j} \right) \\ &= \nabla^2 (v_i - v_j) - 9\mathcal{H} (c_i^2 \mathcal{S}_i - c_j^2 \mathcal{S}_j), \end{aligned} \quad (18.28)$$

where

$$\mathcal{S}_i \equiv \mathcal{H} \left(\frac{\delta p_i}{\bar{p}_i'} - \frac{\delta \rho_i}{\bar{\rho}_i'} \right) \quad (18.29)$$

is the (gauge invariant) internal entropy perturbation of component i . If we have in addition, that for both components the equation of state has the form $p_i = f_i(\rho_i)$, then the internal entropy perturbations vanish³⁹, and we have simply

$$S'_{ij} = \nabla^2 (v_i - v_j). \quad (18.30)$$

In Fourier space Eq. (18.30) reads

$$S'_{ij} = -k(v_i - v_j) \quad \text{or} \quad \mathcal{H}^{-1} S'_{ij} = -\frac{k}{\mathcal{H}} (v_i - v_j), \quad (18.31)$$

showing that *entropy perturbations remain constant at superhorizon scales* $k \ll \mathcal{H}$. From this follows that *adiabatic perturbations* ($S_{ij} = 0$) *stay adiabatic while outside the horizon*.

We can also obtain an equation for S''_{ij} from Eq. (18.27), if also momentum transfer between components (at the perturbation level) can be neglected (and we also keep the assumption $p_i = f_i(\rho_i)$):

$$\begin{aligned} S''_{ij} &= -k(v_i' - v_j') = -k[(v_i^N)' - (v_j^N)'] \\ &= k\mathcal{H} [(1 - 3w_i)v_i^N - (1 - 3w_j)v_j^N] + k \left[\frac{w_i'}{1 + w_i} v_i^N - \frac{w_j'}{1 + w_j} v_j^N \right] \\ &\quad - k^2 \left[\frac{c_i^2 \delta_i^N}{1 + w_i} - \frac{c_j^2 \delta_j^N}{1 + w_j} \right] + \frac{2}{3} k^2 \left[\frac{w_i}{1 + w_i} \Pi_i - \frac{w_j}{1 + w_j} \Pi_j \right]. \end{aligned} \quad (18.32)$$

If one further assumes that both components are perfect fluid, then one can drop the last term; and if one assumes that the equation-of-state parameter is constant for both components, then one can drop the second term.

³⁹For baryons this may require that we ignore baryon pressure, since $p_b = p_b(n_b, T) = n_b T$, and $\rho_b = \rho_b(n_b, T) = n_b(m_b + \frac{3}{2}T)$.

In the synchronous gauge, Eqs. (18.26,18.27) become

$$(\delta_i^Z)' = -(1 + w_i) \left(k v_i^Z + \frac{1}{2} h' \right) + 3\mathcal{H} \left(w_i \delta_i^Z - \frac{\delta p_i^Z}{\bar{\rho}_i} \right) \quad (18.33)$$

$$(v_i^Z)' = -\mathcal{H}(1 - 3w_i)v_i^Z - \frac{w_i'}{1 + w_i}v_i^Z + \frac{k\delta p_i^Z}{\bar{\rho}_i + \bar{p}_i} - \frac{2}{3} \frac{w_i}{1 + w_i} k \Pi_i. \quad (18.34)$$

Since E does not appear in the general gauge scalar continuity equations (A.17), the only difference in them when going from Newtonian to synchronous gauge (as both have $B = 0$) is that $\Psi = D^N$ is replaced by $-\frac{1}{6}h = D^Z$ and $\Phi = A^N$ is dropped as $A^Z = 0$.

19 Simplified Matter+Radiation Universe

Consider the case where the energy tensor consists of two perfect fluid components, matter with $p = 0$ and radiation with $p = \frac{1}{3}\rho$, that do not interact with each other, i.e., there is no energy or momentum transfer between them. (Compared to the real universe, this is a simplification since, while cold dark matter does not much interact with the other fluid components, the baryonic matter does interact with photons. Also, the radiation components of the real universe, neutrinos and photons, behave like a perfect fluid only until their decoupling. We also ignore dark energy.)

19.1 Background solution for radiation+matter

We have now two fluid components,

$$\rho = \rho_r + \rho_m,$$

where

$$\rho_r \propto a^{-4} \quad \text{and} \quad \rho_m \propto a^{-3},$$

and

$$p_m = 0 \quad \text{and} \quad p_r = \frac{1}{3}\rho_r.$$

The equation of state and sound speed parameters are

$$w_m = c_m^2 = 0 \quad \text{and} \quad w_r = c_r^2 = \frac{1}{3}. \quad (19.1)$$

We define

$$y \equiv \frac{a}{a_{\text{eq}}} = \frac{\rho_m}{\rho_r}, \quad (19.2)$$

where

$$a_{\text{eq}} = \frac{\Omega_r}{\Omega_m} \quad (19.3)$$

is the scale factor at matter-radiation equality, so that

$$\frac{\rho_r}{\rho} = \frac{1}{1+y} \quad \frac{\rho_m}{\rho} = \frac{y}{1+y} \quad \frac{\rho_r + p_r}{\rho + p} = \frac{4}{4+3y} \quad \frac{\rho_m + p_m}{\rho + p} = \frac{3y}{4+3y} \quad (19.4)$$

and

$$w = \frac{1}{3(1+y)} \quad 1 + w = \frac{4+3y}{3(1+y)} \quad c_s^2 = \frac{4}{3(4+3y)} \quad 1 - 3c_s^2 = \frac{3y}{4+3y}. \quad (19.5)$$

The Friedmann equation is

$$\begin{aligned} \mathcal{H}^2 &\equiv \frac{1}{a^2} \left(\frac{da}{d\eta} \right)^2 = \frac{8\pi G}{3} \rho a^2 = \frac{8\pi G}{3} (1+y) \rho_{r0} a^{-2} \\ &\Rightarrow \frac{dy}{\sqrt{1+y}} = 2C d\eta, \end{aligned}$$

where

$$C \equiv \sqrt{\frac{2\pi G}{3} \rho_{r0}} \frac{1}{a_{\text{eq}}} = \frac{1}{2} \sqrt{\Omega_r} \frac{H_0}{a_{\text{eq}}} = \frac{1}{2} H_0 \frac{\Omega_m}{\sqrt{\Omega_r}}. \quad (19.6)$$

The solution is

$$y = 2C\eta + C^2\eta^2 \quad \text{or} \quad a(\eta) = \sqrt{\Omega_r} H_0 \eta + \frac{1}{4} \Omega_m H_0^2 \eta^2. \quad (19.7)$$

At the time of matter-radiation equality,

$$y = y_{\text{eq}} = C^2 \eta_{\text{eq}}^2 + 2C\eta_{\text{eq}} = 1 \quad \Rightarrow \quad C\eta_{\text{eq}} = \sqrt{2} - 1, \quad (19.8)$$

so we can write the solution as

$$\boxed{y = 2 \left(\frac{\eta}{\eta_3} \right) + \left(\frac{\eta}{\eta_3} \right)^2}, \quad (19.9)$$

where

$$\eta_3 \equiv \frac{\eta_{\text{eq}}}{\sqrt{2} - 1} = (\sqrt{2} + 1) \eta_{\text{eq}} = \frac{1}{C} = \frac{2}{H_0} \frac{\sqrt{\Omega_r}}{\Omega_m} \quad (19.10)$$

is the time when $y = 3$ ($\rho_m = 3\rho_r$).

The Hubble parameter is

$$\mathcal{H} \equiv \frac{a'}{a} = \frac{y'}{y} = \frac{\eta + \eta_3}{\eta_3 \eta + \frac{1}{2} \eta^2}. \quad (19.11)$$

At matter-radiation equality it has the value

$$\mathcal{H}_{\text{eq}} = \frac{2\sqrt{2}}{\sqrt{2} + 1} \frac{1}{\eta_{\text{eq}}} = \frac{4 - 2\sqrt{2}}{\eta_{\text{eq}}} = \frac{2\sqrt{2}}{\eta_3} = \sqrt{2} \frac{\Omega_m}{\sqrt{\Omega_r}} H_0. \quad (19.12)$$

At early times, $\eta \ll \eta_3$, the universe is radiation dominated, so that

$$y \approx \frac{2\eta}{\eta_3} \ll 1 \quad \Rightarrow \quad a \propto \eta \quad \Rightarrow \quad \mathcal{H} = \frac{1}{\eta} \propto a^{-1}. \quad (19.13)$$

At late times, $\eta \gg \eta_3$, the universe is matter dominated, so that

$$y \approx \left(\frac{\eta}{\eta_3} \right)^2 \gg 1 \quad \Rightarrow \quad a \propto \eta^2 \quad \Rightarrow \quad \mathcal{H} = \frac{2}{\eta} \propto a^{-1/2}. \quad (19.14)$$

When solving for perturbations it turns out to be more convenient to use y (or $\log y$) as time coordinate instead of η . Inverting Eq. (19.9), we have that

$$\eta = \left(\sqrt{1 + y} - 1 \right) \eta_3 = \frac{\sqrt{1 + y} - 1}{\sqrt{2} - 1} \eta_{\text{eq}}, \quad (19.15)$$

and

$$\mathcal{H} = \frac{\sqrt{1 + y}}{y} \frac{2}{\eta_3} = \frac{\sqrt{1 + y}}{y} \frac{\mathcal{H}_{\text{eq}}}{\sqrt{2}}. \quad (19.16)$$

19.2 Perturbations

In terms of the component perturbations the total perturbations are now

$$\delta = \frac{1}{1+y}\delta_r + \frac{y}{1+y}\delta_m \quad (19.17)$$

$$v = \frac{4}{4+3y}v_r + \frac{3y}{4+3y}v_m \quad (19.18)$$

and the relative entropy perturbation is

$$S \equiv S_{mr} = \delta_m - \frac{3}{4}\delta_r. \quad (19.19)$$

From the pair (19.17,19.19) we can solve δ_m and δ_r in terms of δ and S :

$$\begin{aligned} \delta_m &= \frac{3+3y}{4+3y}\delta + \frac{4}{4+3y}S \\ \delta_r &= \frac{4+4y}{4+3y}\delta - \frac{4y}{4+3y}S. \end{aligned} \quad (19.20)$$

Likewise we can express v_m and v_r in terms of the total and relative velocity perturbations, v and $v_m - v_r$:

$$\begin{aligned} v_m &= v + \frac{4}{4+3y}(v_m - v_r) \\ v_r &= v - \frac{3y}{4+3y}(v_m - v_r). \end{aligned} \quad (19.21)$$

We can now also relate the total entropy perturbation \mathcal{S} to S :

$$\mathcal{S} = \frac{1}{3(1+w)} \left(\frac{\delta\rho}{\bar{\rho}} - \frac{1}{c_s^2} \frac{\delta p}{\bar{\rho}} \right) = \dots = \frac{y}{4+3y}S = \frac{1}{3}(1-3c_s^2)S. \quad (19.22)$$

The Bardeen equation (16.42) becomes now

$$\begin{aligned} \mathcal{H}^{-2}\delta_C'' + (1-6w+3c_s^2)\mathcal{H}^{-1}\delta_C' - \frac{3}{2}(1+8w-6c_s^2-3w^2)\delta_C \\ = -c_s^2 \left(\frac{k}{\mathcal{H}} \right)^2 [\delta_C - (1+w)(1-3c_s^2)S] \\ = -c_s^2 \left(\frac{k}{\mathcal{H}} \right)^2 \left(\delta_C - \frac{y}{1+y}S \right). \end{aligned} \quad (19.23)$$

We get the entropy evolution equation by derivating (18.30),

$$S' = -k(v_m - v_r) \Rightarrow S'' = -k(v_m' - v_r'). \quad (19.24)$$

Using the v_i^N evolution equations (18.27), or equivalently, using (18.32) with $w_m = c_m^2 = 0$ and $w_r = c_r^2 = \frac{1}{3}$ (and $\Pi_i = 0$), this becomes (**exercise**)

$$S'' = \mathcal{H}k(v_m - v_r) + \mathcal{H}kv_r^N + \frac{1}{4}k^2\delta_r^N. \quad (19.25)$$

Here the δ_r^N is converted to the comoving gauge using the *total* fluid velocity (see Eq. 18.22),

$$\delta_r^N = \delta_r^C - 3\mathcal{H}(1+w_r)k^{-1}v^N \quad (19.26)$$

so that Eq. (19.24) becomes

$$S'' = \mathcal{H}k \left(\frac{4}{4+3y} \right) (v_m - v_r) + \frac{1}{4}k^2 \delta_r^C. \quad (19.27)$$

Replacing $v_m - v_r$ by $-k^{-1}S'$ we get (**Exercise**) the Kodama-Sasaki equation [8]

$$\mathcal{H}^{-2}S'' + \frac{4}{4+3y}\mathcal{H}^{-1}S' = \left(\frac{k}{\mathcal{H}} \right)^2 \left(\frac{1+y}{4+3y}\delta^C - \frac{y}{4+3y}S \right). \quad (19.28)$$

or

$$\mathcal{H}^{-2}S'' + 3c_s^2\mathcal{H}^{-1}S' = \frac{1}{3} \left(\frac{k}{\mathcal{H}} \right)^2 \left(\frac{1}{1+w}\delta^C - (1-3c_s^2)S \right).$$

The two equations (19.23) and (19.28) form a pair of ordinary differential equations, from which we can solve the evolution of the perturbations $\delta_k^C(\eta)$ and $S_k(\eta)$. Since the coefficient functions of these equations can more easily be expressed in terms of the scale factor y , it may be more convenient to use y as the time coordinate instead of η . The time derivatives are converted with (**exercise**)

$$\mathcal{H}^{-1}f' = y \frac{df}{dy} \quad \text{and} \quad \mathcal{H}^{-2}f'' = y^2 \frac{d^2f}{dy^2} + \frac{1}{2}(1-3w)y \frac{df}{dy} \quad (19.29)$$

and the equations become⁴⁰

$$\begin{aligned} y^2 \frac{d^2\delta^C}{dy^2} + \frac{3}{2}(1-5w+2c_s^2)y \frac{d\delta^C}{dy} - \frac{3}{2}(1+8w-6c_s^2-3w^2)\delta^C &= - \left(\frac{k}{\mathcal{H}} \right)^2 c_s^2 \left(\delta^C - \frac{y}{1+y}S \right) \\ y^2 \frac{d^2S}{dy^2} + \frac{1}{2}(1-3w+6c_s^2)y \frac{dS}{dy} &= \left(\frac{k}{\mathcal{H}} \right)^2 \left(\frac{1+y}{4+3y}\delta^C - \frac{y}{4+3y}S \right). \end{aligned} \quad (19.30)$$

For solving the other perturbation quantities, we collect here the relevant equations:

$$\begin{aligned} \Phi &= -\frac{3}{2} \left(\frac{\mathcal{H}}{k} \right)^2 \delta^C \\ v^N &= \frac{2}{3(1+w)} \left(\frac{k}{\mathcal{H}} \right) (\mathcal{H}^{-1}\Phi' + \Phi) \\ \delta^N &= \delta^C - 3 \left(\frac{\mathcal{H}}{k} \right) (1+w)v^N \\ \mathcal{R} &= -\Phi - \frac{2}{3(1+w)\mathcal{H}} (\Phi' + \mathcal{H}\Phi) \end{aligned} \quad (19.31)$$

When judging which quantities are negligible at superhorizon scales ($k \ll \mathcal{H}$), one has to exercise some care, and not just look blindly at powers of k/\mathcal{H} in equations which contain different perturbation quantities. From Eq. (19.31a) one sees that at superhorizon scales, a small comoving δ^C can still be important and cause a large gravitational potential perturbation Φ . Eq. (16.40),

$$\mathcal{H}^{-1}\mathcal{R}' = -c_s^2 \left(\frac{\delta^C}{1+w} - 3S \right),$$

⁴⁰These are Eqs. (2.12a) and (2.12b) in Kodama&Sasaki[8], when one replaces the photon+baryon fluid in [8] with just radiation with $w_r = c_r^2 = \frac{1}{3}$. The w and c_s^2 in these expressions could be written in terms of y , or the y expressions on the rhs could be written in terms of w , c_s^2 , and $\bar{\rho}_m/\bar{\rho}$; but in this form I found them easy to compare to [8].

may seem to contradict the statement that for adiabatic perturbations \mathcal{R} stays constant at superhorizon scales, but the explanation is that \mathcal{R} is then of the same order of magnitude as Φ , whereas δ^C is suppressed by two powers of k/\mathcal{H} compared to Φ . Using Eqs. (19.31a,19.22) we rewrite (16.40) as

$$\mathcal{H}^{-1}\mathcal{R}' = c_s^2 \left[\frac{1}{1+w} \frac{2}{3} \left(\frac{k}{\mathcal{H}} \right)^2 \Phi + (1 - 3c_s^2)S \right]. \quad (19.32)$$

19.3 Initial Epoch

For $y \ll 1$,

$$\mathcal{H}^2 \approx \frac{\mathcal{H}_{\text{eq}}^2}{2y^2} \quad (19.33)$$

and the equations (19.23,19.28,19.30) can be approximated by

$$\mathcal{H}^{-2}\delta_C'' - 2\delta_C = -\frac{1}{3} \left(\frac{k}{\mathcal{H}} \right)^2 (\delta_C - yS) \quad (19.34)$$

$$\mathcal{H}^{-2}S'' + \mathcal{H}^{-1}S' = \frac{1}{4} \left(\frac{k}{\mathcal{H}} \right)^2 (\delta_C - yS) \quad (19.35)$$

or

$$y^2 \frac{d^2 \delta^C}{dy^2} - 2\delta^C = -\frac{1}{3} \left(\frac{k}{\mathcal{H}} \right)^2 (\delta^C - yS) = -\frac{2}{3} \left(\frac{k}{\mathcal{H}_{\text{eq}}} \right)^2 (y^2 \delta^C - y^3 S) \quad (19.36)$$

$$y^2 \frac{d^2 S}{dy^2} + y \frac{dS}{dy} = \frac{1}{4} \left(\frac{k}{\mathcal{H}} \right)^2 (\delta^C - yS) = \frac{1}{2} \left(\frac{k}{\mathcal{H}_{\text{eq}}} \right)^2 (y^2 \delta^C - y^3 S). \quad (19.37)$$

At early times, all cosmologically interesting scales are outside the horizon, and we start by making the approximation, that the rhs of these equations can be ignored. Then the evolution of δ^C and S decouple and the general solutions are

$$\delta_{\vec{k}}^C = A_{\vec{k}} y^2 + B_{\vec{k}} y^{-1} \quad (19.38)$$

$$S_{\vec{k}} = C_{\vec{k}} + D_{\vec{k}} \ln y. \quad (19.39)$$

Thus, for each Fourier component \vec{k} , there are four independent modes. We identify the adiabatic growing ($A_{\vec{k}}$) and decaying ($B_{\vec{k}}$) modes, and the isocurvature⁴¹ “growing” ($C_{\vec{k}}$) and decaying ($D_{\vec{k}}$) modes. The D -mode is indeed decaying, since $0 < y \ll 1$, so $\ln y$ is large and negative, and gets smaller as y grows.

The decaying modes diverge as $y \rightarrow 0$, but we can suppose that our description of the universe is not valid all the way to $y = 0$, but at very early times there is some process that is responsible for fixing the initial values of $\delta_{\vec{k}}^C$, $\delta_{\vec{k}}^{C'}$, $S_{\vec{k}}$, and $S_{\vec{k}}'$ at some early time $y_{\text{init}} > 0$, and fixing thus $A_{\vec{k}}$, $B_{\vec{k}}$, $C_{\vec{k}}$, and $D_{\vec{k}}$.

Let us now consider the effect of the rhs of the eqs. These couple the δ and S . Thus it is not consistent to assume that the other is exactly zero.

19.3.1 Adiabatic modes

Consider first the adiabatic modes ($C_{\vec{k}} = D_{\vec{k}} = 0$). The coupling forces the existence of a small nonzero S , which at first is $\ll \delta^C$. We can thus keep ignoring S on the rhs, and for the δ^C equation we can keep ignoring the whole rhs. But for the S equation we now have very small S

⁴¹The term “isocurvature” will be explained later.

on the lhs, and therefore we can not ignore the large δ on the rhs, even though it is suppressed by $(k/\mathcal{H})^2$, or y^2 . Thus for adiabatic modes the pair of equations can be approximated as

$$\mathcal{H}^{-2}\delta_C'' - 2\delta_C = 0 \quad (19.40)$$

$$\mathcal{H}^{-2}S'' + \mathcal{H}^{-1}S' = \frac{1}{4}\left(\frac{k}{\mathcal{H}}\right)^2 \delta_C \quad (19.41)$$

or

$$y^2 \frac{d^2 \delta^C}{dy^2} - 2\delta^C = 0 \quad (19.42)$$

$$y^2 \frac{d^2 S}{dy^2} + y \frac{dS}{dy} = \frac{1}{2}\left(\frac{k}{\mathcal{H}_{\text{eq}}}\right)^2 y^2 \delta^C. \quad (19.43)$$

The solution for δ remains Eq. (19.38), but we also get that

$$S_{\vec{k}} = \frac{1}{32}\left(\frac{k}{\mathcal{H}_{\text{eq}}}\right)^2 A_{\vec{k}} y^4 + \frac{1}{2}\left(\frac{k}{\mathcal{H}_{\text{eq}}}\right)^2 B_{\vec{k}} y. \quad (19.44)$$

Thus for the growing adiabatic mode

$$S_{\vec{k}} = \frac{1}{32}\left(\frac{k}{\mathcal{H}_{\text{eq}}}\right)^2 \delta_{\vec{k}}^C y^2 = \frac{1}{64}\left(\frac{k}{\mathcal{H}}\right)^2 \delta_{\vec{k}}^C \quad (19.45)$$

and for the decaying adiabatic mode

$$S_{\vec{k}} = \frac{1}{2}\left(\frac{k}{\mathcal{H}_{\text{eq}}}\right)^2 \delta_{\vec{k}}^C y^2 = \frac{1}{4}\left(\frac{k}{\mathcal{H}}\right)^2 \delta_{\vec{k}}^C. \quad (19.46)$$

So you see that, although these are called adiabatic modes, the perturbations are not exactly adiabatic! The name just means that $S \rightarrow 0$ as $y \rightarrow 0$. The entropy perturbations remain small compared to the density perturbation while the Fourier mode is outside the horizon, but can become large near and after horizon entry.

19.3.2 Isocurvature Modes

Consider then the isocurvature modes ($A_{\vec{k}} = B_{\vec{k}} = 0$). Now the coupling causes a small $\delta^C \ll S$. We can ignore the rhs of the S equation, but the large S cannot be ignored on the rhs of the δ^C equation. Thus for isocurvature modes the pair of equations can be approximated by

$$\mathcal{H}^{-2}\delta_C'' - 2\delta_C = +\frac{1}{3}\left(\frac{k}{\mathcal{H}}\right)^2 yS \quad (19.47)$$

$$\mathcal{H}^{-2}S'' + \mathcal{H}^{-1}S' = 0 \quad (19.48)$$

or

$$y^2 \frac{d^2 \delta^C}{dy^2} - 2\delta^C = +\frac{2}{3}\left(\frac{k}{\mathcal{H}_{\text{eq}}}\right)^2 y^3 S \quad (19.49)$$

$$y^2 \frac{d^2 S}{dy^2} + y \frac{dS}{dy} = 0. \quad (19.50)$$

The solution for S remains Eq. (19.39), but for the density perturbation we get

$$\delta_{\vec{k}}^C = \frac{1}{6}\left(\frac{k}{\mathcal{H}_{\text{eq}}}\right)^2 C_{\vec{k}} y^3 + \frac{1}{6}\left(\frac{k}{\mathcal{H}_{\text{eq}}}\right)^2 D_{\vec{k}} \left(y^3 \ln y - \frac{5}{4}y^3\right). \quad (19.51)$$

(For the decaying isocurvature mode the differential equation for δ is a little more difficult, but you can check (**Exercise**) that the above is the correct solution.) For the growing isocurvature mode we have thus that

$$\delta_{\vec{k}}^C = \frac{1}{6} \left(\frac{k}{\mathcal{H}_{\text{eq}}} \right)^2 S_{\vec{k}} y^3 = \frac{1}{12} \left(\frac{k}{\mathcal{H}} \right)^2 y S_{\vec{k}}. \quad (19.52)$$

So although the relative entropy perturbation stays constant at early times, the density perturbation is growing in this mode. For the decaying isocurvature mode

$$\delta_{\vec{k}}^C = \frac{1}{12} \left(\frac{k}{\mathcal{H}} \right)^2 \left(y - \frac{5}{4} \frac{y}{\ln y} \right) S_{\vec{k}} \approx \frac{1}{12} \left(\frac{k}{\mathcal{H}} \right)^2 y S_{\vec{k}}. \quad (19.53)$$

19.3.3 Other Perturbations

For the initial epoch, $w = \frac{1}{3}$, and Eqs. (19.31) become

$$\begin{aligned} \Phi_{\vec{k}} &= -\frac{3}{2} \left(\frac{\mathcal{H}}{k} \right)^2 \delta_{\vec{k}}^C \\ v_{\vec{k}}^N &= \frac{1}{2} \left(\frac{k}{\mathcal{H}} \right) \left(\mathcal{H}^{-1} \Phi'_{\vec{k}} + \Phi_{\vec{k}} \right) \\ \delta_{\vec{k}}^N &= \delta_{\vec{k}}^C - 4 \left(\frac{\mathcal{H}}{k} \right) v_{\vec{k}}^N \\ \mathcal{R}_{\vec{k}} &= -\frac{3}{2} \Phi_{\vec{k}} - \frac{1}{2} \mathcal{H}^{-1} \Phi'_{\vec{k}} \end{aligned} \quad (19.54)$$

For the growing adiabatic mode, we have

$$\Phi_{\vec{k}} = -\frac{3}{2} \left(\frac{\mathcal{H}}{k} \right)^2 A_{\vec{k}} y^2 = -\frac{3}{4} \left(\frac{\mathcal{H}_{\text{eq}}}{k} \right)^2 A_{\vec{k}} = \text{const.} \quad (19.55)$$

$$\mathcal{R}_{\vec{k}} = -\frac{3}{2} \Phi_{\vec{k}} = \frac{9}{8} \left(\frac{\mathcal{H}_{\text{eq}}}{k} \right)^2 A_{\vec{k}} = \text{const.} \quad (19.56)$$

$$v_{\vec{k}}^N = -\frac{1}{3} \left(\frac{k}{\mathcal{H}} \right) \mathcal{R}_{\vec{k}} = -\frac{\sqrt{2}}{3} \left(\frac{k}{\mathcal{H}_{\text{eq}}} \right) y \mathcal{R}_{\vec{k}} \quad (19.57)$$

$$\delta_{\vec{k}}^N = -2\Phi_{\vec{k}} = \frac{4}{3} \mathcal{R}_{\vec{k}} \quad (19.58)$$

$$S_{\vec{k}} = \frac{1}{144} \left(\frac{k}{\mathcal{H}} \right)^4 \mathcal{R}_{\vec{k}} = \frac{1}{36} \left(\frac{k}{\mathcal{H}_{\text{eq}}} \right)^4 \mathcal{R}_{\vec{k}} y^4. \quad (19.59)$$

For the growing isocurvature mode, where $S = \text{const.}$ we have

$$\Phi_{\vec{k}} = -\frac{1}{8} y S_{\vec{k}} = -\frac{\mathcal{H}_{\text{eq}}}{8\sqrt{2}} \eta S_{\vec{k}} \quad (19.60)$$

$$\mathcal{R}_{\vec{k}} = \frac{1}{4\sqrt{2}} \mathcal{H}_{\text{eq}} S_{\vec{k}} \eta = \frac{1}{4} S_{\vec{k}} y = -2\Phi_{\vec{k}} \quad (19.61)$$

$$v_{\vec{k}}^N = -\frac{1}{4\sqrt{2}} \left(\frac{k}{\mathcal{H}_{\text{eq}}} \right) S_{\vec{k}} y^2 \quad (19.62)$$

$$\delta_{\vec{k}}^N = \frac{1}{6} \left(\frac{k}{\mathcal{H}_{\text{eq}}} \right)^2 S_{\vec{k}} y^3 + \frac{1}{2} S_{\vec{k}} y \approx \frac{1}{2} S_{\vec{k}} y = -4\Phi_{\vec{k}} = 2\mathcal{R}_{\vec{k}}. \quad (19.63)$$

Thus the growing adiabatic mode is characterized by a constant \mathcal{R} and the growing isocurvature mode by a constant S . If both modes are present, and these constants are of a similar

magnitude, then the growing S associated with the adiabatic mode is negligible as long as $k \ll \mathcal{H}$, and the growing \mathcal{R} associated with the isocurvature mode is negligible (hence the name “isocurvature”) as long as $y \ll 1$ ($\rho_m \ll \rho_r$). *Note the asymmetry:* \mathcal{R} evolves when radiation domination begins to give way for matter, whereas S evolves when the scale approaches the horizon!

Thus, after the decaying modes have died out, the general (adiabatic+isocurvature) mode is characterized by these two constants (for each Fourier mode), which we denote by $\mathcal{R}_{\vec{k}}(\text{rad})$ and $S_{\vec{k}}(\text{rad})$. Including the two small growing contributions we have, during the initial epoch,

$$\mathcal{R}_{\vec{k}} = \frac{9}{8} \left(\frac{\mathcal{H}_{\text{eq}}}{k} \right)^2 A_{\vec{k}} + \frac{1}{4} y C_{\vec{k}} = \mathcal{R}_{\vec{k}}(\text{rad}) + \frac{1}{4} y S_{\vec{k}}(\text{rad}) \quad (19.64)$$

$$\begin{aligned} S_{\vec{k}} &= \frac{1}{32} \left(\frac{k}{\mathcal{H}_{\text{eq}}} \right)^2 A_{\vec{k}} y^4 + C_{\vec{k}} = S_{\vec{k}}(\text{rad}) + \frac{1}{36} \left(\frac{k}{\mathcal{H}_{\text{eq}}} \right)^4 \mathcal{R}_{\vec{k}}(\text{rad}) y^4 \\ &= S_{\vec{k}}(\text{rad}) + \frac{1}{144} \left(\frac{k}{\mathcal{H}} \right)^4 \mathcal{R}_{\vec{k}}(\text{rad}). \end{aligned} \quad (19.65)$$

19.4 Full evolution for large scales

The full evolution in the general case is not amenable to analytic solution, and has to be solved numerically (see Sec. 27, we do this in the Newtonian gauge). This is not too difficult since we just have a pair of ordinary differential equations to solve for each k . The results from Sect. 19.3 can be used to set initial values at some small y .

For large scales ($k \ll k_{\text{eq}} \equiv \mathcal{H}_{\text{eq}}$) we can, however, solve the evolution analytically. We now drop the decaying modes and consider what happens to the growing modes after the radiation-dominated epoch.

Our basic equations are (19.32) and (19.28):

$$\mathcal{H}^{-1} \mathcal{R}'_{\vec{k}} = c_s^2 \left[\frac{1}{1+w} \frac{2}{3} \left(\frac{k}{\mathcal{H}} \right)^2 \Phi_{\vec{k}} + (1 - 3c_s^2) S_{\vec{k}} \right] \quad (19.66)$$

$$\mathcal{H}^{-2} S''_{\vec{k}} + 3c_s^2 \mathcal{H}^{-1} S'_{\vec{k}} = \frac{1}{3} \left(\frac{k}{\mathcal{H}} \right)^2 \left[\frac{1}{1+w} \delta_{\vec{k}}^C - (1 - 3c_s^2) S_{\vec{k}} \right]. \quad (19.67)$$

We see that for superhorizon scales, $S_{\vec{k}} = \text{const.}$ remains a solution even when the universe is no longer radiation dominated. Since the other solution has decayed away, we conclude that

$$S_{\vec{k}} = \text{const.} = S_{\vec{k}}(\text{rad}) \quad \text{for} \quad k \ll \mathcal{H}. \quad (19.68)$$

We can also see that, for the *adiabatic mode*, $\mathcal{R}_{\vec{k}}$ stays constant for superhorizon scales, since on the rhs, $S_{\vec{k}}$ remains negligible for $k \ll \mathcal{H}$. For scales that enter the horizon during the matter-dominated epoch, the $\mathcal{R}_{\vec{k}}$ of adiabatic perturbations stays constant even through and after horizon entry, since c_s^2 becomes negligibly small, before k/\mathcal{H} and $S_{\vec{k}}$ become large. We can thus conclude that

$$\mathcal{R}_{\vec{k}} = \text{const.} = \mathcal{R}_{\vec{k}}(\text{rad}) \quad \text{for adiabatic modes with} \quad k \ll k_{\text{eq}}. \quad (19.69)$$

If the isocurvature mode is present, however, we cannot assume that $S_{\vec{k}}$ is negligible for superhorizon scales; it's just constant, and we have, for $k \ll \mathcal{H}$,

$$\mathcal{H}^{-1} \mathcal{R}'_{\vec{k}} = c_s^2 (1 - 3c_s^2) S_{\vec{k}}, \quad (19.70)$$

or

$$\frac{d\mathcal{R}_{\vec{k}}}{dy} = \frac{4}{(4+3y)^2} S_{\vec{k}}. \quad (19.71)$$

Integrating this, we get

$$\mathcal{R}_{\vec{k}} = \mathcal{R}_{\vec{k}}(\text{rad}) + \frac{y}{4+3y} S_{\vec{k}}(\text{rad}) \quad \text{for} \quad k \ll \mathcal{H}. \quad (19.72)$$

For $k \ll k_{\text{eq}}$, this has reached the final value $\mathcal{R}_{\vec{k}}(\text{rad}) + \frac{1}{3} S_{\vec{k}}(\text{rad})$ when the universe has become matter dominated, before horizon entry. After that, $\mathcal{R}_{\vec{k}}$ stays constant even through and after horizon entry by the same argument as for the adiabatic mode. Thus we conclude that

$$\mathcal{R}_{\vec{k}} = \text{const.} = \mathcal{R}_{\vec{k}}(\text{rad}) + \frac{1}{3} S_{\vec{k}}(\text{rad}) \quad \text{for} \quad k \ll k_{\text{eq}} \quad \text{and} \quad \eta \gg \eta_{\text{eq}}. \quad (19.73)$$

For smaller scales, the exact solution has to be obtained numerically. It is customary to represent the solution in terms of transfer functions $T_{RR}(k)$ and $T_{RS}(k)$, which we can define by

$$\mathcal{R}_{\vec{k}} = \text{const.} \equiv T_{RR}(k) \mathcal{R}_{\vec{k}}(\text{rad}) + \frac{1}{3} T_{RS}(k) S_{\vec{k}}(\text{rad}) \quad \text{for} \quad \eta \gg \eta_{\text{eq}}, \quad (19.74)$$

so that $T_{RR}(k) = T_{RS}(k) = 1$ for $k \ll k_{\text{eq}}$.

19.5 Initial Conditions in Terms of Conformal Time

The full (linear) evolution of perturbations at all scales k needs to be calculated numerically with codes such as CAMB.⁴² These codes require theoretically derived initial conditions. CAMB uses conformal time as time variable and works in synchronous gauge. The initial conditions are given as truncated series in powers of conformal time η . Our goal here is to derive the CAMB initial conditions as they are given in “CAMB Notes” by Antony Lewis [9]. These are for the real universe with baryons, CDM, photons, and neutrinos. Our simplified universe of this Section corresponds to the case without neutrinos (who are not perfect) and baryons (who interact with photons). Thus we should compare to the CAMB equations setting $f_\nu = f_b = 0$ (in Lewis’ notation R_ν and R_b), so that photons (γ) represent radiation (r) and CDM (c) represents matter (m). Our strategy is to derive the density perturbations in comoving gauge and then do a gauge transformation to synchronous gauge.

We start from the Bardeen (19.23) and Kosama–Sasaki (19.28) equations

$$\begin{aligned} \delta_C'' + (1 - 6w + 3c_s^2) \mathcal{H} \delta_C' - \frac{3}{2} (1 + 8w - 6c_s^2 - 3w^2) \mathcal{H}^2 \delta_C &= -c_s^2 k^2 [\delta_C - (1 + w)(1 - 3c_s^2) S] \\ S'' + 3c_s^2 \mathcal{H} S' &= \frac{1}{3} k^2 \left(\frac{1}{1 + w} \delta_C - (1 - 3c_s^2) S \right). \end{aligned}$$

Writing the coefficients in terms of y , they read (**exercise**)

$$\begin{aligned} \eta_3^2 \delta_C'' + \frac{2}{\sqrt{1+y}} \frac{5+3y}{4+3y} \eta_3 \delta_C' - \frac{16+38y+30y^2+9y^3}{(1+y)(4+3y)} \frac{2}{y^2} \delta_C &= -\frac{(k\eta_3)^2}{3(1+\frac{3}{4}y)} \left[\delta_C - \frac{y}{1+y} S \right] \\ \eta_3^2 S'' + \frac{\sqrt{1+y}}{1+\frac{3}{4}y} \frac{2}{y} \eta_3 S' &= \frac{(k\eta_3)^2}{4(1+\frac{3}{4}y)} \left[(1+y) \delta_C - y S \right] \end{aligned} \quad (19.75)$$

Defining

$$x \equiv \frac{\eta}{\eta_3} \quad \text{we have} \quad y = 2x + x^2, \quad (19.76)$$

and (19.75) becomes (**exercise**)

$$\begin{aligned} \eta_3^2 x^2 \delta_C'' + \frac{5}{2} \frac{1 + \frac{6}{5}x + \frac{3}{5}x^2}{1 + \frac{5}{2}x + \frac{9}{4}x^2 + \frac{3}{4}x^3} \eta_3 x^2 \delta_C' - 2 \frac{1 + \frac{19}{4}x + \frac{79}{8}x^2 + 12x^3 + \frac{69}{8}x^4 + \frac{27}{8}x^5 + \frac{9}{16}x^6}{1 + \frac{9}{2}x + \frac{17}{2}x^2 + \frac{69}{8}x^3 + \frac{79}{16}x^4 + \frac{3}{2}x^5 + \frac{3}{16}x^6} \delta_C \\ = -\frac{1}{3} (k\eta_3)^2 \frac{x^2}{1 + \frac{3}{2}x + \frac{3}{4}x^2} \delta_C + \frac{2}{3} (k\eta_3)^2 \frac{1 + \frac{1}{2}x}{1 + \frac{7}{2}x + \frac{19}{4}x^2 + 3x^3 + \frac{3}{4}x^4} x^3 S \\ \eta_3^2 x S'' + \frac{1+x}{1 + 2x + \frac{3}{2}x^2 + \frac{3}{8}x^3} \eta_3 S' = \frac{1}{4} (k\eta_3)^2 \frac{1 + 2x + x^2}{1 + \frac{3}{2}x + \frac{3}{4}x^2} x \delta_C - \frac{1}{2} (k\eta_3)^2 \frac{1 + \frac{1}{2}x}{1 + \frac{3}{2}x + \frac{3}{4}x^2} x^2 S \end{aligned} \quad (19.77)$$

⁴²<https://camb.info/>

We are working in the limit $k\eta = k\eta_3 x \ll 1$ and $\eta \ll \eta_3 \Rightarrow x \ll 1$. We search for solutions as series in x , which we will truncate at fourth and fifth order,⁴³ i.e.,

$$\delta_C \approx A + Bx + Cx^2 + Dx^3 + Jx^4 \quad \text{and} \quad S \approx E + Fx + Gx^2 + Hx^3 + Kx^4 + Lx^5. \quad (19.78)$$

We do not allow for negative powers of x in this expansion, since a solution containing a negative power is a decaying mode, and we are not interested in those. To solve all the 11 constants from A to L we need to expand each coefficient expression in (19.77) to sufficient order in x . It turns out that we need all terms in (19.77) up to x^4 . This expansion converts (19.77) into⁴⁴ (**exercise**)

$$\begin{aligned} x^2 \eta_3^2 \delta_C'' + \frac{5}{2} \left(1 - \frac{13}{10}x + \frac{8}{5}x^2\right) x^2 \eta_3 \delta_C' - 2 \left(1 + \frac{1}{4}x + \frac{1}{4}x^2 + \frac{1}{8}x^3 - \frac{37}{32}x^4\right) \delta_C \\ \approx -\frac{1}{3}(k\eta_3)^2 \left(1 - \frac{3}{2}x + \frac{3}{2}x^2\right) x^2 \delta_C + \frac{2}{3}(k\eta_3)^2 (1 - 3x) x^3 S \\ x \eta_3^2 S'' + \left(1 - x + \frac{1}{2}x^2 + \frac{1}{8}x^3 - \frac{5}{8}x^4\right) \eta_3 S' \\ \approx \frac{1}{4}(k\eta_3)^2 \left(1 + \frac{1}{2}x - \frac{1}{2}x^2 + \frac{3}{8}x^3\right) x \delta_C - \frac{1}{2}(k\eta_3)^2 \left(1 - x + \frac{3}{4}x^2\right) x^2 S. \end{aligned} \quad (19.79)$$

Substituting (19.78) into (19.79) and requiring each power of x up to x^4 to agree in (19.79) gives 10 equations for 11 unknowns, but it turns out that one of these equations does not give additional information, leaving two degrees of freedom. We get (**exercise**) $A = B = F = G = 0$, so that C gives the lowest order for δ^C and E gives the lowest order for S . Expressing the other constants in terms of these, we get⁴⁵ (**exercise**)

$$\begin{aligned} D &= -\frac{9}{8}C + \frac{1}{6}(k\eta_3)^2 E \\ J &= \frac{119}{80}C - \frac{1}{30}(k\eta_3)^2 C - \frac{19}{60}(k\eta_3)^2 E \\ H &= -\frac{1}{18}(k\eta_3)^2 E \\ K &= \frac{1}{64}(k\eta_3)^2 C + \frac{1}{48}(k\eta_3)^2 E \\ L &= -\frac{3}{800}(k\eta_3)^2 C - \frac{1}{120}(k\eta_3)^2 E + \frac{1}{600}(k\eta_3)^4 E, \end{aligned} \quad (19.80)$$

so that

$$\begin{aligned} \delta_k^C &= C_k \left\{ x^2 - \frac{9}{8}x^3 + \left[\frac{119}{80} - \frac{1}{30}(k\eta_3)^2 \right] x^4 \right\} + E_k \left\{ \frac{1}{6}(k\eta_3)^2 x^3 - \frac{19}{60}(k\eta_3)^2 x^4 \right\} \\ S_k &= E_k \left\{ 1 - \frac{1}{18}(k\eta_3)^2 x^3 + \frac{1}{48}(k\eta_3)^2 x^4 + \left[-\frac{1}{120}(k\eta_3)^2 + \frac{1}{600}(k\eta_3)^4 \right] x^5 \right\} \\ &\quad + C_k \left\{ \frac{1}{64}(k\eta_3)^2 x^4 - \frac{3}{800}(k\eta_3)^2 x^5 \right\}. \end{aligned} \quad (19.81)$$

We identify an adiabatic mode ($\delta^C = \mathcal{O}(x^2)$ and $S = \mathcal{O}(x^4)$), whose initial amplitude is given by C_k and an isocurvature mode ($S = \mathcal{O}(x^0)$ and $\delta^C = \mathcal{O}(x^3)$), whose initial amplitude is given by E_k . Dropping some higher powers that we will not need, the adiabatic mode has

$$\begin{aligned} \delta^C &\propto x^2 - \frac{9}{8}x^3 + \mathcal{O}(x^4) \\ S &\propto \frac{1}{64}(k\eta_3)^2 x^4 - \frac{3}{800}(k\eta_3)^2 x^5 + \mathcal{O}(x^6) \end{aligned} \quad (19.82)$$

and the isocurvature mode has

$$\begin{aligned} \delta^C &\propto \frac{1}{6}(k\eta_3)^2 x^3 + \mathcal{O}(x^4) \\ S &\propto 1 - \frac{1}{18}(k\eta_3)^2 x^3 + \frac{1}{48}(k\eta_3)^2 x^4 + \mathcal{O}(x^5). \end{aligned} \quad (19.83)$$

For comparison to [9], define

$$\omega \equiv \frac{\Omega_m}{\sqrt{\Omega_r}} H_0 \Rightarrow \eta_3 = \frac{2}{\omega} \quad \text{and} \quad x = \frac{\omega}{2} \eta. \quad (19.84)$$

⁴³The fourth order in δ_C is overkill, but we will need S to fifth order to do the gauge transformation for the adiabatic mode. I originally did this to third order, thus the odd naming of the fourth order coefficients.

⁴⁴I have not checked all the highest order terms, so that the result for J (which we do not use) may not be trusted. The step from (19.77) to (19.79) I have now checked with WolframAlpha.

⁴⁵I have not checked the E part of L , which we will not use.

To get the other perturbations, using results from Sec. 19.2, we need the expansions (**exercise**)

$$\begin{aligned}\mathcal{H} &\approx (1 + \tfrac{1}{2}x - \tfrac{1}{4}x^2) \frac{1}{\eta} = (1 + \tfrac{1}{4}\omega\eta - \tfrac{1}{16}\omega^2\eta^2) \frac{1}{\eta} \\ 1 + w &\approx \tfrac{4}{3} (1 - \tfrac{1}{2}x + \tfrac{3}{4}x^2) = \tfrac{4}{3} (1 - \tfrac{1}{4}\omega\eta + \tfrac{3}{16}\omega^2\eta^2) .\end{aligned}\quad (19.85)$$

Isocurvature mode. Dropping the $\mathcal{O}(x^4)$ part of δ^C and the $\mathcal{O}(x^5)$ part of S , and not writing the initial amplitude $E_{\vec{k}}$ (i.e., all perturbations below are to be multiplied by the value of $E_{\vec{k}}$, i.e., by the initial value of $S_{\vec{k}}$), the isocurvature mode is

$$\begin{aligned}\delta^C &= \tfrac{1}{12}\omega k^2\eta^3 + \mathcal{O}(\eta^4) \\ S &= 1 - \tfrac{1}{36}\omega k^2\eta^3 + \tfrac{1}{192}\omega^2 k^2\eta^4 + \mathcal{O}(\eta^5) .\end{aligned}\quad (19.86)$$

Using (19.31) (**exercise**), we get

$$\begin{aligned}\Phi &= -\tfrac{1}{8}\omega\eta + \mathcal{O}(\eta^2) \\ v^N &= -\tfrac{1}{8}\omega k\eta^2 + \mathcal{O}(\eta^3) \\ \delta^N &= \tfrac{1}{2}\omega\eta + \mathcal{O}(\eta^2) \\ \mathcal{R} &= \tfrac{1}{4}\omega\eta + \mathcal{O}(\eta^2) .\end{aligned}\quad (19.87)$$

Note that we got δ^N entirely from the v^N part of (19.31b); δ^C contributes to it at $\mathcal{O}(\eta^3)$, but to calculate δ^N to $\mathcal{O}(\eta^3)$ we would need v^N to $\mathcal{O}(\eta^4)$, which requires Φ to $\mathcal{O}(\eta^3)$, which requires δ^C to $\mathcal{O}(\eta^5)$. From (19.20) we get (**exercise**)

$$\begin{aligned}\delta_m^C &= 1 - \tfrac{3}{4}\omega\eta + \tfrac{3}{8}\omega^2\eta^2 - \tfrac{9}{64}\omega^3\eta^3 + \tfrac{5}{144}\omega k^2\eta^3 + \mathcal{O}(\eta^4) \\ \delta_r^C &= -\omega\eta + \tfrac{1}{2}\omega^2\eta^2 - \tfrac{3}{16}\omega^3\eta^3 + \tfrac{1}{12}\omega k^2\eta^3 + \mathcal{O}(\eta^4) .\end{aligned}\quad (19.88)$$

The entropy perturbation S and the velocity difference $v_m - v_r$ are gauge invariant, and we have

$$v_m - v_r = -\frac{1}{k}S' = \tfrac{1}{12}\omega k\eta^2 - \tfrac{1}{48}\omega^2 k\eta^3 + \mathcal{O}(\eta^4) .\quad (19.89)$$

Using (19.21) we get (**exercise**)

$$\begin{aligned}v_m^N &= -\tfrac{1}{24}\omega k\eta^2 + \tfrac{1}{60}\omega^2 k\eta^3 + \mathcal{O}(\eta^4) \\ v_r^N &= -\tfrac{1}{8}\omega k\eta^2 + \tfrac{3}{80}\omega^2 k\eta^3 + \mathcal{O}(\eta^4) .\end{aligned}\quad (19.90)$$

Now we want to change to synchronous gauge. We use the remaining gauge freedom to set $v_m^Z \equiv 0$, i.e., the threads, which in synchronous gauge must be geodesics, follow the matter world lines (which are geodesics, since matter here is assumed pressureless and not interacting with the other fluid component). This means that (**exercise**)

$$\begin{aligned}v_r^Z &= v_r^Z - v_m^Z = -\tfrac{1}{12}\omega k\eta^2 + \tfrac{1}{48}\omega^2 k\eta^3 + \mathcal{O}(\eta^4) \\ v^Z &= \frac{4}{4+3y}v_r^Z = -\tfrac{1}{12}\omega k\eta^2 + \tfrac{1}{12}\omega^2 k\eta^3 + \mathcal{O}(\eta^4) .\end{aligned}\quad (19.91)$$

We get from synchronous gauge to comoving gauge by

$$\delta_i^C = \delta_i^Z + 3\frac{\mathcal{H}}{k}(1+w_i)v^Z \quad (19.92)$$

(since $B^Z = 0$), so we have (**exercise**)

$$\begin{aligned}\delta_m^Z &= \delta_m^C - 3\frac{\mathcal{H}}{k}v^Z = 1 - \tfrac{1}{2}\omega\eta + \tfrac{3}{16}\omega^2\eta^2 + \mathcal{O}(\eta^3) \\ \delta_r^Z &= \delta_r^C - 4\frac{\mathcal{H}}{k}v^Z = -\tfrac{2}{3}\omega\eta + \tfrac{1}{4}\omega^2\eta^2 + \mathcal{O}(\eta^3) .\end{aligned}\quad (19.93)$$

These results for Φ , δ_m^Z , δ_r^Z , and v_r^Z agree with [9] (setting there $R_c = 1$ and $R_\nu = 0$).

Adiabatic mode. Calculation of $\mathcal{R}_{\vec{k}}$ for the adiabatic mode gives

$$\mathcal{R}_{\vec{k}} = \frac{9}{16} \frac{\omega^2}{k^2} C_{\vec{k}} + \mathcal{O}(\eta^2). \quad (19.94)$$

Let us normalize the adiabatic mode so that $\mathcal{R}_{\vec{k}} = 1$ initially, i.e, set $C_{\vec{k}} = (16/9)k^2/\omega^2$. This means that all perturbations below are to be multiplied by the true initial value of $\mathcal{R}_{\vec{k}}$. Thus the adiabatic mode is

$$\begin{aligned} \delta^C &= \frac{4}{9}k^2\eta^2 - \frac{1}{4}\omega k^2\eta^3 + \mathcal{O}(\eta^4) \\ S &= \frac{1}{144}k^4\eta^4 - \frac{1}{1200}\omega k^4\eta^5 + \mathcal{O}(\eta^6). \end{aligned} \quad (19.95)$$

Exercise: Calculate Φ to $\mathcal{O}(\eta)$, v^N to $\mathcal{O}(\eta)$, \mathcal{R} to $\mathcal{O}(\eta)$, δ_m^C and δ_r^C to $\mathcal{O}(\eta^3)$, $v_m - v_r$ to $\mathcal{O}(\eta^4)$, v_m^N and v_r^N to $\mathcal{O}(\eta)$, v_r^Z to $\mathcal{O}(\eta^4)$, v^Z to $\mathcal{O}(\eta^4)$, and δ_m^Z and δ_r^Z to $\mathcal{O}(\eta^3)$.

20 Effect of the Cosmological Constant

So far we have ignored dark energy. We consider now the case where dark energy is vacuum energy. Vacuum energy has a constant (homogeneous and time-independent) energy density and pressure, with $p_h = -\rho_h$, i.e., $w_h = -1$ and $1 + w_h = 0$. (I use the subscript h , for “homogeneous”, for vacuum energy.) Vacuum energy is indistinguishable from a cosmological constant $\Lambda = 8\pi G\rho_h$, so we use the two terms here as synonyms, although they represent different ideas (new energy component ρ_h *vs* modification of the Einstein equation by Λ). Vacuum energy cannot have any perturbations, so

$$\delta_h = \delta p_h = \Pi_h = 0. \quad (20.1)$$

The velocity perturbation v_h is not defined, since it comes from $\delta T_{0h}^i \equiv (\rho_h + p_h)v_{h,i}$ and $\rho_h + p_h = 0$ for vacuum energy. But anyway $\delta T_{0h}^i = 0$.

20.1 Λ + Perturbed Component

Consider the fluid divided into two components: the homogeneous (h) vacuum energy component, and a perturbed (p) component, so that for the background quantities (we now drop the bars from background quantities) $\rho = \rho_p + \rho_h$ and $p = p_p + p_h$. We define the density parameter for the perturbed component

$$\Omega \equiv \frac{\rho_p}{\rho}. \quad (0 < \Omega < 1) \quad (20.2)$$

From Sec. 18.1,

$$\begin{aligned} w &= \Omega w_p + (1 - \Omega)w_h = \Omega w_p - (1 - \Omega) = \Omega(1 + w_p) - 1 \\ 1 + w &= \Omega(1 + w_p) \\ \delta &= \Omega \delta_p \\ \delta p &= \delta p_p \\ v &= \frac{1 + w_p}{1 + w} \Omega v_p = v_p \\ \Pi &= \frac{w_p}{w} \Omega \Pi_p = \frac{\Omega w_p}{\Omega w_p - (1 - \Omega)} \Pi_p \Rightarrow \frac{w}{1 + w} \Pi = \frac{w_p}{1 + w_p} \Pi_p. \end{aligned} \quad (20.3)$$

The speed of sound is undefined for vacuum energy but

$$c_s^2 \equiv \frac{p'}{\rho'} = \frac{p'_p}{\rho'_p} = c_p^2. \quad (20.4)$$

From (10.13)–(10.16) the Einstein equations in Newtonian gauge become

$$\begin{aligned} \nabla^2 \Psi &= \frac{3}{2} \mathcal{H}^2 \Omega [\delta_p^N + 3\mathcal{H}(1 + w_p)v^N] \\ \Psi - \Phi &= 3\mathcal{H}^2 \Omega w_p \Pi_p \\ \Psi' + \mathcal{H}\Phi &= \frac{3}{2} \mathcal{H}^2 \Omega (1 + w_p)v^N \\ \Psi'' + \mathcal{H}(\Phi' + 2\Psi') + (2\mathcal{H}' + \mathcal{H}^2)\Phi + \frac{1}{3}\nabla^2(\Phi - \Psi) &= \frac{3}{2} \mathcal{H}^2 \Omega \frac{\delta p_p^N}{\rho_p}. \end{aligned} \quad (20.5)$$

From (18.24) and (18.25), the fluid continuity equations for the p component are

$$\begin{aligned} (\delta_p^N)' &= (1 + w_p)(\nabla^2 v^N + 3\Psi') + 3\mathcal{H} \left(w_p \delta_p^N - \frac{\delta p_p^N}{\rho_p} \right) \\ (v^N)' &= -\mathcal{H}(1 - 3w_p)v^N - \frac{w'_p}{1 + w_p}v^N + \frac{\delta p_p^N}{\rho_p + p_p} + \frac{2}{3} \frac{w_p}{1 + w_p} \nabla^2 \Pi_p + \Phi. \end{aligned} \quad (20.6)$$

Exercise: Ω derivatives. Show that

$$\begin{aligned}\Omega' &= -3\mathcal{H}\Omega(1-\Omega)(1+w_p) \\ \Omega'' &= -\mathcal{H}^2\Omega(1-\Omega)(1+w_p) \left[\frac{15}{2} + \frac{27}{2}w - 9c_s^2 \right],\end{aligned}\quad (20.7)$$

and, if ρ_p is matter,

$$\begin{aligned}\Omega' &= -3\mathcal{H}\Omega(1-\Omega) \\ \Omega'' &= -\mathcal{H}^2\Omega(1-\Omega) \left[\frac{15}{2} - \frac{27}{2}(1-\Omega) \right].\end{aligned}\quad (20.8)$$

We get to the comoving gauge by the usual

$$\begin{aligned}\delta_p^C &\equiv \delta_p^N + 3\mathcal{H}(1+w_p)v^N \\ \delta p_p^C &\equiv \delta p_p^N + 3\mathcal{H}(1+w_p)c_s^2\rho_p v^N.\end{aligned}\quad (20.9)$$

In the mixed gauge (using $2\mathcal{H}' + \mathcal{H}^2 = -3w\mathcal{H}^2$)

$$\begin{aligned}\nabla^2\Psi &= \frac{3}{2}\mathcal{H}^2\Omega\delta_p^C \\ \Psi'' + (2 + 3c_s^2)\mathcal{H}\Psi' + \mathcal{H}\Phi' + 3(c_s^2 - w)\mathcal{H}^2\Phi + \frac{1}{3}\nabla^2(\Phi - \Psi) &= \frac{3}{2}\mathcal{H}^2\Omega\frac{\delta p_p^C}{\rho_p}\end{aligned}\quad (20.10)$$

and

$$\begin{aligned}(\delta_p^C)' - 3\mathcal{H}w_p\delta_p^C &= (1+w_p)\nabla^2v_p^N + 2\mathcal{H}w_p\nabla^2\Pi_p \\ (v^N)' + \mathcal{H}v^N &= \frac{\delta p_p^C}{\rho_p + p_p} + \frac{2}{3}\frac{w_p}{1+w_p}\nabla^2\Pi_p + \Phi.\end{aligned}\quad (20.11)$$

20.2 Λ + Perfect Fluid

(This section generalizes Sec. 16.5 to the presence of Λ .)

For a perfect fluid, $\Pi_p = 0$ and $\Psi = \Phi$. Equations (20.10) and (20.5d) become

$$\begin{aligned}\nabla^2\Phi &= \frac{3}{2}\mathcal{H}^2\Omega\delta_p^C \\ \Phi' + \mathcal{H}\Phi &= \frac{3}{2}\mathcal{H}^2(1+w_p)\Omega v^N \\ \Phi'' + 3(1+c_s^2)\mathcal{H}\Phi' + 3(c_s^2 - w)\mathcal{H}^2\Phi &= \frac{3}{2}\mathcal{H}^2\Omega\frac{\delta p_p^C}{\rho_p} = \frac{3}{2}\mathcal{H}^2\Omega c_s^2 [\delta_p^C - 3(1+w_p)\mathcal{S}],\end{aligned}\quad (20.12)$$

where

$$\mathcal{S} = \mathcal{H}\left(\frac{\delta p}{p'} - \frac{\delta\rho}{\rho'}\right) = \mathcal{H}\left(\frac{\delta p_p}{p_p'} - \frac{\delta\rho_p}{\rho_p'}\right) = \frac{1}{3(1+w_p)}\left(\frac{\delta\rho_p}{\rho_p} - \frac{1}{c_s^2}\frac{\delta p_p}{\rho_p}\right)\quad (20.13)$$

(in any gauge), and (20.11) becomes

$$\begin{aligned}(\delta_p^C)' - 3\mathcal{H}w_p\delta_p^C &= (1+w_p)\nabla^2v_p^N \\ (v^N)' + \mathcal{H}v^N &= \frac{\delta p_p^C}{\rho_p + p_p} + \Phi.\end{aligned}\quad (20.14)$$

In Sec. 16.5 we derived the Bardeen equation (16.42) by taking the Laplacian of the second Einstein evolution equation in the mixed gauge, here (20.12c). Here we can also proceed by inserting $\delta^C = \Omega\delta_p^C$ in (16.42) to get (**exercise**)

$$\begin{aligned}\mathcal{H}^{-2}\delta_p^{C''} + (1 - 6w_p + 3c_s^2)\mathcal{H}^{-1}\delta_p^{C'} - \frac{3}{2}[\Omega + (10 - 2\Omega)w_p - 6c_s^2 - 3\Omega w_p^2]\delta_p^C \\ = c_s^2\mathcal{H}^{-2}\nabla^2[\delta_p^C - 3(1+w_p)\mathcal{S}],\end{aligned}\quad (20.15)$$

the Λ version of the Bardeen equation.

20.3 $\Lambda + \text{Matter}$

Finally, assume the perturbed component is just matter, $\delta p_p = w_p = c_p^2 = 0$, which makes $c_s^2 = 0$ and $w = \Omega - 1$. The Einstein equations become

$$\begin{aligned}\nabla^2 \Phi &= \frac{3}{2} \mathcal{H}^2 \Omega (\delta_m^N + 3\mathcal{H}v^N) = \frac{3}{2} \mathcal{H}^2 \Omega \delta_m^C \\ \Phi' + \mathcal{H}\Phi &= \frac{3}{2} \mathcal{H}^2 \Omega v^N \\ \Phi'' + 3\mathcal{H}\Phi' + 3(1 - \Omega)\mathcal{H}^2 \Phi &= 0,\end{aligned}\tag{20.16}$$

and the fluid equations become

$$\begin{aligned}\delta_m^{N'} &= \nabla^2 v_N + 3\Phi' \\ \delta_m^{C'} &= \nabla^2 v_N \\ v_N' + \mathcal{H}v_N &= \Phi.\end{aligned}\tag{20.17}$$

The Bardeen equation (20.15) becomes

$$\mathcal{H}^{-2} \delta_m^{C''} + \mathcal{H}^{-1} \delta_m^{C'} = \frac{3}{2} \Omega \delta_m^C,\tag{20.18}$$

which can also be easily derived (**exercise**) from the fluid equations (20.17) with the help of the constraint equation (20.16a).⁴⁶ Using t , a , or $\ln a$ as time variable instead of η , (20.18) can be written

$$\begin{aligned}& H^{-2} \ddot{\delta}_m^C + 2H^{-1} \dot{\delta}_m^C \\ &= \frac{d^2 \delta_m^C}{da^2} + \left(\frac{1}{H} \frac{dH}{da} + \frac{3}{a} \right) \frac{d\delta_m^C}{da} \\ &= (\delta_m^C)^{**} + [2 + (\ln H)^*] (\delta_m^C)^* = \frac{3}{2} \Omega \delta_m^C,\end{aligned}\tag{20.19}$$

where I defined the notation $*$ $\equiv d/d \ln a$.⁴⁷ This is the same equation (“Jeans equation”) we derived in Cosmology II for Λ CDM from Newtonian perturbation theory (the first and second forms), except that the Newtonian density perturbation δ_m has been replaced with the comoving density perturbation δ_m^C and the equation is now valid for all distance scales (whereas the Newtonian result was only for subhorizon scales). The third form is Eq. (I.3.20) in [10].

⁴⁶Whereas the alternative way, which starts by taking the Laplacian of the evolution equation (20.16c), is more cumbersome. This, however, was the way we derived the general form of the Bardeen equation (16.42).

⁴⁷This is my own notation. In some modern literature, $'$ is used for this, but I am using $' \equiv d/d\eta$ (with some exceptions).

21 The Real Universe

Note: From here on, the density and velocity perturbations will be in the Newtonian gauge unless otherwise specified. To avoid a clutter of indices we drop the sub- or superscript N. We also drop the bars from the background quantities as is standard practice after the perturbation equation have been derived.⁴⁸

According to present understanding, the universe contains 5 major “fluid” components: “baryons” (including electrons), cold dark matter, photons, neutrinos, and the mysterious dark energy,

$$\rho = \rho_b + \rho_c + \rho_\gamma + \rho_\nu + \rho_{DE}. \quad (21.1)$$

We shall here assume there are no perturbations in the dark energy.

We make the approximation that the pressure of baryons and cold dark matter can be ignored.⁴⁹ Thus $p_b = p_c = 0$ (both for background and perturbations). For photons, $p_\gamma = \rho_\gamma/3$. We assume massless neutrinos, so the same relation holds for them. Thus we have

$$\begin{aligned} w_b &= w_c = c_b^2 = c_c^2 = 0 \\ w_\gamma &= w_\nu = c_\gamma^2 = c_\nu^2 = \frac{1}{3}. \end{aligned} \quad (21.2)$$

Moreover,

$$\begin{aligned} \delta p_b &= \delta p_c = 0 \\ \delta p_\gamma &= \frac{1}{3} \delta \rho_\gamma \\ \delta p_\nu &= \frac{1}{3} \delta \rho_\nu. \end{aligned} \quad (21.3)$$

Thus we have the happy situation, that for each component we have a unique relation $p_i = p_i(\rho_i)$, which moreover is very simple, either $p_i = 0$ or $p_i = \rho_i/3$. (Also, the simplest kind of dark energy, vacuum energy, has $p_{DE} = -\rho_{DE}$).

The components are, however, not all independent. Cold dark matter does not interact with the other components.⁵⁰ We can ignore the interactions of neutrinos, since we are now only interested in times much after neutrino decoupling. But the baryons and photons interact via Thomson scattering.

Our continuity equations for perturbations are thus (for scalar perturbations in the conformal-Newtonian gauge)

$$\begin{aligned} \delta'_c &= \nabla^2 v_c + 3\Psi' \\ v'_c &= -\mathcal{H}v_c + \Phi \\ \delta'_b &= \nabla^2 v_b + 3\Psi' + (\text{collision term}) \\ v'_b &= -\mathcal{H}v_b + \Phi + \text{collision term} \\ \delta'_\gamma &= \frac{4}{3}\nabla^2 v_\gamma + 4\Psi' + (\text{collision term}) \\ v'_\gamma &= \frac{1}{4}\delta_\gamma + \frac{1}{6}\nabla^2 \Pi_\gamma + \Phi + \text{collision term} \\ \delta'_\nu &= \frac{4}{3}\nabla^2 v_\nu + 4\Psi' \\ v'_\nu &= \frac{1}{4}\delta_\nu + \frac{1}{6}\nabla^2 \Pi_\nu + \Phi. \end{aligned} \quad (21.4)$$

We have put the collision terms for δ'_b and δ'_γ in parenthesis, and we drop them from here on, since it will turn out that they can be neglected, and only momentum transfer between photons and baryons is important.

⁴⁸In fact, it would not matter if they were confused with the full (background + perturbation) quantities, since after multiplying with a perturbation, the difference is of second order.

⁴⁹For small distance scales the baryon pressure is important after photon decoupling. If we were interested in small-scale structure formation, we should include it. For the cosmic microwave background it is not needed.

⁵⁰Or we can assume the CDM particles have already decoupled, like neutrinos.

In Fourier space these equations read

$$\begin{aligned}
\delta'_c &= -kv_c + 3\Psi' \\
v'_c &= -\mathcal{H}v_c + k\Phi \\
\delta'_b &= -kv_b + 3\Psi' \\
v'_b &= -\mathcal{H}v_b + k\Phi + \text{collision term} \\
\delta'_\gamma &= -\frac{4}{3}kv_\gamma + 4\Psi' \\
v'_\gamma &= \frac{1}{4}k\delta_\gamma - \frac{1}{6}k\Pi_\gamma + k\Phi + \text{collision term} \\
\delta'_\nu &= -\frac{4}{3}kv_\nu + 4\Psi' \\
v'_\nu &= \frac{1}{4}k\delta_\nu - \frac{1}{6}k\Pi_\nu + k\Phi.
\end{aligned} \tag{21.5}$$

(Remember our Fourier convention for v and Π .)

These equations are supplemented by 2 Einstein equations (there are 4 Einstein equations for perturbations, but since we are also using continuity equations, only two of them remain independent). Thus we have 10 perturbation equations, but there are 12 perturbation quantities to solve. (If we think of the Einstein equations as the equations for Φ and Ψ , the “extra” quantities lacking an equation of their own are the anisotropic stresses Π_γ and Π_ν . In the perfect fluid approximation these vanish, and the number of quantities equals the number of equations.) Also, we do not yet have the collision terms.

Thus more work is needed. This will lead us to the Boltzmann equations which employ a more detailed description of the fluid components, in terms of *distribution functions*.⁵¹ This is done in CMB Physics, from which we will pick a couple of results in the following. The collision terms are given in Sec. 28.

In synchronous gauge one can simplify the cold dark matter equations, since cold dark matter falls freely (in our approximation). Thus we can use cold dark matter particles as the freely falling observers that define the synchronous space coordinate, so that

$$v_c^Z = 0. \tag{21.6}$$

In synchronous gauge Eqs. (21.5) become thus

$$\begin{aligned}
\delta'_c &= -\frac{1}{2}h' \\
v_c &= 0 \\
\delta'_b &= -kv_b - \frac{1}{2}h' \\
v'_b &= -\mathcal{H}v_b + \text{collision term} \\
\delta'_\gamma &= -\frac{4}{3}kv_\gamma - \frac{2}{3}h' \\
v'_\gamma &= \frac{1}{4}k\delta_\gamma - \frac{1}{6}k\Pi_\gamma + \text{collision term} \\
\delta'_\nu &= -\frac{4}{3}kv_\nu - \frac{2}{3}h' \\
v'_\nu &= \frac{1}{4}k\delta_\nu - \frac{1}{6}k\Pi_\nu.
\end{aligned} \tag{21.7}$$

⁵¹The distribution functions $f_i(\eta, \vec{x}, \vec{p})$ give the distribution of particles (of fluid component i) in the six-dimensional phase space $\{(\vec{x}, \vec{p})\}$, i.e., position \vec{x} and momentum \vec{p} . In perturbation theory the distributions are assumed to be close to thermal equilibrium, so that they can be given in terms of a temperature perturbation $\Theta_i(\eta, \vec{x}, \hat{\mathbf{n}})$, which depends on the momentum direction $\hat{\mathbf{n}}$. That the temperature perturbation may be considered independent of $|\vec{p}|$ is an important result derived in CMB Physics. In Fourier space we have the temperature perturbation $\Theta_i(\eta, \vec{k}, \hat{\mathbf{n}})$, and the direction dependence can be expanded in spherical harmonics to be represented by the coefficients $\Theta_\ell^m(\eta, \vec{k})$. When the spherical harmonic expansion is done with respect to the wave vector \vec{k} direction, it turns out that for scalar perturbations only the $\Theta_\ell^0(\eta, \vec{k})$ are nonzero (vector perturbations excite $\Theta_\ell^{\pm 1}$ and tensor perturbations $\Theta_\ell^{\pm 2}$), and we write them as Θ_ℓ (with some normalization factors that we don't spell out here—leaving such detail to CMB Physics.) The lowest *multipoles* or *moments* correspond to familiar quantities: $\Theta_0^i = \frac{1}{4}\delta_i$, $\Theta_1^i = \frac{1}{3}v_i$, and $\Theta_2^i = \frac{1}{12}\Pi_i$, where $i = \gamma$ or ν .

22 Primordial Era

The initial conditions for the evolution of the large scale structure and the cosmic microwave background can be specified during the radiation-dominated epoch, sufficiently early that all scales k of interest are outside the horizon. We do not want to deal with the electron-positron annihilation or big-bang nucleosynthesis (BBN),⁵² so we limit this time period to start after BBN. We assume it is sufficiently far after whatever event created the perturbations in the first place, so that we can assume that all decaying modes have already died out. The comoving horizon at the BBN epoch is

$$\mathcal{H}^{-1} \approx \left(\frac{T}{100 \text{ keV}} \right)^{-1} \cdot 1 \text{ kpc} \quad (22.1)$$

so all cosmological scales are still well outside horizon then.

We are not just specifying “initial values” at some particular instant of time. Rather, we solve the perturbation equations for this particular epoch, and find that the solutions are characterized by quantities that remain constant for the whole epoch. Other perturbation quantities are related to these constant quantities by some powers of $k\eta$. These perturbations during this epoch we call the “primordial perturbations”.

There are different modes of primordial perturbations. In *adiabatic perturbations* all fluid perturbations are determined by the metric perturbations. However, the metric perturbations depend only on the total fluid perturbations δ , δp , v , and Π . Thus there are additional degrees of freedom in the component fluids: *entropy perturbations*. For adiabatic perturbations, all component velocity perturbations are equal, $v_i = v$, and the density perturbations are related

$$\frac{\delta_i}{1 + w_i} = \frac{\delta}{1 + w}. \quad (22.2)$$

The (relative) entropy perturbations are defined

$$S_{ij} \equiv -3\mathcal{H} \left(\frac{\delta\rho_i}{\rho'_i} - \frac{\delta\rho_j}{\rho'_j} \right) = \frac{\delta_i}{1 + w_i} - \frac{\delta_j}{1 + w_j} \quad (22.3)$$

(we assume we can ignore energy transfer between fluid components).

For N fluid components, there are $N - 1$ independent entropy perturbations. Often photons are taken as the reference fluid component for entropy perturbations, so that the independent entropy perturbations are taken to be⁵³

$$S_{i\gamma} \equiv \frac{\delta_i}{1 + w_i} - \frac{3}{4}\delta_\gamma, \quad i \neq \gamma. \quad (22.4)$$

For this section, we shall work mainly in the Newtonian gauge. Unless otherwise specified, $\delta \equiv \delta^N$ and $v = v^N$ (!!!). The relevant equations are the Einstein equations (10.18-10.21):

$$\mathcal{H}^{-1}\Psi' + \Phi + \frac{1}{3} \left(\frac{k}{\mathcal{H}} \right)^2 \Psi = -\frac{1}{2}\delta \quad (22.5)$$

$$\mathcal{H}^{-1}\Psi' + \Phi = \frac{3}{2}(1 + w) \frac{\mathcal{H}}{k} v \quad (22.6)$$

$$\mathcal{H}^{-2}\Psi'' + \mathcal{H}^{-1}(\Phi' + 2\Psi') + \left(1 + \frac{2\mathcal{H}'}{\mathcal{H}^2} \right) \Phi - \frac{1}{3} \left(\frac{k}{\mathcal{H}} \right)^2 (\Phi - \Psi) = \frac{3}{2} \frac{\delta p}{\rho} \quad (22.7)$$

$$\left(\frac{k}{\mathcal{H}} \right)^2 (\Psi - \Phi) = 3w\Pi, \quad (22.8)$$

⁵²It is really the electron-positron annihilation we want to avoid. In BBN the energy transfer from baryons to photons is small from the baryon point of view and negligible from the photon point of view.

⁵³Another possibility is to use the total radiation as the reference fluid, and we will actually do so later.

the fluid equations (21.5), and we also want to refer to the comoving curvature perturbation

$$\begin{aligned}\mathcal{R} &= -\Psi - \frac{2}{3(1+w)}\Phi - \frac{2}{3(1+w)}\mathcal{H}^{-1}\Psi' \\ \Rightarrow \quad \frac{2}{3}\mathcal{H}^{-1}\Psi' + \frac{5+3w}{3}\Psi &= -(1+w)\mathcal{R} + \frac{2}{3}(\Psi - \Phi)\end{aligned}\quad (22.9)$$

(from Eq. 16.27).

The early radiation-dominated era (the *primordial era*) has 4 properties which simplify the solution of the perturbation equations:

1. All scales of interest are outside the horizon, $k \ll \mathcal{H}$. This allows us to drop some (but not all) of the gradient terms (those with k) from the perturbation equations.⁵⁴
2. Radiation domination, $\rho_\gamma, \rho_\nu \gg \rho_b, \rho_c, \rho_{\text{DE}}$

$$\Rightarrow w = c_s^2 = \frac{1}{3} \quad \Rightarrow \quad \text{the background solution is} \quad \mathcal{H} = \frac{1}{\eta}, \quad (22.10)$$

and we can ignore the baryon, CDM, and DE contributions to the total fluid perturbation. We can also ignore the collision term in the photon velocity equation (but not in the baryon velocity equation) since the momentum the baryonic fluid can transfer to the photon fluid is negligible compared to the inertia density of the photon fluid.

3. This era is before recombination and photon decoupling, so baryons and photons are tightly coupled

$$\Rightarrow \quad v_b = v_\gamma \quad (22.11)$$

and the continuous interaction with baryons (really the electrons) keeps the photon distribution isotropic

$$\Rightarrow \quad \Pi_\gamma = 0. \quad (22.12)$$

4. $m_\nu \ll T \Rightarrow$ We can approximate neutrinos to be massless. This helps in solving the evolution of the neutrino momentum distribution. We do not discuss this here; this belongs to the course of CMB Physics, and we need to take one result, Eq. (22.34), from there.

Thus we have

$$\delta = \delta_r = f_\gamma \delta_\gamma + f_\nu \delta_\nu \quad (22.13)$$

$$\Pi = f_\nu \Pi_\nu. \quad (22.14)$$

where⁵⁵

$$f_\nu \equiv \frac{\rho_\nu}{\rho_\gamma + \rho_\nu} = \text{const} = 0.4089 \quad \text{and} \quad f_\gamma \equiv \frac{\rho_\gamma}{\rho_\gamma + \rho_\nu} = 1 - f_\nu, \quad (22.16)$$

so that $f_\nu + f_\gamma = 1$.

⁵⁴For example, if an equation has terms $\alpha + (k/\mathcal{H})^2\alpha$, we can drop the second term, since it is \ll the first. But if we have $\alpha + (k/\mathcal{H})^2\beta$, we cannot drop the second term unless we know that β is not $\gg \alpha$. If an evolution equation for α contains $\mathcal{H}^{-1}\alpha' + (k/\mathcal{H})^2\alpha$, we can drop the second term, since its contribution to the change of α over a cosmological timescale \mathcal{H}^{-1} is negligible.

⁵⁵From Cosmology I,

$$\frac{\rho_\nu}{\rho_\gamma} = N_{\text{eff}} \frac{7}{8} \left(\frac{4}{11} \right)^{4/3} = 0.6918, \quad \text{where } N_{\text{eff}} = 3.046. \quad (22.15)$$

The relevant equations thus become

$$\mathcal{H}^{-1}\delta'_\gamma = -\frac{4}{3}\left(\frac{k}{\mathcal{H}}\right)v_\gamma + 4\mathcal{H}^{-1}\Psi' \quad (22.17)$$

$$\mathcal{H}^{-1}v'_\gamma = \frac{1}{4}\left(\frac{k}{\mathcal{H}}\right)\delta_\gamma + \left(\frac{k}{\mathcal{H}}\right)\Phi \quad (22.18)$$

$$\mathcal{H}^{-1}\delta'_\nu = -\frac{4}{3}\left(\frac{k}{\mathcal{H}}\right)v_\nu + 4\mathcal{H}^{-1}\Psi' \quad (22.19)$$

$$\mathcal{H}^{-1}v'_\nu = \frac{1}{4}\left(\frac{k}{\mathcal{H}}\right)\delta_\nu - \frac{1}{6}\left(\frac{k}{\mathcal{H}}\right)\Pi_\nu + \left(\frac{k}{\mathcal{H}}\right)\Phi. \quad (22.20)$$

and

$$\mathcal{H}^{-1}\Psi' + \Phi = -\frac{1}{2}\delta \quad (22.21)$$

$$\mathcal{H}^{-1}\Psi' + \Phi = 2\frac{\mathcal{H}}{k}v \quad (22.22)$$

$$\mathcal{H}^{-2}\Psi'' + \mathcal{H}^{-1}(\Phi' + 2\Psi') - \Phi = \frac{1}{2}\delta \quad (22.23)$$

$$\left(\frac{k}{\mathcal{H}}\right)^2(\Psi - \Phi) = f_\nu\Pi_\nu \quad (22.24)$$

$$\frac{2}{3}\mathcal{H}^{-1}\Psi' + 2\Psi = -\frac{4}{3}\mathcal{R} + \frac{2}{3}(\Psi - \Phi), \quad (22.25)$$

with $\mathcal{H} = 1/\eta$.

22.1 Neutrino Adiabaticity

We consider now the simpler case, when there are no neutrino entropy perturbations,

$$S_{\nu\gamma} = 0 \quad \Rightarrow \quad \delta_\nu = \delta_\gamma = \delta \quad \text{and} \quad v_\nu = v_\gamma = v. \quad (22.26)$$

We allow for the presence of baryon and CDM entropy perturbations. However, during the radiation-dominated epoch their effect on metric perturbations is negligible. Thus the evolution of metric perturbations are as if the perturbations were adiabatic.

In the simpler case discussed in Sec. 19, where we had $\Phi = \Psi$, we found that the growing adiabatic mode had $\Phi = \Psi = \text{const}$. Guided by that, we now try the ansatz

$$\Phi = \text{const} \quad \text{and} \quad \Psi = \text{const}, \quad (22.27)$$

with Φ and Ψ of the same order of magnitude, and check that it is a solution. The Einstein and \mathcal{R} equations are satisfied with

$$\delta_\nu = \delta_\gamma = \delta = -2\Phi = \text{const} \quad (22.28)$$

$$v_\nu = v_\gamma = v = \frac{1}{2}\left(\frac{k}{\mathcal{H}}\right)\Phi = -\frac{1}{4}\left(\frac{k}{\mathcal{H}}\right)\delta = \frac{1}{2}k\eta\Phi \quad (22.29)$$

$$\Pi_\nu = \frac{1}{f_\nu}\left(\frac{k}{\mathcal{H}}\right)^2(\Psi - \Phi) \quad (22.30)$$

$$\mathcal{R} = -(\Psi + \frac{1}{2}\Phi) = \text{const} \quad (22.31)$$

In the δ'_γ and δ'_ν equations the (k/\mathcal{H}) terms become $(1/3)(k/\mathcal{H})^2\delta \ll \delta$ and can be ignored, and we see that the equations are satisfied (the implied change in δ_i over a Hubble time is negligible).

Using $\mathcal{H} = 1/\eta$ the velocity equations become

$$\begin{aligned}\eta v'_\gamma &= \frac{1}{4}k\eta\delta_\gamma + k\eta\Phi \\ \eta v'_\nu &= \frac{1}{4}k\eta\delta_\nu - \frac{1}{6}k\eta\Pi_\nu + k\eta\Phi.\end{aligned}$$

Here the Π_ν term can be ignored since Π_ν is suppressed by $(k/\mathcal{H})^2$ compared to Ψ and Φ , and we then see that these equations are also satisfied.

We are still missing a piece of information that would tell us what $\Phi - \Psi$, or, in other words, what Π_ν is. The neutrino anisotropy Π_ν depends on the neutrino momentum distribution. Before neutrino decoupling, interactions kept $\Pi_\nu = 0$. After neutrinos decoupled, neutrinos have been “freely streaming”, i.e., moving without collisions through the perturbed universe. In CMB Physics we derive a so-called Boltzmann hierarchy of equations

$$\Theta'_\ell = \frac{\ell}{2\ell+1}k\Theta_{\ell-1} - \frac{\ell+1}{2\ell+1}k\Theta_{\ell+1}, \quad (22.32)$$

which relates the evolution of the different moments of the momentum distribution of freely streaming particles to each other. With $\Theta'_0 = \frac{1}{4}\delta_\nu$, $\Theta'_1 = \frac{1}{3}v_\nu$, and $\Theta'_2 = \frac{1}{12}\Pi_\nu$, these give

$$\begin{aligned}\delta'_\nu &= -\frac{4}{3}kv_\nu \\ v'_\nu &= \frac{1}{4}k\delta_\nu - \frac{1}{6}k\Pi_\nu \\ \Pi'_\nu &= \frac{8}{5}kv_\nu - \frac{36}{5}k\Theta'_3 \\ (\Theta'_3)' &= \frac{1}{28}k\Pi_\nu - \frac{4}{7}k\Theta'_4.\end{aligned} \quad (22.33)$$

The first two equations are familiar, except one still has to add the effect of gravity (the metric perturbations Ψ' , Φ). Metric perturbations do not affect the higher equations in the hierarchy. The moments Θ_ℓ are gauge invariant for $\ell \geq 2$.

Before decoupling all the higher moments, Θ_ℓ for $\ell \geq 2$, vanish. This leads to a “decreasing hierarchy”, where the lower moments seed the higher moments, so that a higher moment Θ_ℓ is one order higher (smaller) in time than the previous moment $\Theta_{\ell-1}$.⁵⁶ Thus for early times (superhorizon scales) we can truncate the hierarchy by ignoring moments higher than some ℓ depending on how accurate we want to be (to how high order in η we want to have the results.) Truncating at $\ell = 2$, the “second moment” Π_ν depends then only on the “first moment” v_ν , and the relevant equation is

$$\mathcal{H}^{-1}\Pi'_\nu = \frac{8}{5}\left(\frac{k}{\mathcal{H}}\right)v_\nu. \quad (22.34)$$

This finally allows us to solve (**exercise**):

$$\Psi = (1 + \frac{2}{5}f_\nu)\Phi \approx 1.164\Phi \quad (22.35)$$

$$\Pi_\nu = \frac{2}{5}(k\eta)^2\Phi \quad (22.36)$$

$$\mathcal{R} = -\frac{3}{2}\left(1 + \frac{4}{15}f_\nu\right)\Phi = \text{const} \quad (22.37)$$

$$\Phi = -\frac{2}{3}\frac{1}{1 + \frac{4}{15}f_\nu}\mathcal{R} \approx -0.6011\mathcal{R} \quad (22.38)$$

$$\Psi = -\frac{2}{3}\frac{1 + \frac{2}{5}f_\nu}{1 + \frac{4}{15}f_\nu}\mathcal{R} \approx -0.6994\mathcal{R}. \quad (22.39)$$

(We see that the perfect fluid approximation, which gave $\Psi = \Phi = -\frac{2}{3}\mathcal{R}$, led to a 10% error in Φ and to a 5% error in Ψ .)

⁵⁶CMB Physics 2004, p. I1.8

22.2 Matter

During the early radiation-dominated era, the metric and the radiation perturbations do not care about matter perturbations, but matter perturbations will become important later, and therefore we are interested in their “primordial” behavior in the radiation-dominated era. The continuity equations for baryons and CDM are

$$\begin{aligned}
\mathcal{H}^{-1}\delta'_c + \left(\frac{k}{\mathcal{H}}\right)v_c - 3\mathcal{H}^{-1}\Psi' &= 0 \\
\mathcal{H}^{-1}\delta'_b + \left(\frac{k}{\mathcal{H}}\right)v_b - 3\mathcal{H}^{-1}\Psi' &= 0 \\
\mathcal{H}^{-1}v'_c + v_c - \left(\frac{k}{\mathcal{H}}\right)\Phi &= 0 \\
\mathcal{H}^{-1}v'_b + v_b - \left(\frac{k}{\mathcal{H}}\right)\Phi &= an_e\sigma_T\frac{4\rho_\gamma}{3\rho_b}(v_\gamma - v_b),
\end{aligned} \tag{22.40}$$

where the collision term in the last equation is derived in CMB Physics, σ_T is the Thomson cross section for photon-electron scattering, and n_e is the free electron number density (there is more discussion about the collision terms in Sec. 28). Well before photon decoupling, $an_e\sigma_T(4\rho_\gamma)/(3\rho_b)$ is very large, and the collision term forces $v_b = v_\gamma$ (baryons are tightly coupled to photons).

For the above photon+neutrino adiabatic growing mode solution, these become

$$\begin{aligned}
\eta\delta'_c + k\eta v_c &= 0 \\
\eta\delta'_b + k\eta v_b &= 0 \\
\eta v'_c + v_c - k\eta\Phi &= 0 \\
\eta v'_b + v_b - k\eta\Phi &= an_e\sigma_T\frac{4\rho_\gamma}{3\rho_b}(v - v_b).
\end{aligned} \tag{22.41}$$

22.2.1 The Completely Adiabatic Solution

One solution for Eq. (22.41) is the completely adiabatic solution:

$$\begin{aligned}
\delta_c = \delta_b = \frac{3}{4}\delta = -\frac{3}{2}\Phi &= \text{const.} \\
v_c = v_b = v = \frac{1}{2}k\eta\Phi.
\end{aligned} \tag{22.42}$$

To check this, substitute Eq. (22.42) into Eq. (22.41). This gives

$$0 + \frac{1}{2}(k\eta)^2\Phi = 0$$

for the δ equations, and

$$\frac{1}{2}k\eta\Phi + \frac{1}{2}k\eta\Phi - k\eta\Phi = 0,$$

for the v equations, so the latter equations are indeed satisfied. The δ equations are satisfied to accuracy $(k\eta)^2 \ll 1$, i.e., in a Hubble time, δ_i will deviate from its initial value $-\frac{3}{2}\Phi$ by about $-\frac{1}{2}(k\eta)^2\Phi$, a negligible change.

22.2.2 Baryon and CDM Entropy Perturbations

There are three independent entropy perturbations: the neutrino, baryon, and CDM entropy perturbations,

$$S_{\nu\gamma} \equiv \frac{3}{4}(\delta_\nu - \delta_\gamma) \quad S_{b\gamma} \equiv \delta_b - \frac{3}{4}\delta_\gamma \quad S_{c\gamma} \equiv \delta_c - \frac{3}{4}\delta_\gamma. \tag{22.43}$$

Their evolution equations are

$$S'_{\nu\gamma} = -k(v_\nu - v_\gamma) \quad S'_{b\gamma} = -k(v_b - v_\gamma) \quad S'_{c\gamma} = -k(v_c - v_\gamma). \quad (22.44)$$

The relative entropy perturbation stays constant unless there is a corresponding relative velocity perturbation.

Entropy perturbations also tend to stay constant at superhorizon scales⁵⁷, as

$$\mathcal{H}^{-1}S'_{i\gamma} = -\left(\frac{k}{\mathcal{H}}\right)(v_i - v_\gamma). \quad (22.45)$$

Assume now the neutrino-adiabatic growing mode solution of Sec. 22.1. This assumes $S_{\nu\gamma} = 0$, but we may still have baryon and CDM entropy perturbations.

The baryon and neutrino density perturbations are

$$\delta_b = \frac{3}{4}\delta + S_{b\gamma} \quad \text{and} \quad \delta_c = \frac{3}{4}\delta + S_{c\gamma}. \quad (22.46)$$

Since baryons are tightly coupled to photons, $v_b = v_\gamma = v$, we have

$$S'_{b\gamma} = 0 \quad \Rightarrow \quad S_{b\gamma} = \text{const.} \quad \Rightarrow \quad \delta_b = \text{const.} \quad (22.47)$$

For CDM we do not have this constraint. Write

$$v_{\text{rel}} \equiv v_c - v \quad \Rightarrow \quad v_c = \frac{1}{2}k\eta\Phi + v_{\text{rel}}. \quad (22.48)$$

Eq. (22.41c) becomes

$$\begin{aligned} \eta\frac{1}{2}k\Phi + \eta v'_{\text{rel}} + \frac{1}{2}k\eta\Phi + v_{\text{rel}} - k\eta\Phi &= 0 \\ \Rightarrow \quad \eta v'_{\text{rel}} &= -v_{\text{rel}} \quad \Rightarrow \quad v_{\text{rel}} \propto \eta^{-1}. \end{aligned}$$

Thus we have

$$v_c = \frac{1}{2}(k\eta)\Phi + C\eta^{-1}. \quad (22.49)$$

from which we identify a growing mode and a decaying mode. As time goes on, the decaying mode decays away, and $v_c \rightarrow v$. Ignoring the decaying mode, we have

$$v_c = v \quad \Rightarrow \quad S_{c\gamma} = \text{const.} \quad \Rightarrow \quad \delta_c = \text{const.} \quad (22.50)$$

Thus (assuming neutrino adiabaticity), the “initial conditions” at the early radiation-dominated epoch can be specified by giving three constants for each Fourier mode \vec{k} : $\Phi_k(\text{rad})$, $S_{c\gamma\vec{k}}(\text{rad})$, and $S_{b\gamma\vec{k}}(\text{rad})$. The general perturbation is a superposition of three modes, where two of these constants are zero:

$$\begin{aligned} (\Phi, S_{c\gamma}, S_{b\gamma}) &= (\Phi, 0, 0) && \text{adiabatic mode (ADI)} \\ (\Phi, S_{c\gamma}, S_{b\gamma}) &= (0, S_{c\gamma}, 0) && \text{CDM density isocurvature mode (CDI)} \\ (\Phi, S_{c\gamma}, S_{b\gamma}) &= (0, 0, S_{b\gamma}) && \text{baryon density isocurvature mode (BDI)}. \end{aligned} \quad (22.51)$$

In these two *isocurvature* modes there are (initially) no metric perturbations, since $\mathcal{R} = \Psi = \Phi = 0$ and no radiation density or velocity perturbations: $\delta \approx \delta_r = \delta_\nu = \delta_\gamma = 0$ and $v \approx v_r = v_\nu = v_\gamma = 0$; but there are nonzero CDM or baryon density perturbations $\delta_c = S_{c\gamma}$, $\delta_b = S_{b\gamma}$. In the following we shall use \mathcal{R} instead of Φ (see Eq. 22.37) as the first initial value constant (since it is better in staying constant also later).

⁵⁷This property is not as general as the constancy of \mathcal{R} at superhorizon scales: The result (22.45) relies on two assumptions: 1) no interaction terms in the component energy continuity equations, and 2) the component fluids have a unique relation $p_i = p_i(\rho_i)$. Note also that this does not hold for the “total entropy perturbation” \mathcal{S} .

22.3 Neutrino perturbations

Bucher, Moodley, and Turok [6] identified 5 different possible modes of primordial perturbations: Adiabatic growing mode (ADI), Baryon density isocurvature mode (BDI), CDM density isocurvature mode (CDI), Neutrino density isocurvature mode (NDI), and Neutrino velocity isocurvature mode (NVI). The three first ones we discussed above; now we shall discuss the NDI mode. The NVI mode, where there is an initial neutrino velocity perturbation, is difficult to motivate, so we do not discuss that.

22.3.1 Neutrinos in the early universe

Before neutrino decoupling ($T \geq 1 \text{ MeV}$), neutrinos were interacting with electrons, positrons, and nucleons, which were interacting with photons. These interactions kept these particles in thermal and kinetic equilibrium, eliminating any neutrino anisotropic pressure and forcing $v_\nu = v_\gamma$. They did not necessarily force $\delta_\nu = \delta_\gamma$. However, they forced $T_\nu = T_\gamma$. The other degrees of freedom for neutrino density in thermal equilibrium are given by the neutrino chemical potentials μ_{ν_e} , μ_{ν_μ} , and μ_{ν_τ} . Thus neutrino entropy perturbations require nonzero neutrino (inhomogeneous) chemical potentials.

The usual assumption is that these chemical potentials are small, just like for electrons and baryons, $\mu_\nu/T_\nu = \mathcal{O}(10^{-9})$, so that any neutrino entropy perturbations would be negligible, but we do not know this, observational upper limits are much weaker.

Thus we could have a primordial NDI mode ($\delta_\nu \neq \delta_\gamma$ initially). To have a primordial NVI mode ($v_\nu \neq v_\gamma$ initially), would require that it was generated after neutrino decoupling.

22.3.2 Primordial neutrino density isocurvature mode

We now find the primordial neutrino density isocurvature mode during the early radiation dominated era. The basic equations are Eqs. (22.13-22.25) and (22.34) with $\mathcal{H} = 1/\eta$.

We used earlier $S_{i\gamma} \equiv \delta_i/(1 + w_i) - \frac{3}{4}\delta_\gamma$ as measures of the different isocurvature modes, which is common in literature. Lyth and Liddle [7] argue that it would be better to use the total radiation perturbation δ_r here instead of δ_γ . This makes no difference when we have neutrino adiabaticity, since then $\delta_\gamma = \delta_\nu = \delta_r$. But in case of the neutrino isocurvature mode these differ. From (22.13), the quantities $S_{\nu\gamma}$ and $S_{\nu r} \equiv \frac{3}{4}(\delta_\nu - \delta_r)$ are related by

$$S_{\nu r} = (1 - f_\nu)S_{\nu\gamma}. \quad (22.52)$$

Lyth and Liddle thus want to define the neutrino isocurvature density mode by the initial condition $\mathcal{R} = S_{br} = S_{cr} = 0$ instead of $\mathcal{R} = S_{b\gamma} = S_{c\gamma} = 0$, arguing that this is the form in which this mode is more likely to be generated. This distinction will matter only when we get to discuss the matter perturbations in this mode; until that the discussion below applies to both cases.

We search for a solution for the primordial perturbation with $\mathcal{R} = 0$ but $S_{\nu r} \neq 0$. Thus we expect nonzero δ_ν and δ_γ . The difference from the previous isocurvature modes is that now the entropy perturbation is between the dominating fluid quantities. This means that it is likely to affect the metric. Although we require $\mathcal{R} = 0$, Φ and Ψ may be nonzero. From (22.18) and (22.20) we expect nonzero v_γ and v_ν , suppressed by $k\eta \ll 1$, and from (22.34) a nonzero Π_ν , suppressed by $(k\eta)^2$, which by (22.24) implies a nonzero $\Psi - \Phi$. Note that because of the $(k\eta)^2$ in Eq. (22.24) there is no superhorizon suppression in $\Psi - \Phi$, although there is in Π_ν , and therefore we cannot ignore this effect of Π_ν . Thus at least one of Φ and Ψ must be nonzero, and there is indeed a metric perturbation in this isocurvature mode.

Guided by the previous cases, we look for a solution with δ_γ , δ_ν , Φ , and Ψ constant, now with $\mathcal{R} = 0$. The terms with derivatives in Eqs. (22.21-22.25) vanish, and we get immediately

$$2\Psi = \frac{2}{3}(\Psi - \Phi) \Rightarrow \Psi = -\frac{1}{2}\Phi \quad (22.53)$$

$$\Phi = -\frac{1}{2}\delta \Rightarrow \delta = -2\Phi \quad (22.54)$$

$$v = \frac{1}{2}k\eta\Phi = -\frac{1}{4}k\eta\delta \quad (22.55)$$

$$f_\nu\Pi_\nu = (k\eta)^2(\Psi - \Phi) = -\frac{3}{2}(k\eta)^2\Phi = \frac{3}{4}(k\eta)^2\delta. \quad (22.56)$$

In (22.20),

$$\eta v'_\nu = \frac{1}{4}k\eta\delta_\nu - \frac{1}{6}k\eta\Pi_\nu + k\eta\Phi, \quad (22.57)$$

the Π_ν term is suppressed by $(k\eta)^2$ compared to the Φ term, so we drop it, and solve, with initial condition $v_\nu = 0$, that

$$v_\nu = \frac{1}{4}k\eta\delta_\nu + k\eta\Phi = \frac{1}{4}k\eta\delta_\nu - \frac{1}{2}k\eta\delta. \quad (22.58)$$

Likewise we get

$$v_\gamma = \frac{1}{4}k\eta\delta_\gamma + k\eta\Phi = \frac{1}{4}k\eta\delta_\gamma - \frac{1}{2}k\eta\delta. \quad (22.59)$$

(Eqs. (22.17) and (22.19) would give time derivatives to δ_γ and δ_ν , that are suppressed by $(k\eta)^2$ so that we can ignore them. Same applies to $S_{\nu\gamma}$, so $S_{\nu\gamma} = \text{const}$, although $v_\nu \neq v_\gamma$.)

Now Eq. (22.34)

$$\eta\Pi'_\nu = \frac{8}{5}k\eta v_\nu \quad \text{becomes} \quad \eta\Pi'_\nu = \frac{2}{5}(k\eta)^2\delta_\nu - \frac{4}{5}(k\eta)^2\delta \quad (22.60)$$

whose solution (with initial condition $\Pi_\nu = 0$) is

$$\Pi_\nu = (k\eta)^2 \left(\frac{1}{5}\delta_\nu - \frac{2}{5}\delta \right). \quad (22.61)$$

Combining the two Eqs., (22.56) and (22.61), relating the neutrino anisotropic pressure Π_ν to density perturbations, one coming from Einstein equations (gravity) and the other from the Boltzmann hierarchy (kinematics), we have

$$\frac{3}{4}\delta = f_\nu \left(\frac{1}{5}\delta_\nu - \frac{2}{5}\delta \right), \quad (22.62)$$

from which we solve (**exercise**)

$$\delta_\nu = \frac{15 + 8f_\nu}{4f_\nu} \delta \quad (22.63)$$

$$S_{\nu r} \equiv \frac{3}{4}(\delta_\nu - \delta) = \frac{45 + 12f_\nu}{16f_\nu} \delta \quad (22.64)$$

$$\delta = \frac{16f_\nu}{45 + 12f_\nu} S_{\nu r} = \frac{4}{3} \frac{4f_\nu}{15 + 4f_\nu} S_{\nu r} \approx 0.1311 S_{\nu r} \quad (22.65)$$

$$\delta_\nu = \frac{60 + 32f_\nu}{45 + 12f_\nu} S_{\nu r} \approx 1.464 S_{\nu r} \quad (22.66)$$

$$\begin{aligned} \delta_\gamma &= \delta_\nu - \frac{4}{3} \frac{S_{\nu r}}{1 - f_\nu} = \left(\frac{60 + 32f_\nu}{45 + 12f_\nu} - \frac{4}{3 - 3f_\nu} \right) S_{\nu r} \\ &= \left[\frac{16}{3} \frac{f_\nu}{4f_\nu + 15} - \frac{4f_\nu}{3(1 - f_\nu)} \right] S_{\nu r} \approx -0.7913 S_{\nu r} \end{aligned} \quad (22.67)$$

$$\Phi = -\frac{1}{2}\delta = -\frac{8f_\nu}{45 + 12f_\nu} S_{\nu r} \quad (22.68)$$

$$\Psi = \frac{1}{4}\delta = \frac{4f_\nu}{45 + 12f_\nu} S_{\nu r} = \frac{4}{3} \frac{f_\nu}{15 + 4f_\nu} S_{\nu r} \quad (22.69)$$

$$v_\nu = \frac{5}{15 + 4f_\nu} k\eta S_{\nu r} \quad (22.70)$$

$$v_\gamma = v_\nu - \frac{1}{3} k\eta S_{\nu\gamma} = -\frac{19f_\nu}{3(1 - f_\nu)(15 + 4f_\nu)} k\eta S_{\nu r} \quad (22.71)$$

$$\Pi_\nu = \frac{4}{15 + 4f_\nu} (k\eta)^2 S_{\nu r}. \quad (22.72)$$

Thus ρ_ν and ρ_γ have perturbations that are in the opposite direction to each other, but do not cancel, leaving a total δ in the same direction as δ_ν .⁵⁸

In the early radiation-dominated era, the CDM and baryon perturbations have no effect on the above solution, and since it shares the results $\Psi' = \Phi' = 0$ and $v = \frac{1}{2}k\eta\Phi$ with the neutrino adiabatic case, we can repeat the arguments in Sec. 22.2.2 so that the CDM and baryon perturbations follow the same equations even if the neutrino adiabaticity condition is relaxed by allowing the presence of the NDI mode, except that now $v_b = v_\gamma \neq v_\nu$ and $\neq v$, because of the tight coupling between baryons and photons (but this difference will not affect the entropy perturbations, since the effect of velocity differences is suppressed by k/\mathcal{H}). For CDM, (22.49) and (22.50) hold, so $v_c = v$.

In the NDI mode, the primordial S_{br} and S_{cr} are zero, so that

$$\delta_b = \delta_c = \frac{3}{4}\delta_r = \frac{3}{4}\delta = \frac{4f_\nu}{15 + 4f_\nu} S_{\nu r} \quad (22.73)$$

during the primordial epoch.

⁵⁸Eq. (22.67) disagrees with Eq. (12.26) of Lyth&Liddle[7]. Eq. (22.69) agrees with Eq. (12.22) of [7], except for the sign. In [7] the sign of δ appears to be the same as δ_γ , opposite to δ_ν and $S_{\nu r}$. Note that in [7] Ψ and Φ have interchanged meaning but have the same sign convention (see their Eq. (8.32) for the metric).

22.4 Real-Universe Initial Conditions as Expansion in Conformal Time

To obtain sufficiently accurate initial conditions for a numerical solution of the full linear evolution of perturbations in the real universe, we want to do a similar thing as we did for the simplified universe in Sec. 19.5. The background equations are

$$y = 2x + x^2, \quad \mathcal{H} = \frac{1}{x} \frac{1+x}{1+\frac{1}{2}x}, \quad w = \frac{1}{3(1+y)} \quad (22.74)$$

Now we use the Newtonian gauge equations. We expand

$$\begin{aligned} \Phi &= A + Bx + Cx^2 \\ \Psi &= D + Ex + Fx^2 \\ \delta_i^N &= G_i + H_i x + J_i x^2 \\ v_i^N &= L_i x + M_i x^2 + N_i x^3 \\ \Pi_\nu &= Qx^2 + Sx^3 + Tx^4, \end{aligned} \quad (22.75)$$

except that for baryons we set $v_b = v_\gamma$, so that $v_b^N = L_\gamma x + M_\gamma x^2 + N_\gamma x^3$.

From (22.38) and (22.68),

$$A = -\frac{2}{3} \frac{1}{1 + \frac{4}{15}f_\nu} \left[\mathcal{R}_{\vec{k}}(\text{rad}) + \frac{4}{15} f_\nu S_{\nu r, \vec{k}}(\text{rad}) \right] \quad (22.76)$$

(here “rad” refers to the constant primordial value during the early radiation-dominated epoch). From (22.33cd), $\Pi'_\nu = \frac{8}{5} k v_\nu^N - \frac{36}{5} k \Theta_3^\nu$ and $(\Theta_3^\nu)' = \frac{1}{28} k \Pi_\nu + \mathcal{O}(x^4)$, and calling $\Theta_3^\nu = Ux^3 + \mathcal{O}(x^4)$, one gets

$$U = \frac{1}{84} k \eta_3 Q, \quad Q = \frac{4}{15} k \eta_3 L_\nu, \quad S = \frac{8}{15} k \eta_3 M_\nu, \quad T = \frac{2}{5} k \eta_3 N_\nu - \frac{9}{5} k \eta_3 U. \quad (22.77)$$

The total densities and velocities are given by

$$\begin{aligned} \delta &= \frac{1}{1+y} (f_\gamma \delta_\gamma + f_\nu \delta_\nu) + \frac{y}{1+y} (f_b \delta_b + f_c \delta_c) \\ v &= \frac{4}{4+3y} (f_\gamma v_\gamma + f_\nu v_\nu) + \frac{y}{1+y} (f_b v_b + f_c v_c). \end{aligned} \quad (22.78)$$

The equations to solve to $\mathcal{O}(x)$ are the fluid density equations

$$\begin{aligned} H_b + 2J_b x &= -k\eta_3 L_\gamma x + 3E + 6Fx \\ H_c + 2J_c x &= -k\eta_3 L_c x + 3E + 6Fx \\ H_\gamma + 2J_\gamma x &= -\frac{4}{3} k\eta_3 L_\gamma x + 4E + 8Fx \\ H_\nu + 2J_\nu x &= -\frac{4}{3} k\eta_3 L_\nu x + 4E + 8Fx \end{aligned} \quad (22.79)$$

and the equations to solve to $\mathcal{O}(x^2)$ are the fluid velocity equations and the Einstein constraint equations

$$\begin{aligned} L_c + 2M_c x + 3N_c x^2 &= -\mathcal{H} \eta_3 v_c^N + k\eta_3 \Phi \\ (1+R)(L_\gamma + 2M_\gamma x + 3N_\gamma x^2) &= \frac{1}{4} k\eta_3 \delta_\gamma^N - R\mathcal{H} \eta_3 v_\gamma^N + (1+R)k\eta_3 \Phi \\ L_\nu c + 2M_\nu x + 3N_\nu x^2 &= \frac{1}{4} k\eta_3 \delta_\nu^N - \frac{1}{6} k\eta_3 Qx^2 + k\eta_3 \Phi \\ k^2(\Phi - \Psi) &= 3\mathcal{H}^2 w f_\nu \Pi_\nu \\ x^2 k^2 \Psi &= -x^2 \frac{3}{2} \mathcal{H}^2 (\delta^N + 3(1+w)(\mathcal{H}/k)v^N). \end{aligned} \quad (22.80)$$

For comparison to [9], we want δ_i and v_i also in the synchronous gauge,

$$\begin{aligned} \delta_i^Z &= G_i^Z + H_i^Z x + J_i^Z x^2 \\ v_i^Z &= L_i^Z x + M_i^Z x^2 + N_i^Z x^3. \end{aligned} \quad (22.81)$$

for which we solve to $\mathcal{O}(x^2)$ the gauge transformation equations

$$\begin{aligned} \frac{1}{x} v_i^Z &= \frac{1}{x} (v_i^N - v_c^N) \\ \delta_i^Z &= \delta_i^N + 3(1+w_i) \frac{\mathcal{H}}{k} (v^N - v^Z) \end{aligned} \quad (22.82)$$

(where we used $v_c^Z = 0$).

The above set seemed too complicated to solve by hand, so I wrote the following Mathematica script to solve them:

```
Clear["Global`,*"]
y = 2x + x^2
hub = (1/x)((1+x)/(1+(1/2)x))
w = 1/(3(1+y))
rcr = 1
semr = 0
senr = 0
a = (-2/3)(1/(1+(4/15)fn))(rcr+(4/15)fn senr)
gb = gm
gc = gm
fc = 1 - fb
fg = 1 - fn
rbg = (3/4)(fb/fg)y
ph = a + b x + c x^2
ps = d + ee x + f x^2
db = gb + hb x + jb x^2
dc = gc + hc x + jc x^2
dg = gg + hg x + jg x^2
dn = gn + hn x + jn x^2
dbz = gbz + hbz x + jbz x^2
dcz = gcz + hcz x + jcz x^2
dgz = ggz + hgz x + jgz x^2
dnz = gnz + hnz x + jnz x^2
vb = lg x + mg x^2 + ng x^3
vc = lc x + mc x^2 + nc x^3
vg = lg x + mg x^2 + ng x^3
vn = ln x + mn x^2 + nn x^3
vgz = lgz x + mgz x^2 + ngz x^3
vnz = lnz x + mnz x^2 + nnz x^3
u = (1/84) k q
q = (4/5)k ln
s = (8/15)k mn
t = (2/5)k nn - (9/5)k u
pin = q x^2 + s x^3 + t x^4
delta = (1/(1+y))(fg dg + fn dn) + (y/(1+y))(fb db + fc dc)
vee = (4/(4+3y))(fg vg + fn vn) + (3y/(4+3y))(fb vb + fc vc)
veez = (4/(4+3y))(fg vgz + fn vnz) + (3y/(4+3y))fb vgz
Solve[{
Series[hb + 2jb x, {x,0,1}] == Series[-k lg x + 3 ee + 6f x, {x,0,1}],
Series[hc + 2jc x, {x,0,1}] == Series[-k lc x + 3 ee + 6f x, {x,0,1}],
Series[hg + 2jg x, {x,0,1}] == Series[-(4/3)k lg x + 4 ee + 8f x, {x,0,1}],
Series[hn + 2jn x, {x,0,1}] == Series[-(4/3)k ln x + 4 ee + 8f x, {x,0,1}],
Series[lc + 2mc x + 3nc x^2, {x,0,2}] == Series[-hub vc + k ph, {x,0,2}],
Series[(1+rbg)(lg + 2mg x + 3ng x^2), {x,0,2}] == Series[(1/4)k dg - rbg hub vg + (1+rbg)k ph, {x,0,2}],
Series[ln + 2mn x + 3nn x^2, {x,0,2}] == Series[(1/4)k dn - (1/6)k q x^2 + k ph, {x,0,2}],
semr == gm - (3/4)(fg gg + fn gn),
senr == (3/4)(1-fn)(gn-gg),
Series[k^2(ps-ph), {x,0,2}] == Series[3hub^2 w fn pin, {x,0,2}],
Series[x^2 k^2 ps, {x,0,2}] == Series[-x^2(3/2)hub^2(delta+3(1+w)(hub/k)vee), {x,0,2}],
Series[vgz/x, {x,0,2}] == Series[(vg - vc)/x, {x,0,2}],
Series[vnz/x, {x,0,2}] == Series[(vn - vc)/x, {x,0,2}],
Series[dbz, {x,0,2}] == Series[db + 3(hub/k)(vee-veez), {x,0,2}],
Series[dcz, {x,0,2}] == Series[dc + 3(hub/k)(vee-veez), {x,0,2}],
Series[dgz, {x,0,2}] == Series[dg + 4(hub/k)(vee-veez), {x,0,2}],
Series[dnz, {x,0,2}] == Series[dn + 4(hub/k)(vee-veez), {x,0,2}]
},
{b,c,d,ee,f,gm,gg,gn,hb,hc,hg,hn,jb,jc,jg,jn,lc,lg,ln,mc,mg,mn,nc,ng,nn,gbz,gcz,ggz,gnz,
```

hbz,hcz,hgz,hnz,jbz,jcz,jgz,jnz,lgz,lnz,mgz,mnz,ngz,nnz}]

Here \mathbf{k} stands for $k\eta_3$ and \mathbf{hub} for $\mathcal{H}\eta_3$. The initial conditions are given by setting the values of $\mathbf{rcr} \equiv \mathcal{R}_{\vec{k}}(\text{rad})$, $\mathbf{senr} \equiv S_{\nu r, \vec{k}}(\text{rad})$, and $\mathbf{semr} \equiv S_{mr, \vec{k}}(\text{rad}) \equiv f_b S_{br, \vec{k}}(\text{rad}) + f_c S_{cr, \vec{k}}(\text{rad})$. An initial relative entropy perturbation between baryons and CDM, $S_{br, \vec{k}}(\text{rad}) - S_{cr, \vec{k}}(\text{rad}) = G_b - G_c$ has no dynamical effect (it affects only the values of G_b and G_c), so for simplicity we set it to zero, $G_b = G_c$.

Adiabatic mode. We get the adiabatic mode by setting $\mathcal{R}_{\vec{k}}(\text{rad}) = 1$, $S_{mr, \vec{k}}(\text{rad}) = S_{\nu r, \vec{k}}(\text{rad}) = 0$. This gives the results

$$\begin{aligned}
\Phi &= -\frac{2}{3} \frac{1}{1 + \frac{4}{15} f_\nu} - \frac{25}{8} \frac{8f_\nu - 3}{(15 + 2f_\nu)(15 + 4f_\nu)} \omega \eta + \mathcal{O}(\eta^2) \\
\Psi &= -\frac{2}{3} \frac{1 + \frac{2}{5} f_\nu}{1 + \frac{4}{15} f_\nu} + \frac{5}{8} \frac{15 + 16f_\nu}{(15 + 2f_\nu)(15 + 4f_\nu)} \omega \eta + \mathcal{O}(\eta^2) \\
\delta_b^N = \delta_c^N &= \frac{1}{1 + \frac{4}{15} f_\nu} + \frac{1}{8} \frac{1 + \frac{16}{15} f_\nu}{(1 + \frac{2}{15} f_\nu)(1 + \frac{4}{15} f_\nu)} \omega \eta + \mathcal{O}(\eta^2) \\
\delta_\gamma^N = \delta_\nu^N &= \frac{4}{3} \frac{1}{1 + \frac{4}{15} f_\nu} + \frac{1}{6} \frac{1 + \frac{16}{15} f_\nu}{(1 + \frac{2}{15} f_\nu)(1 + \frac{4}{15} f_\nu)} \omega \eta + \mathcal{O}(\eta^2) \\
v_c^N &= -\frac{1}{3} \frac{1}{1 + \frac{4}{15} f_\nu} k\eta + \frac{1}{8} \frac{5 - 4f_\nu}{(15 + 2f_\nu)(1 + \frac{4}{15} f_\nu)} \omega k \eta^2 + \mathcal{O}(\eta^3) \\
v_b^N = v_\gamma^N &= v_c^N + \mathcal{O}(\eta^3) \\
v_\nu^N &= v_c^N + \mathcal{O}(\eta^3) \\
\delta_b^Z = \delta_c^Z &= \frac{1}{4} (k\eta)^2 + \mathcal{O}(\eta^3) \\
\delta_\gamma^Z = \delta_\nu^Z &= \frac{1}{3} (k\eta)^2 + \mathcal{O}(\eta^3) \\
v_c^Z &= 0 \\
v_b^Z = v_\gamma^Z &= \frac{1}{36} (k\eta)^3 + \mathcal{O}(\eta^4) \\
v_\nu^Z &= \frac{1}{36} \frac{23 + 4f_\nu}{15 + 4f_\nu} (k\eta)^3 + \mathcal{O}(\eta^4) \\
\Pi_\nu &= -\frac{4}{15} \frac{1}{1 + \frac{4}{15} f_\nu} (k\eta)^2 + \frac{1}{45} \frac{1 - \frac{4}{5} f_\nu}{(1 + \frac{4}{15} f_\nu)(1 + \frac{2}{15} f_\nu)} \omega k^2 \eta^3, \tag{22.83}
\end{aligned}$$

where (see 19.84)

$$\omega \equiv \frac{\Omega_m}{\sqrt{\Omega_r}} H_0 \quad \Rightarrow \quad \eta_3 = \frac{2}{\omega} \quad \text{and} \quad x = \frac{\omega}{2} \eta. \tag{22.84}$$

Equality here means equality to the order included in the expansion in (22.75). Notice how the results are much simpler in synchronous gauge. The $\mathcal{O}(\eta^2)$ terms in Φ , Ψ , and δ_i^N ; and the $\mathcal{O}(\eta^3)$ terms in v_i^N looked so complicated that I did not write them here.

Matter density isocurvature mode. We get the matter density isocurvature mode by setting $\mathcal{R}_{\vec{k}}(\text{rad}) = 0$, $S_{mr,\vec{k}}(\text{rad}) = 1$, $S_{\nu r,\vec{k}}(\text{rad}) = 0$. To get the baryon density isocurvature mode (BDI), multiply all quantities by f_b , except the constant term in δ_b which stays equal to 1, and the constant term in δ_c , which becomes zero. To get the CDM density isocurvature mode (CDI), multiply all quantities by f_c , except the constant term in δ_c which stays equal to 1, and the constant term in δ_b , which becomes zero. We get the results

$$\begin{aligned}
\Phi &= -\frac{1}{8} \frac{1 - \frac{4}{15}f_\nu}{1 + \frac{2}{15}f_\nu} \omega \eta + \mathcal{O}(\eta^2) \\
\Psi &= -\frac{1}{8} \frac{1 + \frac{4}{15}f_\nu}{1 + \frac{2}{15}f_\nu} \omega \eta + \mathcal{O}(\eta^2) \\
\delta_b^N = \delta_c^N &= 1 - \frac{3}{8} \frac{1 + \frac{4}{15}f_\nu}{1 + \frac{2}{15}f_\nu} \omega \eta + \mathcal{O}(\eta^2) \\
\delta_\gamma^N = \delta_\nu^N &= \frac{1}{2} \frac{1 + \frac{4}{15}f_\nu}{1 + \frac{2}{15}f_\nu} \omega \eta + \mathcal{O}(\eta^2) \\
v_c^N &= -\frac{1}{24} \frac{1 - \frac{4}{15}f_\nu}{1 + \frac{2}{15}f_\nu} \omega k \eta^2 + \mathcal{O}(\eta^3) \\
v_\nu^N &= -\frac{1}{8} \frac{1}{1 + \frac{2}{15}f_\nu} \omega k \eta^2 + \mathcal{O}(\eta^3) \\
v_b^N = v_\gamma^N &= v_\nu^N + \mathcal{O}(\eta^3) \\
\delta_b^Z = \delta_c^Z &= 1 - \frac{1}{2} \omega \eta + \frac{3}{16} \omega^2 \eta^2 + \mathcal{O}(\eta^3) \\
\delta_\gamma^Z = \delta_\nu^Z &= -\frac{2}{3} \omega \eta + \frac{1}{4} \omega^2 \eta^2 + \mathcal{O}(\eta^3) \\
v_c^Z &= 0 \\
v_b^Z = v_\gamma^Z &= -\frac{1}{12} \omega k \eta^2 + \frac{1}{48} \frac{1 + 3f_b - f_\nu}{1 - f_\nu} \omega^2 k \eta^3 + \mathcal{O}(\eta^4) \\
v_\nu^Z &= -\frac{1}{12} \omega k \eta^2 + \frac{1}{48} \omega^2 k \eta^3 + \mathcal{O}(\eta^4) \\
\Pi_\nu &= -\frac{1}{15} \frac{1}{1 + \frac{2}{15}f_\nu} \omega k^2 \eta^3. \tag{22.85}
\end{aligned}$$

Neutrino density isocurvature mode. We get the neutrino density isocurvature (NDI) mode by setting $\mathcal{R}_{\vec{k}}(\text{rad}) = 0$, $S_{mr,\vec{k}}(\text{rad}) = 0$, $S_{\nu r,\vec{k}}(\text{rad}) = 3/4$ (to match with the normalization used in CAMB and [9]). This gives the results

$$\begin{aligned}
\Phi &= -\frac{2}{15} \frac{f_\nu}{1 + \frac{4}{15}f_\nu} - \frac{1}{12} \frac{(1 - \frac{2}{15})f_\nu}{(1 + \frac{2}{15}f_\nu)(1 + \frac{4}{15}f_\nu)} \omega\eta + \mathcal{O}(\eta^2) \\
\Psi &= \frac{1}{15} \frac{f_\nu}{1 + \frac{4}{15}f_\nu} - \frac{1}{60} \frac{(1 - \frac{2}{15})f_\nu}{(1 + \frac{2}{15}f_\nu)(1 + \frac{4}{15}f_\nu)} \omega\eta + \mathcal{O}(\eta^2) \\
\delta_b^N &= \frac{3}{15} \frac{f_\nu}{1 + \frac{4}{15}f_\nu} - \frac{f_\nu}{20} \frac{1 - \frac{2}{15}f_\nu}{(1 + \frac{2}{15}f_\nu)(1 + \frac{4}{15}f_\nu)} \omega\eta + \mathcal{O}(\eta^2) \\
\delta_c^N &= \delta_b^N + \mathcal{O}(\eta^3) \\
\delta_\gamma^N &= -\frac{11}{15} \frac{(1 + \frac{8}{11}f_\nu)f_\nu}{(1 + \frac{4}{15}f_\nu)(1 - f_\nu)} - \frac{1}{15} \frac{(1 - \frac{2}{15}f_\nu)f_\nu}{(1 + \frac{2}{15}f_\nu)(1 + \frac{4}{15}f_\nu)} \omega\eta + \mathcal{O}(\eta^2) \\
\delta_\nu^N &= -\frac{1 + \frac{8}{15}f_\nu}{1 + \frac{4}{15}f_\nu} - \frac{1}{15} \frac{(1 - \frac{2}{15}f_\nu)f_\nu}{(1 + \frac{2}{15}f_\nu)(1 + \frac{4}{15}f_\nu)} \omega\eta + \mathcal{O}(\eta^2) \\
v_c^N &= -\frac{1}{15} \frac{f_\nu}{1 + \frac{4}{15}f_\nu} k\eta + \frac{1}{30} \frac{f_\nu}{(1 + \frac{2}{15}f_\nu)(1 + \frac{4}{15}f_\nu)} \omega k\eta^2 + \mathcal{O}(\eta^3) \\
v_\nu^N &= \frac{1}{4} \frac{1}{1 + \frac{4}{15}f_\nu} k\eta + \frac{1}{30} \frac{f_\nu}{(1 + \frac{2}{15}f_\nu)(1 + \frac{4}{15}f_\nu)} \omega k\eta^2 + \mathcal{O}(\eta^3) \\
v_b^N = v_\gamma^N &= -\frac{19}{60} \frac{f_\nu}{(1 - f_\nu)(1 + \frac{4}{15}f_\nu)} k\eta + \mathcal{O}(\eta^2) \\
\delta_b^Z = \delta_c^Z &= \frac{f_\nu}{8(1 - f_\nu)} k^2\eta^2 + \mathcal{O}(\eta^3) \\
\delta_\gamma^Z &= -\frac{f_\nu}{1 - f_\nu} + \frac{f_\nu}{6(1 - f_\nu)} k^2\eta^2 + \mathcal{O}(\eta^3) \\
\delta_\nu^Z &= 1 - \frac{1}{6} k^2\eta^2 + \mathcal{O}(\eta^3) \\
v_c^Z &= 0 \\
v_b^Z = v_\gamma^Z &= -\frac{f_\nu}{4(1 - f_\nu)} k\eta + \frac{3}{16} \frac{f_b f_\nu}{(1 - f_\nu)^2} \omega k\eta^2 + \mathcal{O}(\eta^3) \\
v_\nu^Z &= \frac{1}{4} k\eta - \frac{1}{40} \frac{1 + \frac{4}{27}f_\nu}{1 + \frac{4}{15}f_\nu} (k\eta)^3 + \mathcal{O}(\eta^4) \\
\Pi_\nu &= \frac{1}{5} (1 + \frac{4}{15}f_\nu) (k\eta)^2 + \mathcal{O}(\eta^3).
\end{aligned} \tag{22.86}$$

23 Superhorizon Evolution and Relation to Original Perturbations

The usual thinking nowadays is that the perturbations were generated much, much earlier than neutrino decoupling, e.g., during inflation. Thus we are justified in ignoring the decaying solutions and the NVI mode. While the radiation-dominated period is convenient for specifying the “initial” conditions $\Phi_{\vec{k}}(\text{rad})$, $S_{cr,\vec{k}}(\text{rad})$, $S_{br,\vec{k}}(\text{rad})$, $S_{\nu r,\vec{k}}(\text{rad})$, these values are NOT “truly” initial—instead we call them the “primordial” values. We would like to relate them to the “original” perturbations, i.e., to the situation immediately after the perturbations were generated. We shall denote the generation time by t_* or just $*$. It may be different for different scales k (for inflation it is around the horizon exit time). This relation depends on the assumed generating mechanism (e.g., inflation) and what happened between generation ($*$) and the primordial epoch (rad) (e.g. reheating after inflation).

In Cosmological Perturbation Theory I we do not assume a particular mechanism for the generation of the perturbations (we just assume it was some Gaussian random process)—in Cosmological Perturbation Theory II we discuss inflation as the generating mechanism—so now we make just some more general observations.

In the usual scenarios the perturbations are outside the horizon during the interval between $*$ and rad; but some other properties of the radiation-dominated “primordial” era may not hold. Let us thus collect what general properties follow just from the superhorizon ($k \ll \mathcal{H}$) assumption.

From the general \mathcal{R}' equation (16.30) we see that $\mathcal{R} = \text{const.}$ for adiabatic perturbations ($\mathcal{S} = 0$) at superhorizon scales. The Bardeen potentials Φ and Ψ do not necessarily stay constant. Thus \mathcal{R} is a better quantity to describe the adiabatic mode, and we have

$$\mathcal{R}_{\vec{k}}(\text{rad}) = \mathcal{R}_{\vec{k}}(*) \quad (ADI). \quad (23.1)$$

On the other hand, if $\mathcal{S} \neq 0$, \mathcal{R} will evolve, and thus $\mathcal{R}_{\vec{k}}(\text{rad})$ may have received a contribution from the “original” entropy perturbations at $*$.

While during the primordial era S_{cr} , S_{br} , and $S_{\nu r}$ remain constant, this is not necessarily true at earlier times. The energy components corresponding to CDM, baryons, and neutrinos may have been in some other form (some fields) before they became these particles, and the properties assumed in the derivation of Eq.(18.31), i.e., no energy transfer between these components and no internal entropy perturbation within them, may not have held. Maybe it is not even clear how to separate out the energy components corresponding to the later CDM, baryons, and neutrinos. But to arrive at nonzero primordial S_{cr} , S_{br} , and $S_{\nu r}$ we need to have had some original entropy degrees of freedom responsible for them, which we, for now, denote with $S_{cr,\vec{k}}(*)$, $S_{br,\vec{k}}(*)$, and $S_{\nu r,\vec{k}}(*)$.

Thus, in general, S_{cr} , S_{br} , and $S_{\nu r}$ may have evolved. However, at superhorizon scales they are not affected by \mathcal{R} . (We have not shown this; but a rough argument is that changes in entropy perturbations are “local” physics that are not affected by curvature at superhorizon scales.)

We can thus represent the relation between the primordial (rad) and original ($*$) values formally as

$$\begin{pmatrix} \mathcal{R}_{\vec{k}}(\text{rad}) \\ S_{cr,\vec{k}}(\text{rad}) \\ S_{br,\vec{k}}(\text{rad}) \\ S_{\nu r,\vec{k}}(\text{rad}) \end{pmatrix} = \begin{pmatrix} 1 & T_{\mathcal{R}S_{cr}}(k) & T_{\mathcal{R}S_{br}}(k) & T_{\mathcal{R}S_{\nu r}}(k) \\ 0 & T_{S_{cr}S_{cr}}(k) & T_{S_{cr}S_{br}}(k) & T_{S_{cr}S_{\nu r}}(k) \\ 0 & T_{S_{br}S_{cr}}(k) & T_{S_{br}S_{br}}(k) & T_{S_{br}S_{\nu r}}(k) \\ 0 & T_{S_{\nu r}S_{cr}}(k) & T_{S_{\nu r}S_{br}}(k) & T_{S_{\nu r}S_{\nu r}}(k) \end{pmatrix} \begin{pmatrix} \mathcal{R}_{\vec{k}}(*) \\ S_{cr,\vec{k}}(*) \\ S_{br,\vec{k}}(*) \\ S_{\nu r,\vec{k}}(*) \end{pmatrix}, \quad (23.2)$$

where the $T_{ij}(k) = T_{ij}(\text{rad}, t_*, k)$ are transfer functions giving the relation between quantities at the primordial (rad) era and at the time t_* near origin. They do not depend on the direction of the Fourier mode wave vector \vec{k} since we assume that laws of physics are isotropic.

In words, at superhorizon scales:

1. Curvature perturbations $\mathcal{R}_{\vec{k}}$ remain constant for adiabatic perturbations.
2. Curvature perturbations may be seeded by entropy perturbations.
3. Entropy perturbations are not affected by curvature perturbations.
4. Entropy perturbations may evolve by themselves.

24 Statistically Homogeneous and Isotropic Initial Conditions

In this Section, by “initial conditions” we mean the *primordial* (rad) values of quantities. At the end of this Section we comment on their relation to the “original” (*) values.

If we knew the initial values ($\mathcal{R}_{\vec{k}}(\text{rad})$, $S_{cr,\vec{k}}(\text{rad})$, $S_{br,\vec{k}}(\text{rad})$, $S_{\nu r,\vec{k}}(\text{rad})$) for all Fourier modes \vec{k} , we could calculate from them, using our perturbation equations, the evolution of the universe to obtain the present universe (to the extent first-order perturbation theory holds). Or vice versa, from the observed present universe we could calculate backwards to obtain these initial values.⁵⁹

But this is not a realistic program—how could we know all these initial values! And the reverse program puts way too much demands on observations; and anyway, what use would we have for this huge collection of information $\{(\mathcal{R}_{\vec{k}}(\text{rad}), S_{cr,\vec{k}}(\text{rad}), S_{br,\vec{k}}(\text{rad}), S_{\nu r,\vec{k}}(\text{rad}))\}$?

Instead, we proceed in a statistical manner. Present theories for the origin of the perturbations generate them by a random process. Thus they do not specify initial values; they specify only their probability distributions. Using our perturbation theory we can then calculate probability distributions for observables in the present universe. Comparing then various statistical measures of the observed universe to these probability distributions we can compare theory with observation.

We shall assume that the initial conditions $\mathcal{R}_{\vec{k}}(\text{rad})$, $S_{cr,\vec{k}}(\text{rad})$, $S_{br,\vec{k}}(\text{rad})$, $S_{\nu r,\vec{k}}(\text{rad})$ are the results of some *statistically homogeneous and isotropic random process*.

Note that the Fourier coefficients are complex quantities. The reality of $\mathcal{R}(\vec{x})$ implies that $\mathcal{R}_{-\vec{k}} = \mathcal{R}_{\vec{k}}^*$. For perturbations, the expectation value vanishes:

$$\langle \mathcal{R}_{\vec{k}} \rangle = 0. \quad (24.1)$$

The variance

$$\langle |\mathcal{R}_{\vec{k}}|^2 \rangle \equiv \mathcal{P}_{\mathcal{R}}(\vec{k}) \quad (24.2)$$

we call the *power spectrum* of the perturbation.

From statistical homogeneity follows (as shown in Cosmology II) that the probabilities of different Fourier modes are independent (i.e., not correlated):

$$\langle \mathcal{R}_{\vec{k}} \mathcal{R}_{\vec{k}'}^* \rangle = \langle \mathcal{R}_{\vec{k}} S_{\vec{k}'}^* \rangle = \langle S_{\vec{k}} S_{\vec{k}'}^* \rangle = 0 \quad \text{for } \vec{k} \neq \vec{k}', \quad (24.3)$$

where S is S_{cr} , S_{br} , or $S_{\nu r}$. Because of the complex conjugate $*$ this will hold also for $\vec{k}' = -\vec{k}$ although these two Fourier modes are not independent of each other.

Statistical isotropy means that these probability distributions are independent of the direction of \vec{k} , depending only on its magnitude:

$$\mathcal{P}_{\mathcal{R}}(\vec{k}) = \mathcal{P}_{\mathcal{R}}(k). \quad (24.4)$$

⁵⁹This is just a rhetorical point. If taken seriously, it has some fundamental problems: First order perturbation theory remains reasonably accurate to present times only for the largest scales. For smaller scales, nonlinear evolution has hopelessly messed up the information about the initial values. Also, our observations are not about the present ($t = t_0$) universe. Rather, they are about our past light cone.

When there are several independent perturbation quantities (we have now four: $\mathcal{R}_{\vec{k}}(\text{rad})$, $S_{cr,\vec{k}}(\text{rad})$, $S_{br,\vec{k}}(\text{rad})$, $S_{\nu r,\vec{k}}(\text{rad})$) per Fourier mode \vec{k} , they may be correlated (but the correlation is independent of the direction of \vec{k}). Thus we have $N = 4$ variances and $N(N - 1)/2 = 6$ correlations (covariances):

$$\begin{aligned}
\langle \mathcal{R}_{\vec{k}} \mathcal{R}_{\vec{k}'}^* \rangle &\equiv P_{\mathcal{R}}(k) \delta_{\vec{k}\vec{k}'} \\
\langle S_{cr,\vec{k}} S_{cr,\vec{k}'}^* \rangle &\equiv P_{S_c}(k) \delta_{\vec{k}\vec{k}'} \\
\langle S_{br,\vec{k}} S_{br,\vec{k}'}^* \rangle &\equiv P_{S_b}(k) \delta_{\vec{k}\vec{k}'} \\
\langle S_{\nu r,\vec{k}} S_{\nu r,\vec{k}'}^* \rangle &\equiv P_{S_\nu}(k) \delta_{\vec{k}\vec{k}'} \\
\langle \mathcal{R}_{\vec{k}} S_{cr,\vec{k}'}^* \rangle &\equiv C_{\mathcal{R}S_c}(k) \delta_{\vec{k}\vec{k}'} \\
\langle \mathcal{R}_{\vec{k}} S_{br,\vec{k}'}^* \rangle &\equiv C_{\mathcal{R}S_b}(k) \delta_{\vec{k}\vec{k}'} \\
\langle \mathcal{R}_{\vec{k}} S_{\nu r,\vec{k}'}^* \rangle &\equiv C_{\mathcal{R}S_\nu}(k) \delta_{\vec{k}\vec{k}'} \\
\langle S_{cr,\vec{k}} S_{br,\vec{k}'}^* \rangle &\equiv C_{S_c S_b}(k) \delta_{\vec{k}\vec{k}'} \\
\langle S_{cr,\vec{k}} S_{\nu r,\vec{k}'}^* \rangle &\equiv C_{S_c S_\nu}(k) \delta_{\vec{k}\vec{k}'} \\
\langle S_{br,\vec{k}} S_{\nu r,\vec{k}'}^* \rangle &\equiv C_{S_b S_\nu}(k) \delta_{\vec{k}\vec{k}'} .
\end{aligned} \tag{24.5}$$

We can write this more compactly as

$$\langle A_{i\vec{k}} A_{j\vec{k}'}^* \rangle \equiv C_{ij}(k) \delta_{\vec{k}\vec{k}'} , \tag{24.6}$$

where $A_{i\vec{k}} = \mathcal{R}_{\vec{k}}(\text{rad})$, $S_{cr,\vec{k}}(\text{rad})$, $S_{br,\vec{k}}(\text{rad})$, $S_{\nu r,\vec{k}}(\text{rad})$ for $i = 1, 2, 3, 4$. The diagonal elements of this matrix, $P_i(k) \equiv C_{ii}(k)$, are called primordial power spectra.

The matrix is symmetric and real since, e.g.,

$$\begin{aligned}
C_{\mathcal{R}S}(k) &= C_{\mathcal{R}S}(|\vec{k}|) = \langle \mathcal{R}_{\vec{k}} S_{\vec{k}}^* \rangle = \langle \mathcal{R}_{-\vec{k}}^* S_{-\vec{k}} \rangle = \langle S_{-\vec{k}} \mathcal{R}_{-\vec{k}}^* \rangle = C_{S\mathcal{R}}(|-\vec{k}|) = C_{S\mathcal{R}}(k) = \\
&= \langle S_{\vec{k}} \mathcal{R}_{\vec{k}}^* \rangle = \langle \mathcal{R}_{\vec{k}}^* S_{\vec{k}} \rangle = C_{\mathcal{R}S}(k)^*
\end{aligned} \tag{24.7}$$

Consider now some quantity $f(\vec{x})$ in the present universe (or, more generally, some time later than the primordial time). For as long as linear perturbation theory holds, all Fourier modes \vec{k} evolve independently, and also the ADI, CDI, BDI, and NDI modes evolve independently⁶⁰ of each other. Because of the linearity of the theory, the Fourier component $f_{\vec{k}}$ depends linearly on the initial values $A_{i\vec{k}}$. This dependence can be express as a transfer function

$$f_{\vec{k}} = T_{fi}(\vec{k}) A_{i\vec{k}} = T_{fi}(k) A_{i\vec{k}} \quad (\text{sum over } i) . \tag{24.8}$$

Because our physical laws are isotropic, T_{fi} cannot depend on the direction of \vec{k} .

The variance of $f_{\vec{k}}$ is now

$$\begin{aligned}
\langle f_{\vec{k}} f_{\vec{k}'}^* \rangle &= \langle T_{fi}(k) A_{i\vec{k}} T_{fj}(k') A_{j\vec{k}'}^* \rangle = T_{fi}(k) T_{fj}^*(k') C_{ij}(k) \delta_{\vec{k}\vec{k}'} = \\
&\equiv P_f(k) \delta_{\vec{k}\vec{k}'} ,
\end{aligned} \tag{24.9}$$

where the power spectrum of f is

$$P_f(k) = T_{fi}(k) T_{fj}^*(k') C_{ij}(k) = \Re [T_{fi}(k) T_{fj}(k')^*] C_{ij}(k) . \tag{24.10}$$

(The symmetry of C_{ij} was used for the second equality. \Re stands for the real part.)

⁶⁰Note that the perturbations $\mathcal{R}_{\vec{k}}$, $S_{cr,\vec{k}}$, $S_{br,\vec{k}}$, and $S_{\nu r,\vec{k}}$ do *not* evolve independently. Do not confuse perturbation *modes* with perturbation *quantities*.

The covariance of two quantities f and g is likewise

$$\begin{aligned}\langle f_{\vec{k}} g_{\vec{k}'}^* \rangle &= \langle T_{fi}(k) A_{i\vec{k}} T_{gj}^*(k') A_{j\vec{k}'}^* \rangle = T_{fi}(k) T_{gj}^*(k') C_{ij}(k) \delta_{\vec{k}\vec{k}'} = \\ &\equiv C_{fg}(k) \delta_{\vec{k}\vec{k}'}.\end{aligned}\quad (24.11)$$

From this follows that $C_{gf} = C_{fg}^*$. However, $f(\vec{x})$ and $g(\vec{x})$ are presumably real quantities, so that $f_{\vec{k}}^* = T_{fi}^*(k) A_{i\vec{k}}^* = T_{fi}^*(k) A_{i,-\vec{k}}$ is equal to $f_{-\vec{k}} = T_{fi}(k) A_{i,-\vec{k}}$. Thus the transfer functions $T_{fi}(k)$ (and $T_{gi}(k)$) are real.⁶¹

To summarize,

$$\begin{aligned}\langle f_{\vec{k}} f_{\vec{k}'}^* \rangle &= T_{fi}(k) T_{fj}(k') C_{ij}(k) \delta_{\vec{k}\vec{k}'} \equiv P_f(k) \delta_{\vec{k}\vec{k}'} \\ \langle f_{\vec{k}} g_{\vec{k}'}^* \rangle &= T_{fi}(k) T_{gj}(k') C_{ij}(k) \delta_{\vec{k}\vec{k}'} \equiv C_{fg}(k) \delta_{\vec{k}\vec{k}'}.\end{aligned}\quad (24.12)$$

The primordial power spectrum $C_{ij}(k)$ is given by the theory of the origin of primordial perturbations (e.g., inflation, the topic of the second part of this course). Our task (the first part of this course) is to calculate the transfer functions $T_{fi}(k)$ for all quantities f we are interested in. We obtain them from the linear perturbation theory equations we have already discussed.

In the preceding, we assumed for simplicity that $f(\vec{x})$ referred to some quantity in the present universe, so that the transfer function $T_{fi}(k)$ represented the transfer from the primordial epoch to the present time. More generally, we can consider f at some other time t , writing

$$f_{\vec{k}}(t) = T_{fi}(t, k) A_{i\vec{k}}. \quad (24.13)$$

Even more generally, we can specify transfer functions taking us from some earlier time t_1 to some later time t_2 , writing

$$f_{\vec{k}}(t_2) = T_{fi}(t_2, t_1, k) A_{i\vec{k}}(t_1). \quad (24.14)$$

In the preceding discussion we had $t_2 = t_0$ and “ $t_1 = t_{\text{rad}}$ ” and we did not write these times explicitly in the transfer functions. Likewise, in Sec. 23 we discussed transfer functions with $t_2 = t_{\text{rad}}$ and $t_1 = t_*$ (perturbation generation time).

24.1 Gaussian perturbations

Gaussian perturbations mean that the probability distributions of individual Fourier components are Gaussian:

$$\begin{aligned}\text{Prob}(\mathcal{R}_{\vec{k}}) &= \frac{1}{\sqrt{2\pi}\sigma_{\mathcal{R}\vec{k}}} \exp\left(-\frac{R_{\vec{k}}^2}{2\sigma_{\mathcal{R}\vec{k}}^2}\right) \times \frac{1}{\sqrt{2\pi}\sigma_{\mathcal{R}\vec{k}}} \exp\left(-\frac{I_{\vec{k}}^2}{2\sigma_{\mathcal{R}\vec{k}}^2}\right) \\ &= \frac{1}{2\pi\sigma_{\mathcal{R}\vec{k}}^2} \exp\left(-\frac{|\mathcal{R}_{\vec{k}}|^2}{2\sigma_{\mathcal{R}\vec{k}}^2}\right) \Rightarrow \mathcal{P}_{\mathcal{R}}(\vec{k}) = 2\sigma_{\mathcal{R}\vec{k}}^2,\end{aligned}\quad (24.15)$$

and likewise for $S_{cr,\vec{k}}(\text{rad})$, $S_{br,\vec{k}}(\text{rad})$, $S_{\nu r,\vec{k}}(\text{rad})$. Here $R_{\vec{k}}$ and $I_{\vec{k}}$ denote their real and imaginary parts.

If this holds, then all statistical information is contained in the 4 power spectra $P_i(k)$ and 6 covariances $C_{ij}(k)$ of (24.5). Thus the full (or required) statistical information about the initial conditions is specified by the k -dependent symmetric real 4×4 covariance matrix $C_{ij}(k)$.

(If we only consider “quadratic estimators” (e.g., the angular power spectra C_ℓ of the cosmic microwave background or the present matter power spectrum $P_m(t_0, k)$), they are not affected by

⁶¹Different Fourier conventions for, e.g., scalar parts of vector quantities may lead to complex transfer functions in some cases.

the shape of the probability distribution. These estimators are determined by just the primordial power spectra and covariances.)

Linear evolution preserves (or transmits) the Gaussian nature of the perturbations: If the $A_{i\vec{k}}$ have the statistical properties of statistically isotropic Gaussian random variables, then $f_{\vec{k}}$ and $g_{\vec{k}}$ will have them also.

Let us now consider whether we should expect the primordial perturbations to be Gaussian:

Quantum fluctuations are known to have this Gaussian property. If the original perturbations were produced by quantum fluctuations (as is the case in inflation) and the physics and evolution leading from these quantum fluctuations to the primordial perturbations at t_{rad} is linear, then the primordial perturbations are Gaussian. This linear process can be represented by the transfer functions $T_{i\alpha}(\text{rad}, *, k)$, where α indexes the relevant variables of the generation process. Thus “our” Gaussian random process = original perturbation generation process + subsequent linear evolution from generation time to the primordial epoch. It may be that the relevant variables of the first part are uncorrelated, but correlations $C_{ij}(k)$ appear from the second part via the $T_{i\alpha}(\text{rad}, *, k)$ transfer function.

If one wants to stay agnostic about the original perturbation generation process, one could still motivate the Gaussianity assumption as being a natural one. The Gaussian distribution is particularly simple, since the probability distribution of each Fourier component is fully specified by just a single number, its variance (whose square root is the typical size (the rms value of the distribution) of the perturbation). Many natural processes produce distributions that are close to Gaussian. This is explained by the *Central limit theorem*, which states that the probability distribution of the mean of a set of independent random variables approaches the Gaussian distribution in the limit where the number of these variables becomes large, regardless of what is the probability distribution of each random variable. In cosmology, we are not really able to measure separately each Fourier mode \vec{k} , but rather our observations always average over a large number of modes. Thus the assumption of the independence of the initial values of each Fourier mode already implies that the Gaussianity assumption is likely to work fairly well.

Finally, we can refer to observations. Even if the original perturbations were perfectly Gaussian, we can expect the primordial perturbations to deviate from Gaussianity to the extent that first order perturbation theory gives only an approximation for the relation between the primordial and original perturbations. From observations, the typical magnitude of the primordial perturbations is $10^{-5} \dots 10^{-4}$. Thus a natural expectation for the magnitude of the second order corrections and the error in the linearity assumption would be the square of this, $10^{-10} \dots 10^{-8}$, or a relative error of $10^{-5} \dots 10^{-4}$.

While there is only “one kind” of Gaussianity, possibilities for a deviation from Gaussianity are infinite. A very simple form of non-Gaussianity of primordial perturbations is one where the perturbation is related to a Gaussian perturbation via a simple transformation

$$\Phi(\vec{x}) = \Phi_G(\vec{x}) - f_{NL}\Phi_G(\vec{x})^2, \quad (24.16)$$

where f_{NL} is called the (local) non-linearity parameter.⁶² Its value gives the level of non-Gaussianity. It is customary to define it in terms of the primordial Bardeen potential Φ . Here Φ is the true Bardeen potential, and Φ_G is a quantity with a Gaussian distribution related to Φ in the above way. According to the preceding argument a natural expectation for f_{NL} is that it would be of order 1. From the simplest inflation models we actually get $f_{NL} \ll 1$, whereas there are some interesting models where $f_{NL} \gg 1$. Since the values of Φ are of the order of 10^{-5} to 10^{-4} , an f_{NL} of the order 1 has a very small effect, unobservable with current methods. A non-Gaussianity of a similar magnitude than the primordial perturbation itself requires f_{NL} of the order 10^4 .

⁶²Note that in the literature there is a lot of confusion about the sign of f_{NL} .

The result from Planck is [11]

$$f_{NL} = 0.8 \pm 5.0 \quad 68\% \text{ CL}, \quad (24.17)$$

which is in agreement with naive expectations and most (but not all) inflation models.

25 Large Scales

Note similarity of this section to Sec. 19.4.

In this section we consider the scales $k \ll k_{\text{eq}} \equiv \mathcal{H}_{\text{eq}}$, i.e., scales that enter the horizon during the matter-dominated epoch. Their evolution is much easier to calculate than those of smaller scales. We ignore dark energy, so the calculation will not extend beyond the matter-dominated epoch, when dark energy begins to have an effect. (The effect of dark energy was discussed in Sec. 20 for the case of cosmological constant / vacuum energy.) We also assume that neutrino masses are small enough to be ignored.

We can take a number of results from Sec. 19: the background solution

$$y = \left(\frac{\eta}{\eta_3}\right)^2 + 2\left(\frac{\eta}{\eta_3}\right) \quad \mathcal{H} = \frac{\eta + \eta_3}{\eta_3\eta + \frac{1}{2}\eta^2} \quad w = \frac{1}{3(1+y)} \quad c_s^2 = \frac{4}{3(4+3y)} \quad (25.1)$$

and the relation

$$\mathcal{S} = \frac{y}{4+3y} S_{mr} \quad (25.2)$$

where $y \equiv a/a_{\text{eq}} = \rho_m/\rho_r$. The differences from Sec. 19 are that now the matter component is further divided into baryons and cold dark matter and the radiation component to photons and neutrinos. The radiation part is not a perfect fluid, since neutrinos, and after photon decoupling also photons, have anisotropic pressure, so that $\Phi \neq \Psi$. However, you will notice that the following discussion will be almost the same as in Sec. 19.4, since once the universe becomes matter-dominated, Ψ becomes equal to Φ , and the nature of radiation becomes irrelevant.

We define

$$f_b \equiv \frac{\rho_b}{\rho_m} \quad \text{and} \quad f_c \equiv \frac{\rho_c}{\rho_m} \quad \Rightarrow \quad f_b + f_c = 1 \quad (25.3)$$

so that we have

$$\delta_m = f_b \delta_b + f_c \delta_c \quad \text{and} \quad \delta_r = f_\nu \delta_\nu + f_\gamma \delta_\gamma \quad (25.4)$$

and

$$S_{mr} = f_b S_{br} + f_c S_{cr}. \quad (25.5)$$

25.1 Superhorizon evolution

Consider first the part of the evolution, when k is still outside the horizon, $k \ll \mathcal{H}$. Compared to Sec. 22 we are giving up the assumptions of radiation domination and tight coupling between photons and baryons. Instead we calculate the evolution of the perturbations from the radiation-dominated epoch to the matter-dominated epoch and the discussion applies during and after photon decoupling. Since $\Phi \neq \Psi$, we don't get the Bardeen equation we used in Sec. 19. Instead we use the \mathcal{R} evolution equation (16.31 or C.8)

$$\begin{aligned} \mathcal{H}^{-1} \mathcal{R}' &= y \frac{d\mathcal{R}}{dy} = \frac{2}{3(1+w)} \left(\frac{k}{\mathcal{H}}\right)^2 [c_s^2 \Psi + \frac{1}{3}(\Psi - \Phi)] + 3c_s^2 \mathcal{S} \approx 3c_s^2 \mathcal{S} \\ &= \frac{4}{4+3y} \frac{y}{4+3y} S_{mr}, \end{aligned} \quad (25.6)$$

(same as 19.70) where $S_{mr} = \text{const}$, since $k \ll \mathcal{H}$ (from 18.31 or C.6).

25.1.1 Adiabatic mode

For the adiabatic mode, $S_{mr} = 0$, so $\mathcal{R} = \text{const}$. The Bardeen potentials evolve, so none of the terms in (C.7), which follows from (16.27),

$$\frac{2}{3}\mathcal{H}^{-1}\Psi' + \frac{5+3w}{3}\Psi = -(1+w)\mathcal{R} + \frac{2}{3}(\Psi - \Phi), \quad (25.7)$$

is constant, but once the universe becomes matter dominated, $\Psi \approx \Phi$, since the difference came from the radiation anisotropic pressure, $w \approx 0$, and $\mathcal{H} \approx 2/\eta$, so we are left with

$$\frac{1}{3}\eta\Psi' + \frac{5}{3}\Psi = -\mathcal{R}, \quad (25.8)$$

whose solution is

$$\Psi = -\frac{3}{5}\mathcal{R} + C\eta^{-5}. \quad (25.9)$$

Once the decaying part $C\eta^{-5}$ has died out, we have

$$\Phi = \Psi = -\frac{3}{5}\mathcal{R} = \text{const} = -\frac{3}{5}\mathcal{R}(\text{rad}), \quad (\text{mat.dom}) \quad (25.10)$$

and, from (10.18 and 10.20 or C.1) and (10.20 or C.2)

$$\begin{aligned} \delta_c = \delta_b = \delta_m = \delta &= -2\Phi = \frac{6}{5}\mathcal{R}(\text{rad}) && (\text{mat.dom}, k \ll \mathcal{H}) \\ v_c = v_b = v_m = v &= \frac{2}{3}\left(\frac{k}{\mathcal{H}}\right)\Phi = -\frac{2}{5}\left(\frac{k}{\mathcal{H}}\right)\mathcal{R}(\text{rad}) = -\frac{1}{3}\left(\frac{k}{\mathcal{H}}\right)\delta. \end{aligned} \quad (25.11)$$

25.1.2 Isocurvature modes

(This repeats Sec. 19.4, since the equation to integrate, (25.6) = (19.70), is the same.)

For the matter isocurvature modes, initially $\mathcal{R} = 0$, but $S_{mr} \neq 0$. Integrating (25.6),

$$\mathcal{R}(y) = \int_0^y d\mathcal{R} = \frac{4}{9}S_{mr} \int_0^y \frac{dy}{(y + \frac{4}{3})^2} = -\frac{4}{9}S_{mr} \left[\frac{1}{y + \frac{4}{3}} - \frac{3}{4} \right] = \frac{y}{3y + 4}S_{mr}. \quad (25.12)$$

As the universe becomes matter dominated ($y \rightarrow \infty$),

$$\mathcal{R} \rightarrow \frac{1}{3}S_{mr} = \text{const} = \frac{1}{3}S_{mr}(\text{rad}). \quad (25.13)$$

We have then

$$\Phi = \Psi = -\frac{3}{5}\mathcal{R} = \text{const} = -\frac{1}{5}S_{mr}(\text{rad}), \quad (\text{mat.dom}) \quad (25.14)$$

and

$$\begin{aligned} \delta_m = \delta &= -2\Phi = \frac{2}{5}S_{mr}(\text{rad}). && (\text{mat.dom}, k \ll \mathcal{H}) \\ v_m = v &= \frac{2}{3}\left(\frac{k}{\mathcal{H}}\right)\Phi = -\frac{2}{15}\left(\frac{k}{\mathcal{H}}\right)S_{mr}(\text{rad}) = -\frac{1}{3}\left(\frac{k}{\mathcal{H}}\right)\delta. \end{aligned} \quad (25.15)$$

The individual density perturbations δ_b and δ_c will depend on the S_{br} and S_{cr} (exercise). The $S_{\nu r}$ will not have an effect on them, since it does not contribute to \mathcal{S} .

In the NDI mode, initially $\mathcal{R} = S_{mr} = 0$, $S_{\nu r} \neq 0$. Also Φ and Ψ are initially nonzero. At superhorizon scales $S_{\nu r}$ stays constant, but since it does not contribute to \mathcal{S} , it does not affect \mathcal{R} , which stays zero. Once the universe becomes matter dominated, the significance of $S_{\nu r}$ fades. Since matter is a perfect fluid, Φ and Ψ become equal, they become both zero and the initial matter density perturbations disappear. So in the matter-dominated epoch we have

$$\delta_m = \delta = v_m = v = \Phi = \Psi = \mathcal{R} = 0. \quad (\text{NDI, mat.dom}, k \ll \mathcal{H}) \quad (25.16)$$

Exercise: Solve the matter-dominated era superhorizon δ_b and δ_c in terms of the primordial \mathcal{R} , S_{br} and S_{cr} .

25.2 Through the horizon

When the ($k \ll k_{\text{eq}}$) perturbations approach the horizon, the universe is already matter dominated. This means that $c_s^2 \approx 0$ and $\Phi \approx \Psi$, so that from Eq. (25.6) we have that $\mathcal{R} = \text{const}$ even inside the horizon. We now have Eq. (25.9) for Ψ with $\mathcal{R} = \text{const}$ both for the adiabatic and isocurvature modes. Since Ψ had already settled to a constant before horizon entry, it will stay at this constant solution, and we have, from (25.10)+(25.14) and (25.5)

$$\Phi = \Psi = -\frac{3}{5}\mathcal{R} = -\frac{3}{5}\mathcal{R}(\text{rad}) - \frac{1}{5}f_c S_{cr}(\text{rad}) - \frac{1}{5}f_b S_{br}(\text{rad}). \quad (\text{mat.dom}) \quad (25.17)$$

From (14.21 or C.1) we have that

$$\delta_m = \delta = -2 \left[1 + \frac{1}{3} \left(\frac{k}{\mathcal{H}} \right)^2 \right] \Phi. \quad (25.18)$$

Through and inside the horizon the entropy perturbations S_{ij} do not necessarily stay constant any more: they could be changed by velocity differences. The baryon and CDM velocity equations differ by the baryon-photon collision term. In the NDI mode we had a velocity difference between neutrinos and photons, and the collision term forced $v_b = v_\gamma$ while $v_c = v$ in the primordial era. In the other three modes all velocities were equal in the primordial era. We now consider only these modes:

In ADI, BDI, and CDI modes the collision term disappears in the primordial era, since $v_b = v_\gamma$. The velocity equations depend on the density perturbations only through the common potential. For as long as $v_b = v_\gamma$, the velocity equations for the different fluid components differ only due to the anisotropic pressure terms for photons and neutrinos. These terms are suppressed by k/\mathcal{H} so they will not cause velocity differences at superhorizon scales. When the scales approach the horizon these terms begin to affect v_γ and v_ν . However, for $k \ll k_{\text{eq}}$ the universe is then already matter dominated, so the baryons and CDM no longer care about the neutrinos and photons (for baryons the collision term is suppressed by ρ_γ/ρ_b). Thus we will still have $v_b = v_c = v$. (Here and elsewhere we are ignoring baryon pressure; at these large scales, $k \ll k_{\text{eq}}$, its effect is negligible.)

Therefore, for the adiabatic mode, we still have $\delta_b = \delta_c = \delta_m$.

For the BDI and CDI modes these density perturbations differ, but since $v_b = v_c$, the relative entropy perturbation $S_{bc} = \delta_b - \delta_c = S_{br} - S_{cr}$ stays constant. (The S_{br} and S_{cr} no longer stay constant but we do not care about them any more.)

Thus in the matter-dominated era, when $\mathcal{H} = 2/\eta^2$, we have (using $\delta = f_b \delta_b + f_c \delta_c$)

$$\begin{aligned} \delta_b &= \delta + f_c S_{bc} = \delta + f_c [S_{br}(\text{rad}) - S_{cr}(\text{rad})] \\ \delta_c &= \delta - f_b S_{bc} = \delta - f_b [S_{br}(\text{rad}) - S_{cr}(\text{rad})] \end{aligned} \quad (25.19)$$

where, from (25.17) and (25.18)

$$\delta = \left[1 + \frac{1}{3} \left(\frac{k}{\mathcal{H}} \right)^2 \right] \times \left[\frac{6}{5}\mathcal{R}(\text{rad}) + \frac{2}{5}f_c S_{cr}(\text{rad}) + \frac{2}{5}f_b S_{br}(\text{rad}) \right]. \quad (25.20)$$

and $f_c = 1 - f_b$. Since δ grows as $(k/\mathcal{H})^2$ whereas the differences between δ , δ_c and δ_b stay constant, the relative differences will eventually become negligible, so that at late times $\delta_b \approx \delta_c \approx \delta_m \approx \delta$.

Thus we have that density perturbation transfer functions from the primordial era to the matter-dominated era (when $\mathcal{H} = 2/\eta^2$) are

$$\begin{aligned} T_{\delta\mathcal{R}}(\eta, k) &= \frac{6}{5} \left[1 + \frac{1}{3} \left(\frac{k}{\mathcal{H}} \right)^2 \right] = \frac{6}{5} + \frac{1}{10} (k\eta)^2 \\ T_{\delta S_{cr}}(\eta, k) &= \frac{2}{5} f_c \left[1 + \frac{1}{3} \left(\frac{k}{\mathcal{H}} \right)^2 \right] = \frac{2}{5} f_c + \frac{f_c}{30} (k\eta)^2 = \frac{f_c}{3} T_{\delta\mathcal{R}}(\eta, k) \\ T_{\delta S_{br}}(\eta, k) &= \frac{2}{5} f_b \left[1 + \frac{1}{3} \left(\frac{k}{\mathcal{H}} \right)^2 \right] = \frac{2}{5} f_b + \frac{f_b}{30} (k\eta)^2 = \frac{f_b}{3} T_{\delta\mathcal{R}}(\eta, k) \\ T_{\delta S_{\nu r}}(\eta, k) &= 0. \end{aligned} \tag{25.21}$$

Matter isocurvature perturbations lead to matter perturbations that are a factor 1/3 smaller than those from adiabatic perturbations (from initial \mathcal{R} and S_{mr} of equal size). In the matter power spectrum P_δ this becomes a factor 1/9.

If we have both primordial \mathcal{R} and S_{mr} their correlations are also important. The density power spectrum is

$$\begin{aligned} P_\delta(\eta, k) &= T_{\delta i}(k) T_{\delta j}(k) C_{ij}(k) \\ &= \left[\frac{6}{5} + \frac{1}{10} (k\eta)^2 \right]^2 \times \\ &\times \left[P_{\mathcal{R}}(k) + \frac{f_c^2}{9} P_{S_{cr}}(k) + \frac{f_b^2}{9} P_{S_{br}}(k) + \frac{2f_c}{3} C_{\mathcal{R}S_{cr}}(k) + \frac{2f_b}{3} C_{\mathcal{R}S_{br}}(k) + \frac{2f_c f_b}{9} C_{S_{cr}S_{br}}(k) \right]. \end{aligned} \tag{25.22}$$

The power spectra $P_i(k)$ are necessarily nonnegative, but the correlations $C_{ij}(k)$ for $i \neq j$ may be positive or negative indicating correlation or anticorrelation. We see that (positively) correlated curvature and entropy perturbations enhance the density power spectrum $P_\delta(k)$ whereas anticorrelated⁶³ curvature and entropy perturbations weaken it.

(While for these large scales the NDI mode has no late-time density or metric perturbations, this is not true for the smaller scales, which are inside the horizon when the universe becomes matter dominated. Here the initial matter density perturbation will have a similar later evolution as in the adiabatic mode [7], although we do not include the calculation in this course.)

26 Sachs–Wolfe Effect

Consider photon travel in the perturbed universe. The geodesic equation is

$$\frac{d^2 x^\mu}{du^2} + \Gamma_{\alpha\beta}^\mu \frac{dx^\alpha}{du} \frac{dx^\beta}{du} = 0, \tag{26.1}$$

where u is an affine parameter of the geodesic. For photons, we choose u so that the photon 4-momentum is

$$p^\mu = \frac{dx^\mu}{du}, \tag{26.2}$$

which allows us to write the geodesic equation as

$$\frac{dp^\mu}{du} + \Gamma_{\alpha\beta}^\mu p^\alpha p^\beta = 0. \tag{26.3}$$

⁶³Beware: Some authors have the opposite sign conventions for entropy and/or curvature perturbations, changing our correlation to anticorrelation and vice versa (unless both sign conventions are opposite!). One actually gets this opposite sign convention if you take the “entropy perturbation” $S_{i\gamma}$ to literally mean “relative perturbation in photon entropy per i particle”. Our sign convention for relative entropy perturbations seems to be the more common in literature.

Dividing by $p^0 = d\eta/du$, this becomes

$$\frac{dp^\mu}{d\eta} + \Gamma_{\alpha\beta}^\mu \frac{p^\alpha p^\beta}{p^0} = 0. \quad (26.4)$$

In the following, we need only the time component of this equation,

$$\frac{dp^0}{d\eta} + \Gamma_{00}^0 p^0 + 2\Gamma_{0k}^0 p^k + \Gamma_{ij}^0 \frac{p^i p^j}{p^0} = 0. \quad (26.5)$$

Assuming scalar perturbations and using the Newtonian gauge (the $\Gamma_{\alpha\beta}^\mu$ from Eq. (8.7)), this becomes

$$\frac{dp^0}{d\eta} + (\mathcal{H} + \Phi') p^0 + 2\Phi_{,k} p^k + [\mathcal{H} - 2\mathcal{H}(\Phi + \Psi) - \Psi'] \frac{\delta_{ij} p^i p^j}{p^0} = 0. \quad (26.6)$$

These 4-momentum components p^μ are in the coordinate frame. What the observer interprets as the photon energy and momentum are the components $p^{\hat{\mu}}$ in his local orthonormal frame. Since the metric is diagonal, the conversion is easy, $p^{\hat{\mu}} = \sqrt{|g_{\mu\mu}|} p^\mu$ (for a comoving observer):

$$\begin{aligned} E \equiv p^{\hat{0}} &= a\sqrt{1+2\Phi} p^0 = a(1+\Phi)p^0 \\ p^{\hat{i}} &= a\sqrt{1-2\Psi} p^i = a(1-\Psi)p^i. \end{aligned} \quad (26.7)$$

Since photons are massless, $E^2 = \delta_{ij} p^{\hat{i}} p^{\hat{j}}$.

In the background universe, the photon energy redshifts as $\bar{E} \propto a^{-1} \Leftrightarrow \bar{q} \equiv a\bar{E} = \text{const.}$ In the presence of perturbations, $q \equiv aE \neq \text{const.}$ Thus we define q and \vec{q} ,

$$\begin{aligned} q \equiv aE &= a^2(1+\Phi)p^0 \Rightarrow p^0 = a^{-2}(1-\Phi)q \\ q^i \equiv ap^{\hat{i}} &= a^2(1-\Psi)p^i \Rightarrow p^i = a^{-2}(1+\Psi)q^i \end{aligned} \quad (26.8)$$

where $q^2 = \delta_{ij} q^i q^j$, as suitable quantities to track the perturbation in the redshift.

Rewriting Eq. (26.6) in terms of q and \vec{q} (and dropping 2nd order terms) gives (**exercise**)

$$(1-\Phi) \frac{dq}{d\eta} = q \frac{d\Phi}{d\eta} - q\Phi' - 2q^k \Phi_{,k} + q\Psi'. \quad (26.9)$$

Here the rhs is 1st order small, therefore $dq/d\eta$ is also 1st order small, and we can drop the factor $(1-\Phi)$. Dividing by q we get

$$\frac{1}{q} \frac{dq}{d\eta} = \frac{d\Phi}{d\eta} - \Phi' + \Psi' - 2 \frac{\vec{q} \cdot \nabla \Phi}{q}. \quad (26.10)$$

Here the total derivative along the photon geodesic is

$$\frac{d}{d\eta} = \frac{\partial}{\partial \eta} + \frac{dx^k}{d\eta} \frac{\partial}{\partial x^k} \quad (26.11)$$

and

$$\frac{\vec{q}}{q} = \frac{(1-\Psi)p^k}{(1+\Phi)p^0} \approx \frac{p^k}{p^0} = \frac{dx^k}{d\eta} \quad \text{to 0th order} \quad (26.12)$$

so that

$$-2 \frac{\vec{q} \cdot \nabla \Phi}{q} = -2 \frac{dx^k}{d\eta} \frac{\partial \Phi}{\partial x^k} = -2 \left(\frac{d\Phi}{d\eta} - \frac{\partial \Phi}{\partial \eta} \right), \quad (26.13)$$

so that Eq. (26.10) becomes

$$\frac{1}{q} \frac{dq}{d\eta} = -\frac{d\Phi}{d\eta} + \Phi' + \Psi' \approx \frac{1}{\bar{q}} \frac{dq}{d\eta}. \quad (26.14)$$

The relative perturbation in the photon energy, $\delta(E/\bar{E})$, that the photon accumulates when traveling from \mathbf{x}_* to \mathbf{x}_{obs} in the perturbed universe is thus

$$\begin{aligned}\delta\left(\frac{E}{\bar{E}}\right) &= \frac{\delta q}{\bar{q}} = \int \frac{dq}{q} = -\int d\Phi + \int (\Phi' + \Psi') d\eta \\ &= \Phi(\mathbf{x}_*) - \Phi(\mathbf{x}_{\text{obs}}) + \int_{\eta_*}^{\eta_{\text{obs}}} \left(\frac{\partial\Phi}{\partial\eta} + \frac{\partial\Psi}{\partial\eta}\right) d\eta,\end{aligned}\quad (26.15)$$

where the integrals are along the photon path. Here $\mathbf{x}_* = (\eta_*, \vec{x}_*)$ denotes the location \vec{x}_* at the last scattering surface from where the photon originated at the time η_* of photon decoupling.

For a thermal distribution of photons, a uniform relative photon energy perturbation corresponds to a temperature perturbation of the same amount:

$$\left(\frac{\delta T}{T}\right)_{\text{jour}} = \delta\left(\frac{E}{\bar{E}}\right). \quad (26.16)$$

Here “jour” refers to the temperature perturbation the photon distribution accumulates on the journey between \mathbf{x}_* and \mathbf{x}_{obs} .

The other contributions to the observed CMB temperature anisotropy are due to the local photon energy density perturbation and photon velocity perturbation at the origin of the photon, on the last scattering surface:

$$\left(\frac{\delta T}{T}\right)_{\text{intr}} = \frac{1}{4}\delta_\gamma(\mathbf{x}_*) - \vec{v}_\gamma(\mathbf{x}_*) \cdot \hat{n}, \quad (26.17)$$

where \hat{n} is the direction the observer is looking at.

For a given observer, the $\Phi(\mathbf{x}_{\text{obs}})$ part is common to photons from all directions, and the observer interprets it as part of the mean (background) photon temperature. Thus the observed CMB temperature anisotropy is

$$\frac{\delta T}{T} = \frac{1}{4}\delta_\gamma^N(\mathbf{x}_*) - \vec{v}_\gamma^N(\mathbf{x}_*) \cdot \hat{n} + \Phi(\mathbf{x}_*) + \int_{\eta_*}^{\eta_{\text{obs}}} \left(\frac{\partial\Phi}{\partial\eta} + \frac{\partial\Psi}{\partial\eta}\right) d\eta. \quad (26.18)$$

(There is also a contribution from the motion of the observer that causes a dipole pattern in the observed anisotropy. To get rid of that, the dipole of the observed anisotropy is subtracted away from the observations before any cosmological analysis.)

For large scales, much larger than the horizon size at photon decoupling, the Doppler effect $-\vec{v}_\gamma^N(\mathbf{x}_*) \cdot \hat{n}$ is small compared to the other terms: We see from (25.11) and (25.15) that v^N is suppressed by k/\mathcal{H} compared to Φ (v_γ^N at photon decoupling is not exactly the same as the total v^N but is of the same magnitude; before photon decoupling $v_\gamma = v_b$, but the decoupling is not instantaneous). The contribution

$$\left(\frac{\delta T}{T}\right)_{\text{SW}} = \frac{1}{4}\delta_\gamma^N(\mathbf{x}_*) + \Phi(\mathbf{x}_*) \quad (26.19)$$

is called the *ordinary Sachs–Wolfe effect*, and the contribution

$$\left(\frac{\delta T}{T}\right)_{\text{ISW}} = \int_{\eta_*}^{\eta_{\text{obs}}} \left(\frac{\partial\Phi}{\partial\eta} + \frac{\partial\Psi}{\partial\eta}\right) d\eta \quad (26.20)$$

is called the *integrated Sachs–Wolfe effect* (ISW). During the matter-dominated epoch $\Phi = \Psi = \text{const}$, so there are two contributions to ISW: the *early Sachs–Wolfe effect* (ESW) from the time after photon decoupling when the universe was not yet completely matter dominated, and the

late Sachs-Wolfe effect (LSW) from the time when dark energy began to have an effect on the expansion of the universe. LSW becomes important at the largest angular scales on the sky and ESW at scales that are comparable to the horizon at photon decoupling.

For $k \ll k_{\text{dec}} < k_{\text{eq}}$ (or $\ell \ll \ell_{\text{dec}} < \ell_{\text{eq}}$) the ordinary Sachs-Wolfe effect is the dominant contribution. We now give the result for this in the (not very good) approximation that the universe was already matter-dominated at the time of photon decoupling. We need Φ and δ_γ at η_* , and we consider only scales that were superhorizon at that time. For these scales the equations that we need have no k -dependence, so we can apply them directly at coordinate space (assuming that we ignore all smaller-scale contributions, e.g., by making observations with a coarse resolution or smoothing the observations afterwards).

For the *adiabatic mode*, $\delta_\gamma = \frac{4}{3}\delta_m = \frac{4}{3}\delta$ (since entropy perturbations have remained zero at superhorizon scales), and

$$\begin{aligned} \delta &= -2\Phi = \frac{6}{5}\mathcal{R}(\text{rad}) \\ \Rightarrow \frac{\delta T}{T} &= \frac{1}{3}\delta + \Phi = \frac{1}{3}\Phi = -\frac{1}{5}\mathcal{R}(\text{rad}). \end{aligned} \quad (26.21)$$

For the *BDI and CDI modes*, the entropy perturbations have remained constant at superhorizon scales, and

$$\delta_\gamma = \delta_r = \frac{4}{3}(\delta_m - S_{mr}). \quad (26.22)$$

From Sec. 25.1.2 we have

$$\delta_m = \delta = -2\Phi = \frac{2}{5}S_{mr}. \quad (26.23)$$

Thus

$$\delta_\gamma = \frac{4}{3}\left(\frac{2}{5} - 1\right)S_{mr} = -\frac{4}{5}S_{mr} = -2\delta_m, \quad (26.24)$$

so that

$$\frac{\delta T}{T} = \frac{1}{4}\delta_\gamma + \Phi = -\delta_m = -\frac{2}{5}S_{mr}, \quad (26.25)$$

both terms contributing equally.

In total, the ordinary Sachs-Wolfe effect becomes

$$\frac{\delta T}{T} \approx -\frac{1}{5}\mathcal{R}(\text{rad}) - \frac{2}{5}S_{mr}(\text{rad}) \quad (26.26)$$

Primordial isocurvature perturbations lead to an ordinary Sachs-Wolfe effect that is twice as large as that from primordial adiabatic perturbations (from initial S_{mr} and \mathcal{R} of equal size). In the angular power spectrum C_ℓ this becomes a factor 4.

(Note: NDI mode added in Sec. 26.2.)

26.1 Ordinary Sachs–Wolfe Effect and Primordial Correlations

(Note: NDI is missing from this subsection; it is discussed in Sec. 26.2.) Proper CMB analysis is done in terms of the angular power spectrum C_ℓ (which is the analog of power spectrum on the celestial sphere). But for now, let us do this in coordinate space, as the ordinary Sachs-Wolfe effect refers to just the position on the last scattering (photon decoupling) surface, from which the CMB photons we observe originate. For this purpose we define a covariance matrix of primordial perturbations

$$C_{ij} \equiv \langle A_i(\vec{x}) A_j(\vec{x}) \rangle \quad (26.27)$$

where the $A_i(\vec{x})$ are “smoothed out to leave only the large scales”. The C_{ij} are expectation values, so they are the same for all \vec{x} . Define also the transfer functions $T_{\text{SW},i}$ so that

$$\frac{\delta T(\vec{x})}{T} = T_{\text{SW},i} A_i(\vec{x}). \quad (26.28)$$

Thus we have (in our approximation)

$$T_{\text{SW},\mathcal{R}} = -\frac{1}{5} \quad T_{\text{SW},S_{cr}} = -\frac{2}{5}f_c \quad T_{\text{SW},S_{br}} = -\frac{2}{5}f_b. \quad (26.29)$$

The expectation value of the variance of $\delta T/T$ due to the SW effect is

$$\begin{aligned} \langle (\delta T/T)^2 \rangle_{\text{SW}} &= T_{\text{SW},i} T_{\text{SW},j} \langle A_i A_j \rangle \\ &= \frac{1}{25} \langle \mathcal{R}^2 \rangle + \frac{4f_c^2}{25} \langle S_{cr}^2 \rangle + \frac{4f_b^2}{25} \langle S_{br}^2 \rangle + \frac{4f_c}{25} \langle \mathcal{R} S_{cr} \rangle + \frac{4f_b}{25} \langle \mathcal{R} S_{br} \rangle + \frac{8f_b f_c}{25} \langle S_{cr} S_{br} \rangle \end{aligned} \quad (26.30)$$

Uncorrelated curvature and entropy perturbations add up in the rms sense,

$$\sqrt{\langle (\delta T/T)^2 \rangle_{\text{SW}}} = \sqrt{\langle (\delta T/T)^2 \rangle_{\mathcal{R}} + \langle (\delta T/T)^2 \rangle_{S_{mr}}},$$

whereas positively correlated ones lead to a stronger effect and anticorrelated ones lead to a weaker effect.

Perfectly anticorrelated perturbations could in principle cancel the SW effect completely, i.e., if

$$\mathcal{R}(\text{rad}) = -2S_{mr}(\text{rad}) \quad (26.31)$$

everywhere, then $(\delta T/T)_{\text{SW}} = 0$ everywhere. These would still give rise to matter density perturbations. For them to cancel (at large scales), we need

$$\mathcal{R}(\text{rad}) = -\frac{1}{3}S_{mr}(\text{rad}). \quad (26.32)$$

26.2 Ordinary Sachs–Wolfe Effect in the NDI Mode

This subsection was added later; should be consolidated with the above. In the approximation where the universe is already matter dominated at photon decoupling, the NDI mode has then $\delta = \delta_m = \Psi = \Phi = 0$ at superhorizon scales. However, we do get a $\frac{1}{4}\delta_\gamma^N$ contribution. Since we are considering superhorizon scales, $S_{\nu r}$ and S_{mr} have remained constant. In the NDI mode, $S_{mr} = 0$, so $\delta_m = 0$ implies that $\delta_r = 0$ and we get

$$\delta_r = f_\nu \delta_\nu + f_\gamma \delta_\gamma = 0 \quad \Rightarrow \quad \delta_\gamma = -\frac{f_\nu}{f_\gamma} \delta_\nu \quad (26.33)$$

and

$$S_{\nu r} \equiv \frac{3}{4}(\delta_\nu - \delta_r) = \frac{3}{4}\delta_\nu \quad (26.34)$$

so that

$$\delta_\gamma = -\frac{4f_\nu}{3f_\gamma} S_{\nu r} \quad \text{and} \quad \frac{\delta T}{T} \approx -\frac{f_\nu}{3f_\gamma} S_{\nu r}(\text{rad}). \quad (26.35)$$

Exercise: Modify (26.31) by adding the effect of $S_{\nu r}$ and its correlations.

27 Approximate Treatment of the Smaller Scales

(I follow Dodelson[5], Secs. 7.1.2, 7.3, and 7.4.)

Smaller scales are more difficult to solve than the large $k \ll k_{\text{eq}}$ scales. To be able to obtain some relatively easy results we now make some approximations:

1. We ignore neutrino and photon anisotropy, so that $\Pi = 0$ and $\Psi = \Phi$, which we call the gravitational potential; and assume neutrino adiabaticity (i.e., no NDI mode). In reality neutrino anisotropy has about a 5–10 % effect, as we saw in Sec. 22.1, so ignoring it is not an excellent approximation. On the other hand, the photon distribution remains isotropic until we approach photon decoupling, which happens after matter-radiation equality, so the photon anisotropy will have a smaller effect on the metric. Together these assumptions and approximations mean that there is no difference in the initial values or the evolution equations of photons and neutrinos, so that $\delta_\gamma = \delta_\nu = \delta_r = \delta$ and $v_\gamma = v_\nu = v_r = v$.
2. We ignore baryons completely, and treat matter as if it consisted of CDM only. Since in reality, baryons make about one sixth of all matter, this approximation should be good to about 20 %. (What happens with the baryons is that, because of their tight coupling with photons, baryon perturbations do not grow during the radiation-dominated epoch, instead they oscillate with the radiation; but after photon decoupling they fall into the gravitational wells of the CDM, so that the baryon perturbation becomes almost the same as the CDM perturbation).

Together these approximations mean that there is no difference between photons and neutrinos so that we have a single radiation component. Thus we are *back to the simplified universe of Sec. 19*, so this can be seen as a continuation of that section, *except that we work now fully in the Newtonian gauge*.

With the above approximations the two first Einstein equations (10.18 or C.1) and (10.20 or C.2) become

$$\mathcal{H}^{-1}\Phi' + \Phi + \frac{1}{3}\left(\frac{k}{\mathcal{H}}\right)^2\Phi = -\frac{1}{2}\delta \quad (27.1)$$

$$\mathcal{H}^{-1}\Phi' + \Phi = \frac{3}{2}(1+w)\frac{\mathcal{H}}{k}v, \quad (27.2)$$

and the fluid equations (C.5) become

$$\begin{aligned} \delta'_m + kv_m &= 3\Phi' \\ v'_m + \mathcal{H}v_m &= k\Phi \\ \delta'_r + \frac{4}{3}kv_r &= 4\Phi' \\ v'_r - \frac{1}{4}k\delta_r &= k\Phi. \end{aligned} \quad (27.3)$$

Multiplying the first Einstein equation with $3\mathcal{H}^2 = 8\pi G\rho a^2$ we get

$$k^2\Phi + 3\mathcal{H}(\Phi' + \mathcal{H}\Phi) = -4\pi G a^2 \delta\rho = -4\pi G a^2(\rho_m\delta_m + \rho_r\delta_r). \quad (27.4)$$

Here (27.3 and 27.4) we have five differential equations and five unknowns: δ_m , v_m , δ_r , v_r , and Φ . It is easy to write a simple computer code to solve these equations numerically (**exercise**). For background quantities use the analytic solution from Sec. 19. Because of the approximations, the late time evolution (after photon decoupling) of the radiation component we get is not useful; but we get a reasonable (better than 20%) approximation of the dark matter

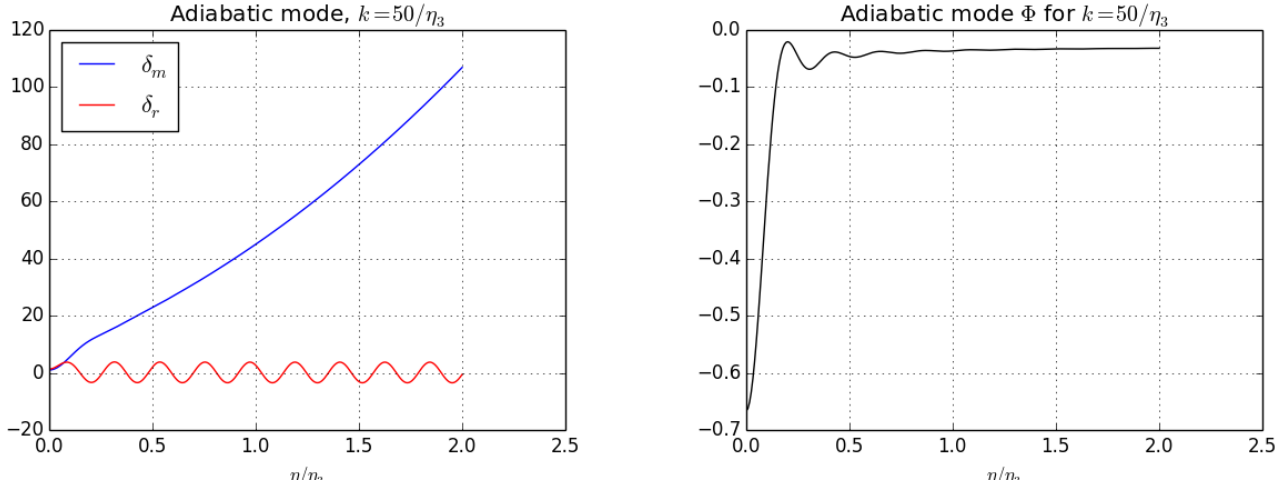


Figure 4: Output from a numerical solution of (27.3) and (27.4) for adiabatic initial conditions. The quantity given by the horizontal axis is η/η_3 . Matter-radiation equality is at $\eta/\eta_3 = \sqrt{2} - 1 \approx 0.4142$. Left: The density perturbations δ_m^N and δ_r^N . Right: The gravitational potential Φ .

evolution. We did not make any assumption about scales, so this will apply for all scales—but because of the approximations we made, the earlier analytic treatment of the large scales is more accurate.

Subtracting the second Einstein equation from the first one we get

$$k^2 \Phi = -4\pi G a^2 \left(\delta\rho + \frac{3\mathcal{H}}{k}(\rho + p)v \right) = -4\pi G a^2 \left[\rho_m \delta_m + \rho_r \delta_r + \frac{3\mathcal{H}}{k}(\rho_m v_m + \frac{4}{3}\rho_r v_r) \right] \quad (27.5)$$

which is not an evolution equation (no time derivatives), but a constraint equation. It can be used instead of the Φ evolution equation (27.4) for some purposes.

Numerical solution. I wrote a `python` script to solve the set of differential equations (27.3 and 27.4) and ran it with adiabatic and isocurvature initial conditions for $k = 50/\eta_3$, a scale which enters during the radiation-dominated epoch ($k_{\text{eq}} = 2\sqrt{2}/\eta_3$ from (19.12)). I started at an initial time $\eta = 10^{-4}$, i.e., $x = 10^{-4}$. In Sec. 19.5 we didn't do the δ_i^N , but we can get the initial conditions from Sec. 22.4 by setting $f_b = f_\nu = 0$. Thus, to get the adiabatic mode I set the initial condition

$$\begin{aligned} \delta_m &= 1 + \frac{1}{8}\omega\eta = 1 + \frac{1}{4}x \\ v_m &= -\frac{1}{3}k\eta + \frac{1}{24}\omega k\eta^2 = -\frac{1}{3}k\eta_3 x + \frac{1}{12}k\eta_3 x^2 \\ \delta_r &= \frac{4}{3}\delta_m \\ v_r &= v_m \\ \Phi &= -\frac{2}{3} + \frac{1}{24}\omega\eta = -\frac{2}{3} + \frac{1}{12}x \end{aligned} \quad (27.6)$$

and to get the isocurvature mode I set the initial condition

$$\begin{aligned} \delta_m &= 1 - \frac{3}{8}\omega\eta = 1 - \frac{3}{4}x \\ v_m &= -\frac{1}{24}\omega k\eta^2 = -\frac{1}{12}k\eta_3 x^2 \\ \delta_r &= \frac{1}{2}\omega\eta = x \\ v_r &= -\frac{1}{8}\omega k\eta^2 = -\frac{1}{4}k\eta_3 x^2 \\ \Phi &= -\frac{1}{8}\omega\eta = -\frac{1}{4}x. \end{aligned} \quad (27.7)$$

Output from these two runs is shown in Figs. 4 and 5.

My `python` script⁶⁴:

⁶⁴I learned coding using programming languages such as Algol and Fortran. I am not good in python, so this is probably not a good example of python coding.

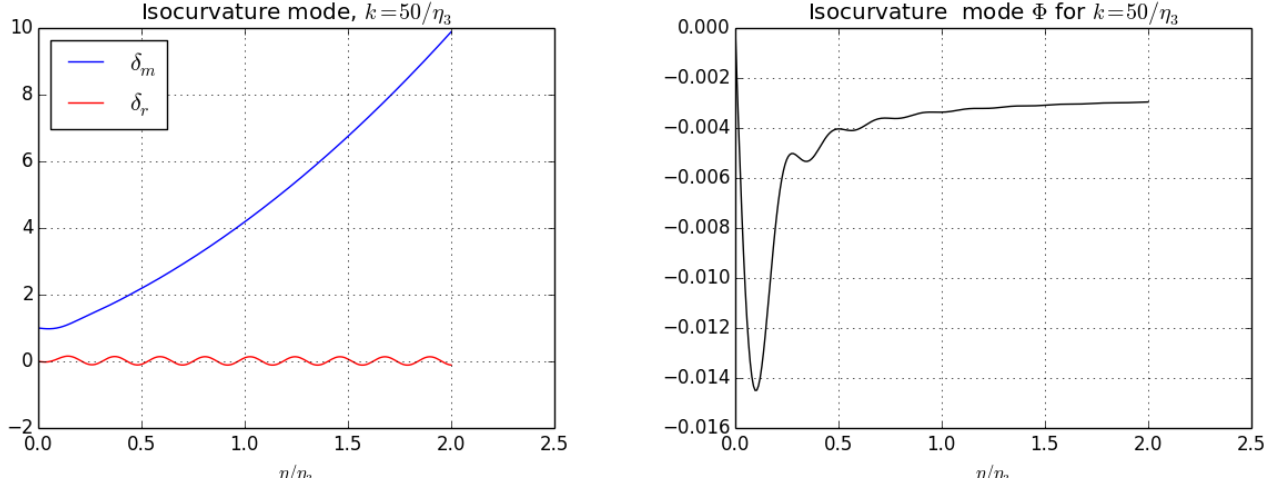


Figure 5: Same as Fig. 4, but for isocurvature initial conditions. Note how, compared to the adiabatic mode, the matter density perturbation misses the initial growth by about a factor of 10 during the radiation dominated era.

```

from scipy.integrate import odeint
from numpy import loadtxt
from pylab import figure, plot, xlabel, grid, hold, legend, title, savefig
from matplotlib.font_manager import FontProperties

def Phid(x, k, delta_m, delta_r, Phi):
    d = -(1./x)*(1.+x)/(1.+0.5*x)*Phi - (1./(1.+x))*delta_m - (1./((2*x+x**2)*(1.+x)))*delta_r \
        - (x/3.)*((1.+0.5*x)/(1.+x))*k**2*Phi
    return d

def vectorfield(w,x,p):
    delta_m, v_m, delta_r, v_r, Phi = w
    k, dummy = p
    f = [-k*v_m + 3*Phid(x, k, delta_m, delta_r, Phi),
        -(1./x)*((1.+x)/(1.+0.5*x))*v_m + k*Phi,
        -(4./3.)*k*v_r + 4*Phid(x, k, delta_m, delta_r, Phi),
        +0.25*k*delta_r + k*Phi,
        Phid(x, k, delta_m, delta_r, Phi)]
    return f

# Parameter values
k = 50.
dummy = 1.

# Initial conditions
x_in = 0.0001

# Adiabatic mode
delta_m = 1.+0.25*x_in
v_m = -(1./3.)*k*x_in + (1./12.)*k*x_in**2
delta_r = 4./3. + x_in/3.
v_r = v_m
Phi = -(2./3.)*x_in/12.

# Isocurvature mode
# delta_m = 1. - 0.75*x_in
# v_m = -(1./12.)*k*x_in**2
# delta_r = x_in

```

```

# v_r = -0.25*k*x_in**2
# Phi = -0.25*x_in

# ODE solver parameters
abserr = 1.0e-7
relerr = 1.0e-5
stoptime = 2.0
numpoints = 250

x = [x_in + stoptime * float(i) / (numpoints - 1) for i in range(numpoints)]

p = [k, dummy]
w0 = [delta_m, v_m, delta_r, v_r, Phi]

wsol = odeint(vectorfield, w0, x, args=(p,), atol=abserr, rtol=relerr)

with open('simpl_univ_k_50.dat', 'w') as f:
    for x1, w1 in zip(x, wsol):
        print >> f, x1, w1[0], w1[1], w1[2], w1[3], w1[4]

x, delta_m, v_m, delta_r, v_r, Phi = loadtxt('simpl_univ_k_50.dat', unpack=True)

figure(1, figsize=(6, 4.5))
xlabel(r'$\eta/\eta_3$')
grid(True)
hold(True)
lw = 1
plot(x, delta_m, 'b', linewidth=lw)
plot(x, delta_r, 'r', linewidth=lw)
legend((r'$\delta_m$', r'$\delta_r$'), loc=2, prop=FontProperties(size=16))
title(r'Adiabatic mode, $k = 50/\eta_3$')
savefig('simpl_univ_adi_k_50.png', dpi=100)
# title(r'Isocurvature mode, $k = 50/\eta_3$')
# savefig('simpl_univ_iso_k_50.png', dpi=100)

figure(2, figsize=(6, 4.5))
xlabel(r'$\eta/\eta_3$')
grid(True)
hold(True)
lw = 1
plot(x, Phi, 'k', linewidth=lw)
title(r'Adiabatic mode $\Phi$ for $k = 50/\eta_3$')
savefig('simpl_univ_adi_Phi_k_50.png', dpi=100)
# title(r'Isocurvature mode $\Phi$ for $k = 50/\eta_3$')
# savefig('simpl_univ_iso_Phi_k_50.png', dpi=100)

```

27.1 Small Scales during Radiation-Dominated Epoch

Next we consider the small scales, $k \gg k_{\text{eq}}$, i.e., scales that enter the horizon during the radiation-dominated epoch. Now we can make the approximation of radiation domination while we follow the perturbations through the horizon. In this subsection we consider only the part of the evolution during the radiation-dominated epoch. Radiation domination means that we can ignore the matter components while we calculate the evolution of the radiation components and the metric perturbations Φ .⁶⁵

⁶⁵In this section we follow Chapter 7 of Dodelson[5]. Note that the multipole moments of the photon brightness function Θ_ℓ in Dodelson's notation are related to the Θ_ℓ^m of CMB Physics 2007 and to perturbations of the photon energy tensor by $\Theta_0 \equiv \Theta_0^0 \equiv \frac{1}{4}\delta_\gamma$ and $\Theta_1 \equiv \frac{1}{3}\Theta_1^0 \equiv \frac{1}{3}v_\gamma$.

For the background we have now

$$y \equiv \frac{a}{a_{\text{eq}}} = 2 \frac{\eta}{\eta_3} \ll 1 \quad \mathcal{H} = \frac{1}{\eta} \quad \text{and} \quad w = c_s^2 = \frac{1}{3} \quad (27.8)$$

and for the perturbations (of the total fluid)

$$\delta p = \frac{1}{3} \delta \rho. \quad (27.9)$$

27.1.1 Radiation

The fluid perturbation equations (C.5) for radiation become

$$\eta \delta' + \frac{4}{3} k \eta v = 4 \eta \Phi' \quad (27.10)$$

$$\eta v' - \frac{1}{4} k \eta \delta = k \eta \Phi. \quad (27.11)$$

The Einstein equations (27.1) and (27.2) become

$$\eta \Phi' + \Phi + \frac{1}{3} (k \eta)^2 \Phi = -\frac{1}{2} \delta \quad (27.12)$$

$$\eta \Phi' + \Phi = \frac{2}{k \eta} v, \quad (27.13)$$

which immediately give

$$\delta = -\frac{4}{k \eta} v - \frac{2}{3} (k \eta)^2 \Phi, \quad (27.14)$$

which we use to eliminate δ from (27.11) so that it becomes

$$\eta v' + v = \left[1 - \frac{1}{6} (k \eta)^2 \right] k \eta \Phi. \quad (27.15)$$

We then derivate (27.13) to arrive at

$$\Phi'' + \frac{2}{\eta} \Phi' = \frac{2}{k \eta^3} (\eta v' - v) = \frac{2}{k \eta^3} (\eta v' + v) - \frac{4}{k \eta^3} v = -\frac{1}{3} k^2 \Phi - \frac{2}{\eta} \Phi', \quad (27.16)$$

so that the final differential equation for Φ in the radiation-dominated era is

$$\boxed{\Phi'' + \frac{4}{\eta} \Phi' + \frac{1}{3} k^2 \Phi = 0.} \quad (27.17)$$

This is the radiation-dominated universe case, which we already did in Sec. 15, where we got the growing mode solution

$$\boxed{\Phi(\eta) = \frac{1}{\eta} u = 3 \Phi(\text{rad}) \frac{\sin\left(\frac{k\eta}{\sqrt{3}}\right) - \frac{k\eta}{\sqrt{3}} \cos\left(\frac{k\eta}{\sqrt{3}}\right)}{\left(\frac{k\eta}{\sqrt{3}}\right)^3}} \quad (27.18)$$

to (27.17).

For superhorizon scales ($k\eta \ll 1$, i.e., at early times $\eta \ll k^{-1}$), we have $\Phi(\eta) \approx \Phi(\text{rad}) = \text{const.}$

For subhorizon scales, i.e., at later times, after horizon entry ($\eta \gg k^{-1}$ so that $k\eta \gg 1$), the cosine part dominates, so that for the gravitational potential we have

$$\Phi(\eta) \approx -9 \Phi(\text{rad}) \frac{\cos(k\eta/\sqrt{3})}{(k\eta)^2}, \quad (27.19)$$

which *oscillates* with frequency $\omega = 2\pi f = (1/\sqrt{3})k = c_s k$ and a *decaying* amplitude

$$\frac{9\Phi(\text{rad})}{(k\eta)^2}. \quad (27.20)$$

The radiation density δ and velocity v perturbations at these late (but still radiation-dominated) times *oscillate with constant amplitude*:

$$\begin{aligned} v &= \frac{1}{2}(k\eta^2\Phi' + k\eta\Phi) \approx \frac{9}{2}\Phi(\text{rad})c_s \sin(c_s k\eta) \\ \delta &\approx -\frac{2}{3}(k\eta)^2\Phi \approx 6\Phi(\text{rad}) \cos(c_s k\eta). \end{aligned}$$

27.1.2 Matter

While matter is subdominant during the radiation-dominated epoch, we still want to know what happens to it during that epoch, since it becomes important later.

The matter fluid equations (from 27.3ab) are thus

$$\eta\delta'_m + k\eta v_m = 3\eta\Phi' \quad (27.21)$$

$$\eta v'_m + v_m = k\eta\Phi. \quad (27.22)$$

Derivating the first one and then using the second to get rid of v_m gives the *matter perturbation equation*

$$\delta''_m + \frac{1}{\eta}\delta'_m = 3\Phi'' + \frac{3}{\eta}\Phi' - k^2\Phi \equiv F(k, \eta), \quad (27.23)$$

where the *source function* $F(k, \eta)$ is a known function that we get from the solution (27.18). The general solution will be the solution

$$\delta_m = C_1 + C_2 \ln k\eta \quad (27.24)$$

of the homogeneous equation plus a special solution of the full equation.

A special solution (**exercise**) is

$$\delta_m(\eta) = - \int_0^\eta d\eta' F(k, \eta') \eta' (\ln k\eta' - \ln k\eta) \quad (27.25)$$

(constructed from the homogeneous equation solutions and the source function). Thus the general solution will be

$$\boxed{\delta_m(\eta) = C_1 + C_2 \ln k\eta - \int_0^\eta d\eta' F(k, \eta') \eta' (\ln k\eta' - \ln k\eta).} \quad (27.26)$$

Note first that the source function and its integral are proportional to the initial value of Φ , i.e., $\Phi_{\vec{k}}(\text{rad})$. Otherwise we care now only about their limiting behavior as $k\eta \ll 1$ and $k\eta \gg 1$.

As $\eta \rightarrow 0$ the integral $\rightarrow 0$ (**exercise**). For the solution to stay finite, we thus must have $C_2 = 0$ (i.e., we reject the decaying mode). Thus

$$C_1 = \delta_m(0) = \delta_m(\text{rad}) = -\frac{3}{2}\Phi_{\vec{k}}(\text{rad}) + S_{m,\vec{k}}(\text{rad}), \quad (27.27)$$

where $S_m = \delta_m - \frac{4}{3}\delta_r$ is the matter entropy perturbation. For the rest of this subsection we consider only the adiabatic mode, so $S_{m,\vec{k}}(\text{rad}) = 0$ and

$$C_1 = \delta_m(0) = -\frac{3}{2}\Phi_{\vec{k}}(\text{rad}). \quad (27.28)$$

After horizon entry Φ , and therefore also $F(k, \eta)$, decays. Thus, if we rewrite the integral term in (27.26) as

$$-\int_0^\eta d\eta' F(k, \eta') \eta' \ln k\eta' + \ln k\eta \int_0^\eta d\eta' F(k, \eta') \eta' \rightarrow B_1 \Phi_{\vec{k}}(\text{rad}) + B_2 \Phi_{\vec{k}}(\text{rad}) \ln k\eta, \quad (27.29)$$

both integrals stop changing as a function of the upper limit η for $k\eta \gg 1$, and we can capture the effect of $F(k, \eta)$ in these two constants

$$B_1 \Phi_{\vec{k}}(\text{rad}) \equiv -\int_0^\infty d\eta' F(k, \eta') \eta' \ln k\eta' \quad \text{and} \quad B_2 \Phi_{\vec{k}}(\text{rad}) \equiv \int_0^\infty d\eta' F(k, \eta') \eta', \quad (27.30)$$

i.e.,

$$\delta_m \approx \left(-\frac{3}{2} + B_1 + B_2 \ln k\eta\right) \Phi_{\vec{k}}(\text{rad}) \quad \text{for} \quad k\eta \gg 1. \quad (27.31)$$

Rewrite

$$-\frac{3}{2} + B_1 + B_2 \ln k\eta \equiv -A \ln(Bk\eta) = -A \ln B - A \ln(k\eta), \quad (27.32)$$

so that

$$\begin{aligned} A &= -B_2 \approx 9.0 \\ B &= \exp[(B_1 - \frac{3}{2})/B_2] \approx 0.62 \end{aligned} \quad (27.33)$$

where the numerical results can be obtained by numerical integration of the solution (27.18). Thus we have the final result

$$\boxed{\delta_m \approx -A \Phi_{\vec{k}}(\text{rad}) \ln(Bk\eta) \approx -9.0 \Phi_{\vec{k}}(\text{rad}) \ln(0.62k\eta) \quad (k\eta \gg 1)} \quad (27.34)$$

for the growth of matter perturbations inside the horizon during the radiation-dominated epoch. Since in this epoch $\mathcal{H} = 1/\eta$ and $y \propto \eta$ we can write the $k\eta$ also as

$$k\eta = \frac{y}{y_k} \quad (\text{rad.dom}) \quad (27.35)$$

where $y_k \ll 1$ is the value of y at horizon entry ($k = \mathcal{H}$).

After horizon entry (as $a \propto \eta$),

$$\frac{d\delta_m}{d \ln a} = \frac{d\delta_m}{d \ln(k\eta)} \rightarrow -A \Phi_{\vec{k}}(\text{rad}) = -9.0 \Phi_{\vec{k}}(\text{rad}), \quad (27.36)$$

i.e., for each e -folding δ_m picks another $-9\Phi_{\vec{k}}(\text{rad})$ (for each 10-folding another $-20.3\Phi_{\vec{k}}(\text{rad})$).

A more detailed look at the evolution of δ_m through the horizon would show that the oscillations in Φ (see Eq. 27.18) are reflected in $F(k, \eta)$ and cause small oscillations around (27.34) that gradually fade into insignificance. For radiation perturbations we have gravity contending with pressure, causing the radiation density perturbation to oscillate around zero with a constant amplitude. Cold dark matter sees only the gravity (caused by the radiation): it begins to fall towards the initial gravity wells; when the gravity wells begin to oscillate, these will alternately slow down and accelerate this fall, but since this oscillation amplitude is decreasing it is not able to stop the fall—the initial kick the CDM got before the first reversal of Φ carries it on towards the bottom of the initial potential wells. Thus the CDM density perturbation keeps growing, albeit only logarithmically.

27.2 Small Scales to the Matter-Dominated Epoch

In Sec. 27.1 we found that during the radiation-dominated epoch the radiation (and baryon) density perturbation δ_r oscillates without growing, whereas the CDM perturbation δ_c grows logarithmically. This will have the effect that $\delta\rho_m = \delta_m\rho_m$ will become larger than $\delta\rho_r = \delta_r\rho_r$ while ρ_m is still smaller than ρ_r , i.e., the perturbation becomes matter dominated at some $y = y_{\text{deq}} \ll 1$ while the background is still radiation dominated. The gravitational potential Φ will then be determined by matter perturbations. In this section we solve the evolution of matter perturbations in the approximation that we ignore the contribution of the radiation perturbation, while we use the background solution that includes both matter and radiation contributions. This leaves a gap between Sec. 27.1 and this section for going from $\delta_m \ll \delta_r$ to $\delta_m \gg \delta_r$. We will bridge the gap at the end of this section by matching the two solutions.

We work now in the combined 1) subhorizon $k \gg \mathcal{H}$ and 2) perturbations matter-dominated $\delta\rho_m \gg \delta\rho_r$ ($y \gg y_{\text{deq}}$) limit but with the full matter+radiation background solution. The relevant perturbation equations, from Eqs. (27.3) and (27.5), are now

$$\delta'_m + kv_m = 3\Phi' \quad (27.37)$$

$$v'_m + \mathcal{H}v_m = k\Phi \quad (27.38)$$

$$k^2\Phi = -4\pi G a^2 \rho_m \delta_m = 4\pi G a^2 \rho \frac{y}{y+1} \delta_m = \frac{3}{2} \mathcal{H}^2 \frac{y}{y+1} \delta_m \quad (27.39)$$

Derivating (27.37) and using (27.38) and (27.39) we get

$$\delta''_m + \mathcal{H}\delta'_m = 3\Phi'' + 3\mathcal{H}\Phi' - k^2\Phi \quad (27.40)$$

or

$$\mathcal{H}^{-2}\delta''_m + \mathcal{H}^{-1}\delta'_m = 3\mathcal{H}^{-2}\Phi'' + 3\mathcal{H}^{-1}\Phi' - \left(\frac{k}{\mathcal{H}}\right)^2 \Phi, \quad (27.41)$$

where on the rhs the last term dominates as $k \gg \mathcal{H}$. We thus have

$$\mathcal{H}^{-2}\delta''_m + \mathcal{H}^{-1}\delta'_m = \frac{3}{2} \frac{y}{y+1} \delta_m. \quad (27.42)$$

We now change to using y as the time coordinate (see Appendix B) to get the *Meszaros equation*

$$\boxed{y^2 \frac{d^2 \delta_m}{dy^2} + \frac{3y+2}{2(y+1)} \frac{d\delta_m}{dy} = \frac{3}{2} \frac{y}{y+1} \delta_m.} \quad (27.43)$$

This equation applies when $y \gg y_{\text{deq}}$ and $y \gg y_k$ and will remain valid until dark energy begins to affect the expansion of the universe. It is a second order differential equation so it will have two independent solutions. They are (**exercise**)

$$\begin{aligned} D_1(y) &= y + \frac{2}{3} \\ D_2(y) &= \left(y + \frac{2}{3}\right) \ln \left(\frac{\sqrt{1+y} + 1}{\sqrt{1+y} - 1} \right) - 2\sqrt{1+y}. \end{aligned} \quad (27.44)$$

Their late-time ($y \gg 1$) behavior is (**exercise**)

$$\begin{aligned} D_1(y) &\propto y \\ D_2(y) &\propto y^{-3/2}, \end{aligned} \quad (27.45)$$

so $D_1(y)$ is a growing mode and $D_2(y)$ is a decaying mode.

The general solution is thus

$$\delta_m(y) = C_1 D_1(y) + C_2 D_2(y). \quad (27.46)$$

To find C_1 and C_2 we match this solution with the $y_k \ll y \ll y_{\delta\text{eq}}$ solution (27.34)

$$\delta_m = -A\Phi_{\vec{k}}(\text{rad}) \ln \left(B \frac{y}{y_k} \right). \quad (27.47)$$

at some time $y = y_m$ which is near $y_{\delta\text{eq}}$, or at least $y_k \ll y_m \ll 1$. Neither solution is valid near $y_{\delta\text{eq}}$, but we expect that the behavior near $y_{\delta\text{eq}}$ is not dramatically different. (We want to check how much the matching depends on y_m so therefore we do not fix it to be, e.g., equal to $y_{\delta\text{eq}}$.)

By matching we mean that we require both solutions to give the same value for $\delta_m(y_m)$ and $d\delta_m/dy(y_m)$, i.e., we solve (**exercise**) C_1 and C_2 from the two matching conditions:

$$\begin{aligned} -A\Phi_{\vec{k}}(\text{rad}) \ln \left(B \frac{y_m}{y_k} \right) &= C_1 D_1(y_m) + C_2 D_2(y_m) \\ -\frac{A\Phi_{\vec{k}}(\text{rad})}{y_m} &= C_1 \frac{dD_1}{dy}(y_m) + C_2 \frac{dD_2}{dy}(y_m). \end{aligned} \quad (27.48)$$

Actually, if we are only interested in the late-time behavior, we only need to solve C_1 , since the decaying mode will not be important later. The matching will initially give a combination of the growing mode and the decaying mode, but the decaying mode will decay away after some time. C_1 will pick up some y_m -dependence, but this disappears in the limit $y_m \ll 1$ and we get the approximate result (**exercise**)

$$C_1 = \frac{3}{2} \left[3 - \ln \left(\frac{4B}{y_k} \right) \right] A\Phi_{\vec{k}}(\text{rad}). \quad (27.49)$$

Thus our final result, the density perturbation growing mode at $y \gg y_{\delta\text{eq}}$, is

$$\delta_m \approx C_1 D_1(y) \approx -\frac{3}{2} A\Phi_{\vec{k}}(\text{rad}) \ln \left(\frac{4Be^{-3}}{y_k} \right) \left(y + \frac{2}{3} \right). \quad (27.50)$$

27.3 Transfer Function

The transfer function $T_\Phi(k)$ relates the gravitational potential during the matter-dominated epoch to its primordial value. For small scales, we get it from Eq. (27.50). Since now $k \ll \mathcal{H}$ the last term on the lhs of (27.1) dominates and using

$$\delta\rho \approx \delta\rho_m \Rightarrow \delta \approx \frac{y}{y+1} \delta_m \quad (27.51)$$

we have

$$\Phi(y) = -\frac{3}{2} \left(\frac{\mathcal{H}}{k} \right)^2 \delta = \frac{9}{4} A \left(\frac{\mathcal{H}}{k} \right)^2 \Phi_{\vec{k}}(\text{rad}) \ln \left(\frac{4Be^{-3}}{y_k} \right) \frac{y(y + \frac{2}{3})}{y+1}. \quad (27.52)$$

This equation has both time-dependent and scale-dependent quantities. We want to express these in terms of $y \equiv a/a_{\text{eq}}$ and k/k_{eq} . (Note that we have not fixed the normalization of the scale factor a and the normalization of k depends on the normalization of a , but these ratios are dimensionless quantities independent of this normalization.)

So what are \mathcal{H} and y_k ? Use the result (19.16) from Sec. 19,

$$\mathcal{H}^2 = \frac{1+y}{y^2} \frac{\mathcal{H}_{\text{eq}}^2}{2} = \frac{1+y}{y^2} \frac{k_{\text{eq}}^2}{2} \quad (27.53)$$

(the definition of k_{eq} is $k_{\text{eq}} = \mathcal{H}_{\text{eq}}$.) This gives

$$\left(\frac{\mathcal{H}}{k}\right)^2 = \frac{1+y}{2y^2} \left(\frac{k_{\text{eq}}}{k}\right)^2 \quad (27.54)$$

and for y_k , i.e., y at the time the scale k enters the horizon,

$$k^2 = \mathcal{H}_k^2 = \frac{1+y_k}{y_k^2} \frac{k_{\text{eq}}^2}{2} \approx \frac{k_{\text{eq}}^2}{2y_k^2} \Rightarrow y_k = \frac{1}{\sqrt{2}} \frac{k_{\text{eq}}}{k} \quad (27.55)$$

(since this happens during the radiation-dominated epoch, $y_k \ll 1$).

Thus Eq. (27.52) is

$$\Phi(y) = \frac{9}{8} \frac{y + \frac{2}{3}}{y} \left(\frac{k_{\text{eq}}}{k}\right)^2 A \ln \left(4\sqrt{2} B e^{-3} \frac{k}{k_{\text{eq}}}\right) \Phi_{\vec{k}}(\text{rad}), \quad (27.56)$$

where in the matter-dominated epoch $(y + \frac{2}{3})/y \approx 1$, so that Φ becomes time-independent. The small-scale behavior of the transfer function $T_\Phi(k)$ is thus

$$T_\Phi(k) = \frac{9}{8} \left(\frac{k_{\text{eq}}}{k}\right)^2 A \ln \left(4\sqrt{2} B e^{-3} \frac{k}{k_{\text{eq}}}\right) = 12.6 \left(\frac{k_{\text{eq}}}{k}\right)^2 \ln \left(0.17 \frac{k}{k_{\text{eq}}}\right). \quad (k \gg k_{\text{eq}}) \quad (27.57)$$

(Note that the logarithm is negative if $k \lesssim 6k_{\text{eq}}$; Eq. (27.57) is not supposed to apply for this small k .)

In the literature this result is usually given as

$$T(k) = \frac{5}{4} \left(\frac{k_{\text{eq}}}{k}\right)^2 A \ln \left(4\sqrt{2} B e^{-3} \frac{k}{k_{\text{eq}}}\right) = 11.3 \left(\frac{k_{\text{eq}}}{k}\right)^2 \ln \left(0.17 \frac{k}{k_{\text{eq}}}\right), \quad (k \gg k_{\text{eq}}) \quad (27.58)$$

since in this simplified universe the large-scale transfer function is

$$T_\Phi(k) = \frac{9}{10} \quad (k \ll k_{\text{eq}}) \quad (27.59)$$

(\mathcal{R} stays constant but Φ changes from its radiation-dominated value of $-\frac{2}{3}\mathcal{R}$ to the matter-dominated value $-\frac{3}{5}\mathcal{R}$, see Sec. 16.5.1) and this transfer function is customarily normalized to unity at large scales, i.e.

$$T(k) \equiv \frac{T_\Phi(k)}{T_\Phi(k \ll k_{\text{eq}})}. \quad (27.60)$$

At scales closer to k_{eq} the transfer function changes smoothly from the $k \ll k_{\text{eq}}$ solution to the $k \gg k_{\text{eq}}$ solution. To find how, one has to solve the equations (27.3 and 27.4) numerically. (The logarithm in Eq. (27.57) changes sign at $k = 5.9k_{\text{eq}}$, but the $k \gg k_{\text{eq}}$ result is supposed to apply only at larger k than this, so the transfer function is everywhere positive.) Bardeen, Bond, Kaiser, and Szalay [12] gives a fitting formula, the BBKS transfer function

$$T(k) = \frac{\ln(1 + 2.34q)}{2.34q} \frac{1}{[1 + 3.89q + (16.1q)^2 + (5.64q)^3 + (6.71q)^4]^{1/4}}, \quad (27.61)$$

where $q = 0.073(k/k_{\text{eq}})$, to such numerical results. See Fig. 6 for these results. The slope of the BBKS transfer function is

$$\frac{d \ln T}{d \ln q} = \frac{2.34q}{(1 + 2.34q) \ln(1 + 2.34q)} - \frac{1}{4} \frac{3.89q + 2(16.1q)^2 + 3(5.64q)^3 + 4(6.71q)^4}{1 + 3.89q + (16.1q)^2 + (5.64q)^3 + (6.71q)^4} - 1. \quad (27.62)$$

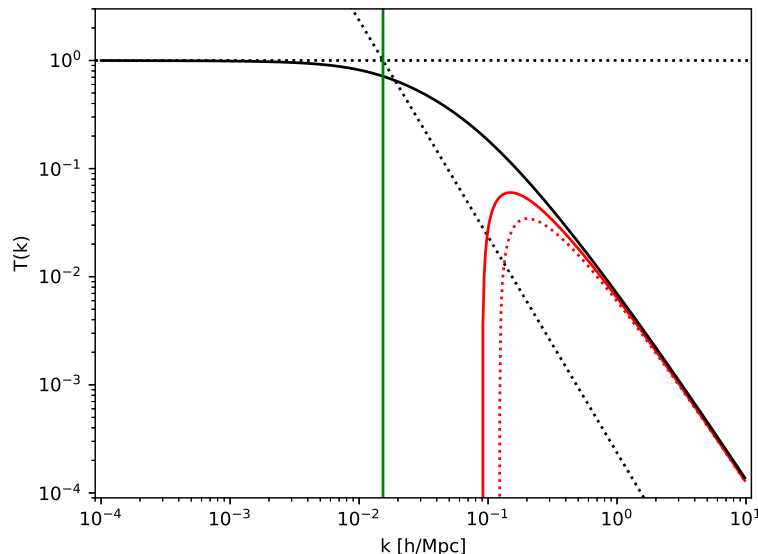


Figure 6: Transfer function $T(k)$ for CDM, adiabatic primordial fluctuations. The black curve is the BBKS transfer function (27.61), the red curve is the small-scale approximate analytical result (27.58) (the dotted red curve is Dodelson (7.69), which for some reason has slightly different numerical factors), the two black dotted lines correspond to $T(k) \equiv 1$ and $T(k) = (k/k_{\text{eq}})^2$, and the green vertical line gives $k = k_{\text{eq}}$. The k scale is for a reference model with $\Omega_m = 0.3$, $h = 0.7$, for which $k_{\text{eq}} = 0.0153 h/\text{Mpc} = 1/(65 h^{-1}\text{Mpc})$.

For an accurate (linear perturbation theory) calculation of the true transfer function of the real universe, there are publicly available computer programs, such as CMBFAST, CAMB⁶⁶, and CLASS⁶⁷; you give your favorite values for the cosmological parameters as input. They represent the current state of the art. The exact result can be given in form of the transfer function $T(k)$ we defined above. We show in Fig. 7 a transfer function calculated with CAMB. The effect of *baryon acoustic oscillations* (i.e., the oscillations of $\delta_{b\gamma}$ before decoupling, which leave a trace in δ_b) shows up as a small-amplitude wavy pattern in the $k > k_{\text{eq}}$ part of the transfer function, since different modes k were at a different phase of the oscillation when that ended around t_{dec} .

Here we studied only the adiabatic mode. The logarithmic part is due to the growth of the CDM perturbation while the universe is still radiation dominated, i.e., δ_m was growing due to the potential Φ due to the radiation perturbation. In the isocurvature mode this effect will be missing as the potential (now due to δ_m) will only appear once the universe becomes matter dominated.

Should add here also the calculation of the isocurvature mode. This should be much easier than the adiabatic mode. Is it too much for a homework problem? **Exercise:** Find the isocurvature mode transfer function $T_{\Phi S_m}$ for small scales.

⁶⁶<https://camb.info/>

⁶⁷<http://class-code.net/>

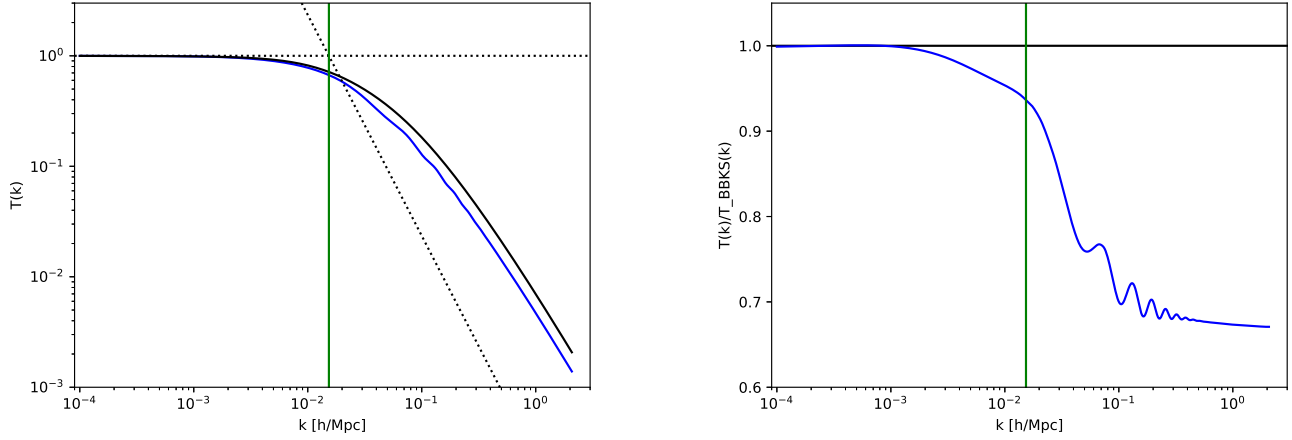


Figure 7: Left: Transfer function $T(k)$ calculated with CAMB (blue curve) for adiabatic primordial fluctuations in the flat Λ CDM model with $\omega_b = 0.023$, $\omega_c = 0.124$, $h = 0.7$ (so that $\Omega_m = 0.3$), and massless neutrinos (a neutrino mass 0.06 eV for one neutrino species changes the transfer function by less than the width of the curve). The black curve is the BBKS transfer function (27.61), the black dotted lines and the green vertical line are as in Fig. 6. The main difference from BBKS is due to baryons. Right: The ratio (blue) of the $T(k)$ from CAMB to the BBKS transfer function.

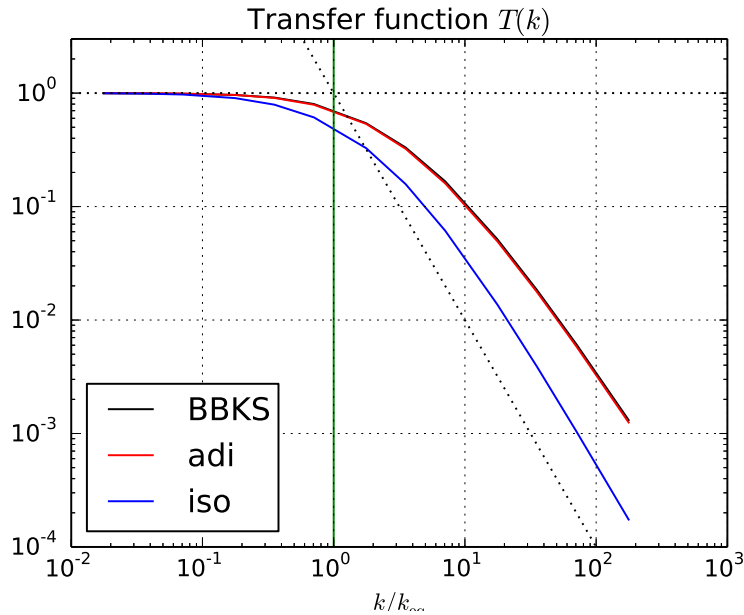


Figure 8: Transfer functions $T(k)$ for matter, adiabatic and isocurvature modes, calculated using a modified version of the python script from the beginning of this section (added a loop over different k values and compared the final Φ value to what that was for the lowest k (“large scales”)). The sloping black dotted line corresponds to $T(k) = (k/k_{\text{eq}})^2$, and the green vertical line gives $k = k_{\text{eq}}$. The adiabatic $T(k)$ agrees with the BBKS transfer function to about 5%; I suppose the up-to-5% difference is due to neutrinos, which BBKS includes. For small scales the isocurvature $T(k)$ is smaller than the adiabatic one, because in the isocurvature mode we miss the initial growth during the radiation-dominated epoch.

28 Acoustic Oscillations of the Baryon-Photon Fluid

In the preceding discussion of subhorizon evolution of small scales (those that enter during radiation domination) our main focus was on the CDM, and we ignored baryons, so that the effect on the CDM of the baryon-photon fluid oscillations before photon decoupling was obtained in the approximation where we had just the photon fluid oscillating.

Let us discuss the baryon-photon fluid without ignoring baryons. Now our main interest is in what happens to photons until photons decouple from baryons, since this is essential for the CMB. Thus we now care about the effect of the baryons on the photons, but not so much about what happens to the baryons, since after decoupling the baryons will anyway fall into the CDM potential wells, so that $\delta_b \rightarrow \delta_c$.

From (21.5) we have the fluid equations for baryons and photons. We write now explicitly also the collision terms in the baryon and photon velocity equations, which we get from CMB Physics, and which are proportional to the velocity difference $v_\gamma - v_b$:

$$\delta'_b = -kv_b + 3\Psi' \quad (28.1)$$

$$v'_b = -\mathcal{H}v_b + k\Phi + \frac{1}{R\eta_{\text{coll}}}(v_\gamma - v_b) \quad (28.2)$$

$$\delta'_\gamma = -\frac{4}{3}kv_\gamma + 4\Psi' \quad (28.3)$$

$$v'_\gamma = \frac{1}{4}k\delta_\gamma - \frac{1}{6}k\Pi_\gamma + k\Phi + \frac{1}{\eta_{\text{coll}}}(v_b - v_\gamma) , \quad (28.4)$$

where

$$R \equiv \frac{3\rho_b}{4\rho_\gamma} \propto a \quad \Rightarrow \quad R' = \mathcal{H}R \quad (28.5)$$

and

$$\eta_{\text{coll}} = \frac{1}{an_e\sigma_T} \quad (28.6)$$

is the mean (conformal) time between collisions with electrons for a photon. Here a is the scale factor, n_e is the number density of free electrons, and $\sigma_T = (8\pi/3)(\alpha^2/m_e^2) = 66.5 \text{ fm}^2$ is the Thomson cross section (α is the fine structure constant and m_e the electron mass). Since the collision cross section is inversely proportional to the square of the charged particle mass, we can ignore the contribution of protons to photon scattering.

We work in the *tight coupling* limit: $\eta_{\text{coll}} \ll \mathcal{H}^{-1}$, which will keep $v_\gamma - v_b$ and Π_γ small. We ignore the latter, but keep the former, but only to lowest order in the tight coupling limit. This allows us to write (28.2) as

$$v_b = v_\gamma - R\eta_{\text{coll}}(v'_b + \mathcal{H}v_b - k\Phi) \approx v_\gamma - R\eta_{\text{coll}}(v'_\gamma + \mathcal{H}v_\gamma - k\Phi) \quad (28.7)$$

and (28.4) becomes

$$v'_\gamma \approx \frac{1}{4}k\delta_\gamma + k\Phi - R(v'_\gamma + \mathcal{H}v_\gamma - k\Phi) \quad (28.8)$$

or

$$[(1+R)v_\gamma]' = k\left[\frac{1}{4}\delta_\gamma + (1+R)\Phi\right] . \quad (28.9)$$

From here on, we write the above tight-coupling approximations as equalities. The tight-coupling approximation fails when photons decouple at t_{dec} .

Multiplying (28.3) by $1+R$ and derivating gives

$$\delta''_\gamma + \frac{R'}{1+R}\delta'_\gamma + c_s^2 k^2 \delta_\gamma = -\frac{4}{3}k^2\Phi + 4\Psi'' + \frac{4R'}{1+R}\Psi' , \quad (28.10)$$

where

$$c_s^2 = \frac{1}{3(1+R)} \quad (28.11)$$

is the speed of sound in the baryon-photon fluid. Defining the temperature perturbation

$$\Theta \equiv \frac{1}{4}\delta_\gamma, \quad (28.12)$$

(28.10) becomes⁶⁸

$$\Theta'' + \frac{R'}{1+R}\Theta' + c_s^2 k^2 \Theta = F_{\vec{k}}(\eta), \quad (28.14)$$

where

$$F_{\vec{k}}(\eta) \equiv \frac{1}{3}k^2\Phi + \Psi'' + \frac{R'}{1+R}\Psi'. \quad (28.15)$$

Equation (28.14) has the form of harmonic oscillator with a friction term due to change of R and a forcing term $F_{\vec{k}}(\eta)$ due to the potentials. Without the friction and forcing terms, and ignoring the time dependence of c_s we would get sinusoidal oscillations $\Theta_{\vec{k}} \propto \exp(ikc_s\eta)$, giving an oscillation period $2\pi/(kc_s) > 1/k$, i.e., larger than the time of horizon entry. The friction term damps the amplitude of the oscillations.

The forcing term comes from the gravitational potentials to which all forms of matter (neutrinos, CDM, baryons, photons) contribute. After neutrino decoupling the neutrino density becomes practically homogeneous (for as long as neutrinos remain relativistic), so their contribution is negligible. When the universe becomes matter dominated (and actually already a bit earlier) CDM dominates the Φ part, but since the CDM perturbation grows slowly compared to the time scale of the baryon-photon oscillation, once the perturbation is inside the horizon, baryons and photons remain important for Ψ' and Ψ'' .

Approximating $v_b \approx v_\gamma$, we have $\delta'_b = \frac{3}{4}\delta'_\gamma$, so if initially we have the adiabatic relation $\delta_b = \frac{3}{4}\delta_\gamma$, this relation is maintained in the tight-coupling approximation.

28.1 Inside the horizon

Inside the horizon, the solution of (28.14) will be oscillating. When we are well inside the horizon, $k \gg \mathcal{H}$, the oscillation will be rapid compared to the change of the background quantities. Let us first ignore the forcing term $F_{\vec{k}}(\eta)$. Write

$$\Theta = Ae^{iB}, \quad (28.16)$$

where A and B are real, $A(\eta)$ representing the slowly changing amplitude and $B(\eta)$ giving the rapid oscillation. The real and imaginary parts of the homogeneous (no $F_{\vec{k}}$) version of (28.14) become

$$\begin{aligned} (B')^2 - \frac{A''}{A} - \frac{R'}{1+R} \frac{A'}{A} &= \frac{k^2}{3(1+R)} \\ \frac{A'}{A} &= -\frac{1}{2} \frac{B''}{B'} - \frac{1}{2} \frac{R'}{1+R}. \end{aligned} \quad (28.17)$$

Neglecting the derivatives of A in the first equation gives

$$B' = c_s k \quad \Rightarrow \quad B(\eta) = k \int^\eta c_s(\eta) d\eta = kr_s(\eta) + \phi, \quad (28.18)$$

⁶⁸Noting the symmetry between the derivatives of δ_γ and Ψ , this can be written, e.g., in the form

$$[(1+R)(\Theta - \Psi)']' = -\frac{1}{3}k^2[\Theta + (1+R)\Phi]. \quad (28.13)$$

where $r_s(\eta)$ is the sound horizon $\int_0^\eta c_s d\eta$ and ϕ is an integration constant. Inserting this in the second equation gives

$$\frac{A'}{A} = -\frac{1}{4} \frac{R'}{1+R} \Rightarrow A \propto (1+R)^{-1/4}. \quad (28.19)$$

Thus

$$\begin{aligned} \Theta_{\vec{k}}(\eta) &= A_{\vec{k}}(1+R)^{-1/4} e^{i(kr_s+\phi)} \\ &= B_{\vec{k}}(1+R)^{-1/4} \cos kr_s + C_{\vec{k}}(1+R)^{-1/4} \sin kr_s. \end{aligned} \quad (28.20)$$

Ignoring Φ' in (28.3) gives

$$\begin{aligned} v_\gamma &= -\frac{3}{k} \Theta' = -\frac{3}{k} A_{\vec{k}} \left[-\frac{1}{4} (1+R)^{-5/4} R' + \frac{ik}{\sqrt{3}(1+R)^{3/4}} \right] e^{i(kr_s+\phi)} \\ &\approx -i\sqrt{3} A_{\vec{k}} (1+R)^{-3/4} e^{i(kr_s+\phi)}. \end{aligned} \quad (28.21)$$

Let us then consider the effect of the forcing term (28.15). From the constraint equation (10.18),

$$\Psi = -\frac{3}{2} \left(\frac{\mathcal{H}}{k} \right)^2 \left[\delta + 3 \frac{\mathcal{H}}{k} (1+w)v \right] \quad (28.22)$$

and Φ is not very different from Ψ . The potentials receive contributions from CDM, baryons, and photons. For $k \gg \mathcal{H}$ the δ contribution dominates. Once the universe becomes matter-dominated, CDM dominates the contribution to the Φ term in (28.15); but well inside the horizon, the derivatives of Ψ will be dominated by the rapid oscillation of the baryon-photon contributions to δ' ,

$$\delta' \approx \frac{\rho_b}{\rho} \delta'_b + \frac{\rho_\gamma}{\rho} \delta'_\gamma \approx (1+R) \frac{\rho_\gamma}{\rho} \delta'_\gamma. \quad (28.23)$$

Divide (28.14) by \mathcal{H}^2 to make it dimensionless and facilitate comparison of magnitudes:

$$\mathcal{H}^{-2} \Theta'' + \frac{R}{1+R} \mathcal{H}^{-1} \Theta' + \frac{1}{3(1+R)} \left(\frac{k}{\mathcal{H}} \right)^2 \Theta = -\frac{1}{3} \left(\frac{k}{\mathcal{H}} \right)^2 \Phi + \mathcal{H}^{-2} \Psi'' + \frac{R}{1+R} \mathcal{H}^{-1} \Psi'. \quad (28.24)$$

The time derivatives come mainly from the oscillation $\propto e^{ikr_s} \sim e^{ikc_s\eta}$, so

$$\mathcal{H}^{-1} \frac{d}{d\eta} \sim c_s \frac{k}{\mathcal{H}} \quad \text{and} \quad \mathcal{H}^{-2} \frac{d^2}{d\eta^2} \sim c_s^2 \left(\frac{k}{\mathcal{H}} \right)^2. \quad (28.25)$$

Thus both the Φ and Ψ'' contribution have the same power of k/\mathcal{H} ; but the Φ term is more important by the factor $(\rho_c/\rho)\delta_c$ is larger than $(1+R)(\rho_\gamma/\rho)\delta_\gamma$. This factor also compensates the $(\mathcal{H}/k)^2$ suppression of the potentials compared to the Θ terms in (28.24). Thus for the end stages of the acoustic oscillation, when CDM perturbations have become dominant, but photons have not yet decoupled, we ignore the less important Ψ' and Ψ'' contributions to the forcing, and (28.14) becomes

$$\Theta'' + \frac{R'}{1+R} \Theta' + \frac{k^2}{3(1+R)} [\Theta + (1+R)\Phi] = 0. \quad (28.26)$$

We can likewise ignore derivatives of Φ or $(1+R)\Phi$, so we can rewrite this as

$$[\Theta + (1+R)\Phi]'' + \frac{R'}{1+R} [\Theta + (1+R)\Phi]' + \frac{k^2}{3(1+R)} [\Theta + (1+R)\Phi] = 0, \quad (28.27)$$

which is the equation we already solved, except now for $\Theta + (1 + R)\Phi$ instead of Θ . Thus the solution is

$$\Theta_{\vec{k}}(\eta) = -(1 + R)\Phi_{\vec{k}} + A_{\vec{k}}(1 + R)^{-1/4}e^{i(kr_s + \phi)}, \quad (28.28)$$

where $A_{\vec{k}}$ and ϕ are constants.

The oscillation ends when photons decouple at t_{dec} . What happens in detail to the photons is discussed in CMB Physics; but from the point of view of considering the possible effect of the photons on matter perturbations, we can summarize it by saying that the photon fluid becomes homogeneous but anisotropic.

29 Tensor Perturbations

29.1 Einstein equation

For tensor perturbations, we set the scalar and vector parts to zero in the metric perturbation,

$$A = D = B_i = 0, \quad E_{ij} = E_{ij}^T, \quad (29.1)$$

and in the energy tensor perturbation,

$$\delta\rho = \delta p = v_i = 0, \quad \Pi_{ij} = \Pi_{ij}^T. \quad (29.2)$$

Thus the metric is

$$ds^2 = a^2(\eta) [-d\eta^2 + \delta_{ij} dx^i dx^j + 2E_{ij}^T dx^i dx^j], \quad (29.3)$$

where

$$\begin{aligned} \delta^{ij} E_{ij}^T \equiv E_{ii}^T &= 0 & (\text{traceless}) \\ \delta^{ij} E_{ij,k}^T \equiv E_{ij,i}^T &= 0 & (\text{transverse}). \end{aligned} \quad (29.4)$$

We get the Christoffel symbols either directly from (29.3) (**exercise**), or from the general perturbation Christoffel symbols (A.2) by setting the scalar and the vector parts to zero:

$$\begin{aligned} \Gamma_{00}^0 &= \mathcal{H} \\ \Gamma_{0i}^0 &= 0 \\ \Gamma_{ij}^0 &= \mathcal{H} (\delta_{ij} + 2E_{ij}^T) + E_{ij}^{T'} \\ \Gamma_{00}^i &= 0 \\ \Gamma_{0j}^i &= \mathcal{H} \delta_{ij} + E_{ij}^{T'} \\ \Gamma_{jk}^i &= E_{ij,k}^T + E_{ik,j}^T - E_{jk,i}^T, \end{aligned} \quad (29.5)$$

where $' \equiv \partial/\partial\eta$. Likewise we get the Ricci tensor (**exercise**):

$$\begin{aligned} R_{00} &= -3\mathcal{H}' = \bar{R}_{00} \\ R_{0i} &= E_{ik,k}^T = 0 \\ R_{ij} &= (\mathcal{H}' + 2\mathcal{H}^2) \delta_{ij} + E_{ij}^{T''} - \nabla^2 E_{ij}^T + 2\mathcal{H} E_{ij}^{T'} + (2\mathcal{H}' + 4\mathcal{H}^2) E_{ij}^T. \end{aligned} \quad (29.6)$$

We raise the index of the Ricci tensor by the inverse metric

$$g^{\mu\nu} = a^{-2} \begin{bmatrix} -1 & 0 \\ 0 & \delta_{ij} - 2E_{ij}^T \end{bmatrix} \quad (29.7)$$

to get

$$\begin{aligned} R_0^0 &= 3a^{-2}\mathcal{H}' \\ R_0^i &= R_i^0 = 0 \\ R_j^i &= a^{-2} (\mathcal{H}' + 2\mathcal{H}^2) \delta_{ij} + a^{-2} \left[E_{ij}^{T''} - \nabla^2 E_{ij}^T + 2\mathcal{H} E_{ij}^{T'} \right]. \end{aligned} \quad (29.8)$$

The Ricci curvature scalar becomes

$$R = 6a^{-2} (\mathcal{H}' + \mathcal{H}^2) = \bar{R}, \quad (29.9)$$

so that the curvature scalar is not affected by the tensor perturbation. The Einstein tensor $G_\nu^\mu = R_\nu^\mu - \frac{1}{2}\delta_\nu^\mu R$ is

$$\begin{aligned} G_0^0 &= -3a^{-2}\mathcal{H}^2 = \bar{G}_0^0 \\ G_0^i &= G_i^0 = 0 \\ G_j^i &= a^{-2}(-2\mathcal{H}' - \mathcal{H}^2)\delta_{ij} + a^{-2}\left[E_{ij}^{T''} - \nabla^2 E_{ij}^T + 2\mathcal{H}E_{ij}^{T'}\right]. \end{aligned} \quad (29.10)$$

Likewise, we drop the scalar and vector perturbation parts from the energy tensor, and keep only the background and tensor perturbation parts:

$$T_\nu^\mu = \bar{T}_\nu^\mu + \delta T_\nu^\mu = \begin{bmatrix} -\bar{\rho} & 0 \\ 0 & \bar{p}\delta_j^i \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \bar{p}\Pi_{ij}^T \end{bmatrix}, \quad (29.11)$$

where $\bar{p}\Pi_{ij}^T$ is the tensor part of the (anisotropic) pressure/stress perturbation.

We can now write the Einstein equations $G_\nu^\mu = 8\pi GT_\nu^\mu$:

$$\begin{aligned} 3a^{-2}\mathcal{H}^2 &= 8\pi G\bar{\rho} \\ a^{-2}(-2\mathcal{H}' - \mathcal{H}^2)\delta_{ij} + a^{-2}\left[E_{ij}^{T''} - \nabla^2 E_{ij}^T + 2\mathcal{H}E_{ij}^{T'}\right] &= 8\pi G(\bar{p}\delta_j^i + \bar{p}\Pi_{ij}^T). \end{aligned} \quad (29.12)$$

From the second one we subtract the background equation $2\mathcal{H}' + \mathcal{H}^2 = -8\pi G\bar{p}a^2$ to get the Einstein equation for tensor perturbations:

$$E_{ij}^{T''} + 2\mathcal{H}E_{ij}^{T'} - \nabla^2 E_{ij}^T = 8\pi Ga^2\bar{p}\Pi_{ij}^T. \quad (29.13)$$

This is a wave equation, with a damping term $2\mathcal{H}E_{ij}^{T'}$ due to expansion of the universe, and a coupling term to matter, $\bar{p}\Pi_{ij}^T$. Thus gravitational waves couple to matter only through anisotropic pressure/stress (anisotropic momentum distribution) in the matter, at least to first order in perturbation theory. In the cosmological context (large scales and the perturbative regime) this generation or damping/absorption of gravitational waves by matter can usually be ignored. (There are exceptions to this: for example, gravitational waves can be generated in first order phase transitions which proceed through bubble nucleation/growth/merging and may involve large density inhomogeneities and fluid flow velocities). We assume a primordial origin (e.g., quantum metric fluctuations during inflation) for the cosmological tensor perturbations we will discuss.

Ignoring the coupling to matter, we have

$$E_{ij}^{T''} + 2\mathcal{H}E_{ij}^{T'} - \nabla^2 E_{ij}^T = 0 \quad (29.14)$$

or, in Fourier space,

$$E_{ij}^{T''} + 2\mathcal{H}E_{ij}^{T'} + k^2 E_{ij}^T = 0. \quad (29.15)$$

The equation is the same for all components E_{ij}^T . Remember that there are just two independent components in E_{ij}^T , and that if we choose $\hat{z} \parallel \vec{k}$, they read

$$E_{ij}^T = \begin{bmatrix} E_{11}^T & E_{12}^T & 0 \\ E_{12}^T & -E_{11}^T & 0 \\ 0 & 0 & 0 \end{bmatrix} \equiv \frac{1}{2} \begin{bmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \delta g_{\mu\nu} = a^2 \begin{bmatrix} 0 & & \\ & h_+ & h_\times \\ & h_\times & -h_+ \\ & & & 0 \end{bmatrix} \quad (29.16)$$

Thus the gravity wave equation for the expanding universe is

$$\boxed{h'' + 2\mathcal{H}h' + k^2 h = 0} \quad (29.17)$$

for both h_+ and h_\times . This equation for $h(\eta, \vec{x})$ depends on the background universe through $\mathcal{H}(\eta)$.

29.2 Evolution in a matter-dominated universe

We solve (29.17) first for the case of the matter-dominated universe, where

$$a(\eta) \propto \eta^2 \quad \Rightarrow \quad \mathcal{H} \equiv \frac{a'}{a} = \frac{2}{\eta}. \quad (29.18)$$

Thus (29.17) becomes

$$h'' + \frac{4}{\eta}h' + k^2h = 0 \quad \Rightarrow \quad \eta^2 \frac{d^2h}{d\eta^2} + 4\eta \frac{dh}{d\eta} + k^2\eta^2h = 0. \quad (29.19)$$

This looks a lot like the Bessel equation

$$r^2 \frac{d^2}{dr^2} Z(kr) + r \frac{d}{dr} Z(kr) + (k^2r^2 - \nu^2) Z(kr) = 0 \quad (29.20)$$

or like the equation for the spherical Bessel functions

$$r^2 \frac{d^2}{dr^2} R(kr) + 2r \frac{d}{dr} R(kr) + [k^2r^2 - n(n+1)] R(kr) = 0, \quad (29.21)$$

but is not quite either of them. We can, however, easily convert it to the Bessel equation. Note also that if I write $x = kr$, then

$$x \frac{d}{dx} Z(x) = kr \frac{d}{d(kr)} Z(kr) = r \frac{d}{dr} Z(kr). \quad (29.22)$$

(Note: This paragraph is more complicated than needed for just getting the solutions (29.29) and (29.38), but it is general enough that we can use it also for the vacuum-dominated case in Sec. 29.6.) The more general equation

$$x^2 R'' + (2\alpha + 1)xR' + [x^2 - n(n+1)] R = 0 \quad (29.23)$$

can be converted to

$$x^2 Z'' + xZ' + [x^2 - \alpha^2 - n(n+1)] Z = 0 \quad (29.24)$$

by the substitution

$$R = \frac{Z}{x^\alpha}. \quad (29.25)$$

In (29.19) we have the case $n = 0$ and $2\alpha + 1 = 4 \Rightarrow \alpha = \frac{3}{2}$. This way (**exercise**) (29.19) becomes

$$x^2 Z'' + xZ' + \left(x^2 - \frac{9}{4}\right) Z = 0, \quad (29.26)$$

which is Bessel's equation with $\nu = \frac{3}{2}$, whose solution is

$$Z(x) = \tilde{A} J_{3/2}(x) + \tilde{B} N_{3/2}(x), \quad (29.27)$$

where the J_ν and N_ν are the Bessel and Neumann functions. For $\nu = n + \frac{1}{2}$ half-integer, they are related to the spherical Bessel and Neumann functions

$$j_n(x) \equiv \sqrt{\frac{\pi}{2x}} J_{n+1/2}(x), \quad n_n(x) \equiv \sqrt{\frac{\pi}{2x}} N_{n+1/2}(x). \quad (29.28)$$

Thus the general solution to (29.19) is

$$h(x) = A \frac{1}{x} j_1(x) + B \frac{1}{x} n_1(x), \quad (29.29)$$

where $x = k\eta$, and the two independent solutions are

$$\begin{aligned} h_A(x) &= \frac{1}{x} j_1(x) = \frac{\sin x}{x^3} - \frac{\cos x}{x^2} \\ h_B(x) &= \frac{1}{x} n_1(x) = -\frac{\cos x}{x^3} + \frac{\sin x}{x^2}. \end{aligned} \quad (29.30)$$

To relate these to initial conditions at $\eta \approx 0$ (more specifically, at $x = k\eta \ll 1$, *i.e.*, $\eta \ll k^{-1}$, the age of the universe much less than the wavelength of the Fourier mode), we expand

$$\begin{aligned} \sin x &\approx x - \frac{1}{6}x^3 + \frac{1}{120}x^5 \\ \cos x &\approx 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 \end{aligned} \quad (29.31)$$

so that

$$\begin{aligned} h_A(x) &\approx \frac{1}{3} - \frac{1}{30}x^2 \\ h_B(x) &\approx -x^{-3} \quad \text{diverges for } x \rightarrow 0. \end{aligned} \quad (29.32)$$

We can ignore the diverging solution, since if such a component were generated by inflation, when $x \ll 1$, it would have become negligible by the time of our interest. So the solution is

$$h(k\eta) = \frac{A}{k\eta} j_1(k\eta) = A \left[\frac{\sin k\eta}{(k\eta)^3} - \frac{\cos k\eta}{(k\eta)^2} \right] \quad (29.33)$$

with $h(k\eta \rightarrow 0) = \frac{1}{3}A$, so that the constant A is determined by initial conditions as $A = 3h(0)$. The “initial conditions” refer to the time when the scales of interest well outside the horizon, $k^{-1} \gg \mathcal{H}^{-1} = \frac{1}{2}\eta$, and h remains constant. We call perturbations at that time the *primordial perturbations*.

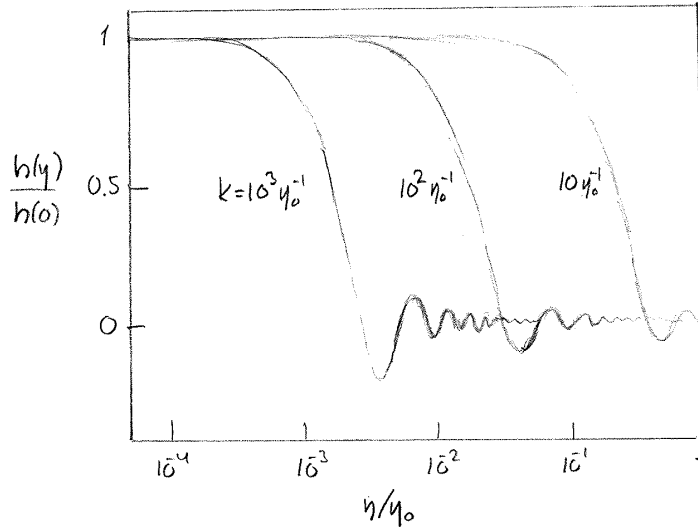


Figure 9: Evolution of tensor perturbations in the matter-dominated universe, for three different scales k . Adapted from Fig. 5.1 in [5].

The evolution of tensor perturbations is thus given by

$$\begin{aligned} h_+(\eta, \vec{k}) &= 3h_+(0, \vec{k}) \left[\frac{\sin k\eta}{(k\eta)^3} - \frac{\cos k\eta}{(k\eta)^2} \right] \\ h_\times(\eta, \vec{k}) &= 3h_\times(0, \vec{k}) \left[\frac{\sin k\eta}{(k\eta)^3} - \frac{\cos k\eta}{(k\eta)^2} \right]. \end{aligned} \quad (29.34)$$

The perturbations stay constant, $h \approx h(0)$, while outside the horizon ($k\eta \ll 1$) but begin to oscillate and are damped when they enter the horizon ($k\eta \sim 1$), see Fig. 9. At late times, deep inside the horizon ($k\eta \gg 1$), the cosine term dominates,

$$h \approx -3h(0) \frac{\cos k\eta}{(k\eta)^2} \Rightarrow \text{the amplitude goes as } \frac{3h(0)}{(k\eta)^2} \propto a^{-1} \propto \mathcal{H}^2. \quad (29.35)$$

This calculation was for a matter-dominated universe; but at early times the universe was radiation dominated, and at late times it is becoming dark energy dominated. Next we shall solve the radiation-dominated case and then combine it with the matter-dominated solution. Finally we shall look at the effect of dark energy, in the case it is vacuum energy (cosmological constant).

29.3 Evolution in the radiation-dominated universe

In the radiation-dominated universe

$$a(\eta) \propto \eta \Rightarrow \mathcal{H} \equiv \frac{a'}{a} = \frac{1}{\eta}, \quad (29.36)$$

so that (29.17) becomes

$$h'' + \frac{2}{\eta}h' + k^2h = 0 \Rightarrow x^2 \frac{d^2h}{dx^2} + 2x \frac{dh}{dx} + x^2h = 0, \quad (29.37)$$

where $x \equiv k\eta$. We see immediately that this is the spherical Bessel equation for $n = 0$, so that

$$h(x) = Aj_0(x) + Bn_0(x). \quad (29.38)$$

Again, we can ignore the solution $n_0(x)$, which diverges at early times $x \rightarrow 0$. The evolution of tensor perturbations is thus given by

$$\begin{aligned} h_+(\eta, \vec{k}) &= h_+(0, \vec{k}) \frac{\sin k\eta}{k\eta} \\ h_\times(\eta, \vec{k}) &= h_\times(0, \vec{k}) \frac{\sin k\eta}{k\eta} \end{aligned} \quad (29.39)$$

during the radiation-dominated epoch.

Again, the perturbations stay constant, $h \approx h(0)$, outside the horizon, but begin to oscillate and get damped as they enter the horizon. At late times, deep inside the horizon, the solution oscillates and the amplitude goes as

$$\frac{h(0)}{k\eta} \propto a^{-1} \propto \mathcal{H}. \quad (29.40)$$

29.4 Radiation+matter universe and the transfer function

Consider now the real-universe case, where there is both matter and radiation (we address the effect of dark energy in Sec. 29.6; ignore it for now). The background solution is, from (19.9),

$$y(\eta) \equiv \frac{a(\eta)}{a_{\text{eq}}} = \frac{\rho_m}{\rho_r} = \frac{2\eta}{(1 + \sqrt{2})\eta_{\text{eq}}} + \left[\frac{\eta}{(1 + \sqrt{2})\eta_{\text{eq}}} \right]^2 \quad (29.41)$$

(so that $a \propto \eta$ for $\eta \ll \eta_{\text{eq}}$ and $a \propto \eta^2$ for $\eta \gg \eta_{\text{eq}}$), and

$$\mathcal{H}(\eta) = \frac{(1 + \sqrt{2})\eta_{\text{eq}} + \eta}{(1 + \sqrt{2})\eta_{\text{eq}}\eta + \frac{1}{2}\eta^2}, \quad (29.42)$$

where η_{eq} is the time of matter-radiation equality ($\rho_m = \rho_r$), and $a_{\text{eq}} \equiv a(\eta_{\text{eq}})$. We also have

$$\mathcal{H}_{\text{eq}} \equiv \mathcal{H}(\eta_{\text{eq}}) = \frac{4 + 2\sqrt{2}}{3 + 2\sqrt{2}} \frac{1}{\eta_{\text{eq}}} = \frac{2\sqrt{2}}{1 + \sqrt{2}} \frac{1}{\eta_{\text{eq}}} \approx 1.1716 \eta_{\text{eq}}^{-1}. \quad (29.43)$$

Since $\mathcal{H}(\eta)$ is now a more complicated equation, (29.17) must be solved numerically (integration of ordinary differential equation, a relatively simple task). But here we look for understanding through analytical approximation. Since the matter- and radiation-dominated solutions behaved the same way in the two regimes:

1. well outside the horizon, $k \ll \mathcal{H}$, we got: $h \approx h(0) = \text{const}$
2. well inside the horizon, $k \gg \mathcal{H}$, we got: $h(\eta)$ oscillates sinusoidally with conformal frequency $k/2\pi$ and amplitude $\propto a^{-1}$

we can expect this to be a good approximation also for the matter+radiation case.

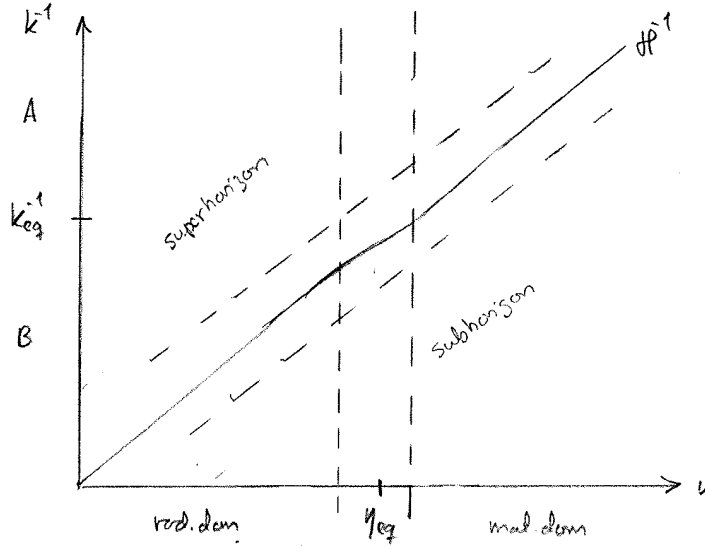


Figure 10: Superhorizon and subhorizon scales at different times.

Look at the situation in the (η, k) -plane (Fig. 10) and consider the situation for different scales k :

- A) $k \ll k_{\text{eq}}$, large scales, enter during matter-dominated epoch: Stay constant while outside the horizon; the universe is already matter dominated when the scale approaches the horizon and something begins to happen. Thus we get just the matter-dominated result (29.34). Focusing our attention to the times and scales where we are already well inside the horizon, the oscillation amplitude goes as

$$h \sim \frac{3h(0)}{(k\eta)^2} \propto a^{-1} \quad (29.44)$$

- B) $k \gg k_{\text{eq}}$, small scales, enter during radiation-dominated epoch: While the universe is radiation dominated, we have (29.39). As we approach matter-radiation equality, the oscillation is already rapid, with the amplitude falling as

$$h \sim \frac{h(0)}{k\eta} \propto a^{-1}. \quad (29.45)$$

We can expect that the amplitude keeps falling approximately as $\propto a^{-1}$ also while the universe becomes matter dominated.

Since the amplitude has a simple behavior in terms of a (rather than η), we shall work through the scale factor. Consider now the question we would like to answer: What is the amplitude of tensor perturbations today (or, at this point, at some time η_m during the late matter-dominated epoch) in terms of the initial value $h(0, \vec{k})$ as a function of k ? Denote the time when scale k entered the horizon ($k = \mathcal{H}$) by η_k and write $a_k \equiv a(\eta_k)$, $\mathcal{H}_k \equiv \mathcal{H}(\eta_k)$, $a_m \equiv a(\eta_m)$, $\mathcal{H}_m \equiv \mathcal{H}(\eta_m)$.

- A) $\mathcal{H}_m \ll k \ll k_{\text{eq}}$, scales which entered during matter domination, but are already well inside the horizon at $\eta = \eta_m$:

$$h(\eta_m, \vec{k}) \approx \frac{3}{(k\eta_m)^2} h(0, \vec{k}). \quad (29.46)$$

Relate $k\eta_m$ to a_m/a_k : Since η_k is during the matter-dominated epoch, we can use the matter-dominated approximation for a and \mathcal{H} : $k = \mathcal{H}_k \approx 2/\eta_k \Rightarrow k\eta_m \approx 2(\eta_m/\eta_k)$ and $a_m/a_k \approx (\eta_m/\eta_k)^2$ to rewrite (29.46) as

$$h(\eta_m, \vec{k}) \approx \frac{3}{4} h(0, \vec{k}) \left(\frac{\eta_k}{\eta_m} \right)^2 \approx \frac{3a_k}{4a_m} h(0, \vec{k}). \quad (29.47)$$

- B) $k \gg k_{\text{eq}}$, small scales, which entered during radiation domination: After horizon entry, but while still radiation dominated, we had (29.45). We want to relate $k\eta$ to a/a_k . Since η_k is during radiation domination we can approximate (29.42) and (29.41) by

$$\begin{aligned} k &\equiv \mathcal{H}_k \approx \frac{1}{\eta_k} \Rightarrow k\eta \approx \frac{\eta}{\eta_k} \\ a &\propto \eta \Rightarrow \frac{a}{a_k} \end{aligned} \quad (29.48)$$

so that (29.45) becomes

$$h(\eta, \vec{k}) \approx h(0, \vec{k}) \frac{\eta_k}{\eta} \approx \frac{a_k}{a} h(0, \vec{k}) \quad (29.49)$$

(so far shown to be valid for η still in the radiation-dominated epoch). Since the h oscillation amplitude keeps falling as a^{-1} , we can extend the last result through matter-radiation equality to the matter-dominated epoch:

$$h(\eta_m, \vec{k}) \approx \frac{a_k}{a_m} h(0, \vec{k}). \quad (29.50)$$

We want to express this in terms of k and η_m to compare it to (29.46). So what is a_k/a_m ? Here a_k is in the rad.dom, a_m in the mat.dom epoch. We go via a_{eq} and use (29.41) in both approximations:

$$\begin{aligned} y(\eta_k) &\equiv \frac{a_k}{a_{\text{eq}}} \approx \frac{2}{1 + \sqrt{2}} \frac{\eta_k}{\eta_{\text{eq}}} && \text{(rad.dom approx)} \\ y(\eta_m) &\equiv \frac{a_m}{a_{\text{eq}}} \approx \left[\frac{\eta_m}{(1 + \sqrt{2})\eta_{\text{eq}}} \right]^2 && \text{(mat.dom approx)}, \end{aligned} \quad (29.51)$$

so that

$$\frac{a_k}{a_m} \approx 2(1 + \sqrt{2})\eta_{\text{eq}} \frac{\eta_k}{\eta_m^2} = 2(1 + \sqrt{2})k\eta_{\text{eq}} \frac{1}{(k\eta_m)^2}. \quad (29.52)$$

Using (29.43),

$$k\eta_{\text{eq}} = \frac{2\sqrt{2}}{1 + \sqrt{2}} \frac{k}{k_{\text{eq}}}, \quad (29.53)$$

where $k_{\text{eq}} = \mathcal{H}_{\text{eq}}$ is the scale that enters at η_{eq} (see Fig. 10), and we have finally

$$h(\eta_m, \vec{k}) \approx \frac{a_k}{a_m} h(0, \vec{k}) \approx 4\sqrt{2} \frac{k}{k_{\text{eq}}} \frac{1}{(k\eta_m)^2} h(0, \vec{k}). \quad (29.54)$$

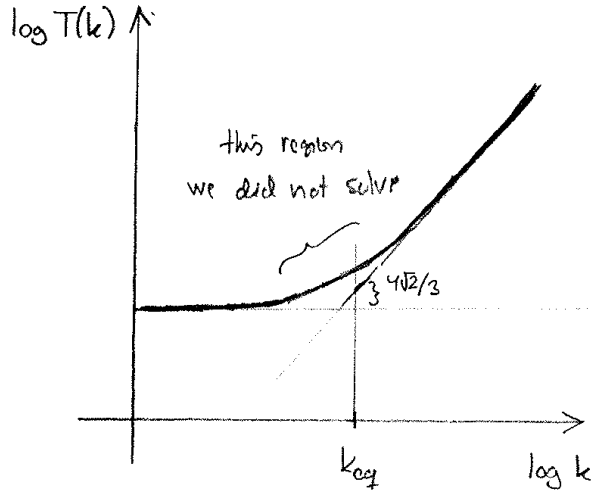


Figure 11: Transfer function for tensor perturbation amplitude.

Note that (29.46) and (29.54) have the same time (η_m) dependence, but different scale (k) dependence. It is customary to use the simpler case (29.46) as a reference, and to define a *transfer function* $T(k)$ to denote the difference, i.e.,

$$h(\eta_m, \vec{k}) \equiv \frac{3h(0, \vec{k})}{(k\eta_m)^2} T(k) \quad (29.55)$$

so that we have

$$\begin{aligned} T(k) &= 1 && \text{for } k \ll k_{\text{eq}} \\ T(k) &= \frac{4\sqrt{2}}{3} \frac{k}{k_{\text{eq}}} \approx 1.9 \frac{k}{k_{\text{eq}}} && \text{for } k \gg k_{\text{eq}}, \end{aligned} \quad (29.56)$$

see Fig. 11. (This captures just the amplitude of the oscillations, not their phase.) We see that the small scales get a relative boost. Why? All scales are damped after horizon entry as $\propto a^{-1}$, but during rad.dom $k_{\text{entry}} = \mathcal{H} \propto a^{-1}$, whereas during mat.dom $k_{\text{entry}} = \mathcal{H} \propto a^{-1/2}$; so for $k \gg k_{\text{eq}}$ more scales enter per log interval in a . Thus for $k \gg k_{\text{eq}}$, we need to cover twice as long interval in $\log k$ of entering scales, to cover the same range in $\log a$ (expansion while these scales enter), i.e., to get the same drop in amplitude. See Fig. 12, where we have plotted the ratio

$$\frac{\text{amplitude of } h(\eta_m, \vec{k})}{h(0, \vec{k})}. \quad (29.57)$$

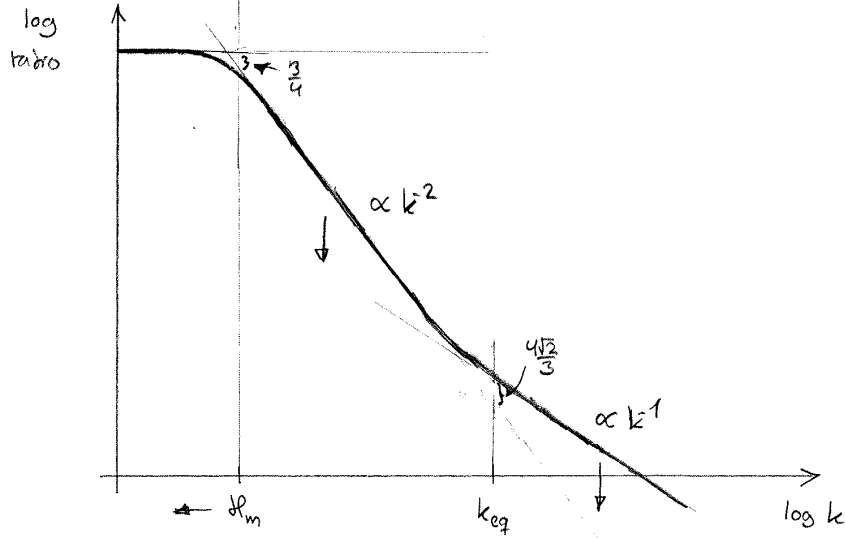


Figure 12: Damping of tensor perturbations. As time (η_m) goes on, \mathcal{H}_m moves to the left, and the part of the curve to the right of \mathcal{H}_m moves down.

29.5 Power spectrum

It is usually thought that primordial tensor perturbations (gravitational waves) were generated by some statistically isotropic and homogeneous random process (e.g., quantum fluctuations during inflation); so that the primordial Fourier amplitudes $h_+(0, \vec{k})$, $h_\times(0, \vec{k})$ have randomly acquired values with some primordial power spectrum $\mathcal{P}(k)$:

$$\langle h_{a\vec{k}}(0)^* h_{b\vec{k}}(0) \rangle = \frac{2\pi^2}{V k^3} \delta_{ab} \delta_{\vec{k}\vec{k}'} \mathcal{P}(k), \quad \text{where } a, b \in \{+, \times\}. \quad (29.58)$$

It is often assumed that $\mathcal{P}(k)$ can be approximated by a power law

$$\mathcal{P}(k) = A_t^2 \left(\frac{k}{k_p} \right)^{n_t}, \quad (29.59)$$

where A_t is the primordial perturbation amplitude at some reference (“pivot”) scale k_p and n_t is the primordial spectral index for tensor perturbations.

For inflation, we expect a weak scale dependence, so that n_t is close to zero. In fact, slow-roll inflation predicts $n_t = -2\varepsilon$, where ε is a slow-roll parameter (see Cosmology II or CPT II). For scale-invariant perturbations ($n_t = 0$), $\mathcal{P}(k) = A_t^2 = \text{const}$, so that the normalization is independent of the choice of pivot scale.

The power spectrum at later times is

$$\mathcal{P}(\eta, k) \equiv \frac{V}{2\pi^2} k^3 \langle |h_{\vec{k}}(\eta)|^2 \rangle. \quad (29.60)$$

For scales that are still outside horizon, $\mathcal{P}(k)$ has still the original value,

$$\mathcal{P}(\eta, k) = \mathcal{P}(k), \quad k \ll \mathcal{H}. \quad (29.61)$$

For scales that are well inside horizon, from (29.55),

$$\mathcal{P}(\eta_m, k) = \frac{1}{2} \frac{9}{(k\eta_m)^4} |T(k)|^2 \mathcal{P}(k), \quad (\text{during mat.dom}), \quad (29.62)$$

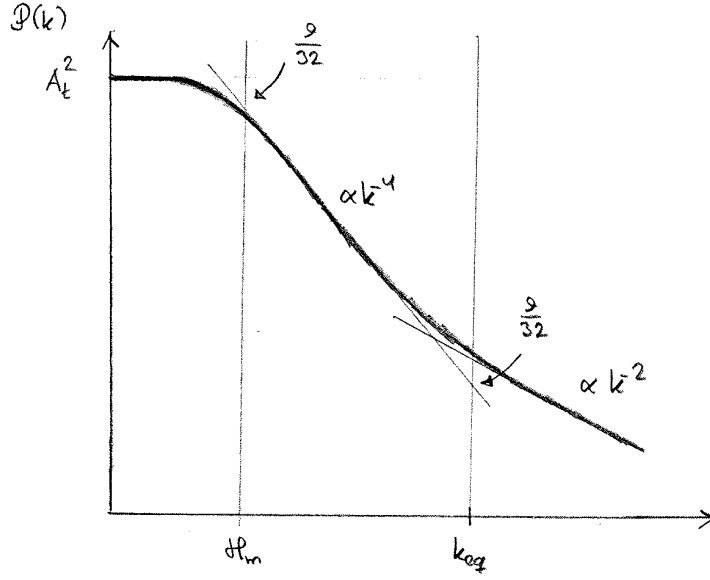


Figure 13: Tensor power spectrum for scale-invariant primordial perturbations. (Note that this is just the square of Fig. 12, except for the extra factor $\frac{1}{2}$ drop at \mathcal{H}_m introduced in (29.62).)

with $T(k)$ from (29.56). We put in the factor $\frac{1}{2} = \langle \cos^2 \rangle$ to represent the effect of oscillations averaged over a range of k . To be more exact, $\mathcal{P}(\eta_m, k)$ has these oscillations, so we are giving just a “smoothed” version of the power spectrum.

For a power-law spectrum (29.59),

$$\mathcal{P}(\eta_m, k) \propto \begin{cases} k^{n_t} & (k \ll \mathcal{H}) \\ k^{n_t-4} & (\mathcal{H} \ll k \ll k_{\text{eq}}) \\ k^{n_t-2} & (k \gg k_{\text{eq}}). \end{cases} \quad (29.63)$$

For the scale-invariant case, $n_t = 0$,

$$\mathcal{P}(\eta_m, k) = \begin{cases} A_t^2 & (k \ll \mathcal{H}) \\ \frac{9}{2} \frac{1}{(k\eta_m)^4} A_t^2 = \frac{9}{32} \left(\frac{\mathcal{H}_m}{k}\right)^4 A_t^2 & (\mathcal{H} \ll k \ll k_{\text{eq}}) \\ 16 \left(\frac{k}{k_{\text{eq}}}\right)^2 \frac{1}{(k\eta_m)^4} A_t^2 = \left(\frac{k}{k_{\text{eq}}}\right)^2 \left(\frac{\mathcal{H}_m}{k}\right)^4 A_t^2 & (k \gg k_{\text{eq}}), \end{cases} \quad (29.64)$$

see Fig. 13.

29.6 Effect of the late-time acceleration (vacuum energy)

We are no longer in the matter-dominated epoch, since the expansion of the universe is accelerating. This is often attributed to some mysterious dark energy. Observations suggest that the equation of state for dark energy is close to that of vacuum energy, $w = -1$. Thus we consider just the case of vacuum energy (or a cosmological constant) as the only new ingredient.

In the vacuum-dominated case the background solution is

$$a = e^{Ht} \quad \text{with } H = \text{const} \quad \Rightarrow \quad \eta = -\frac{1}{H} e^{-Ht} = -\frac{1}{aH} = -\frac{1}{\mathcal{H}} \quad \Rightarrow \quad \mathcal{H} = -\frac{1}{\eta}. \quad (29.65)$$

As t goes from $-\infty$ to ∞ , η goes from $-\infty$ to 0. (Here we choose when $t = 0$ and $\eta = 0$ for convenience; if we would want to match these with the time coordinates for the earlier epochs, we would have to add some constants to them.)

Solving the gravitational wave equation (29.17) for the vacuum-dominated case (**exercise**) we find the familiar behavior,

1. Well outside horizon, $k \ll \mathcal{H}$, $h(\eta) \approx \text{const}$
2. Well inside horizon, $k \gg \mathcal{H}$, $h(\eta)$ oscillates, amplitude $\propto a^{-1}$,

but again with different $a(\eta)$, $\mathcal{H}(\eta)$.

The main difference to the previous cases is, however, that now the comoving horizon \mathcal{H}^{-1} is shrinking, and that scales, starting from the largest, are exiting the horizon. As they do that, $h(\vec{k}, \eta)$ freezes to constant value, different from $h(\vec{k}, 0)$, dependent on the history of the mode inside the horizon. The change of the power spectrum from the late matter-dominated epoch to the early vacuum-dominated epoch is represented in Fig. 14. As the universe expands, power at subhorizon scales keeps falling as $\propto a^{-2}$. The horizon scale $k = \mathcal{H}$ moves to the right at the rate $\mathcal{H} \propto a \Rightarrow$ the power spectrum between \mathcal{H}_{\min} (the minimum \mathcal{H} , i.e., the maximum comoving horizon \mathcal{H}^{-1} , reached during the mat.dom \rightarrow transition) and \mathcal{H} steepens by another factor k^2 (factor k for amplitude). The shape for $k \gg \mathcal{H}$ is unaffected (just falls down with time).

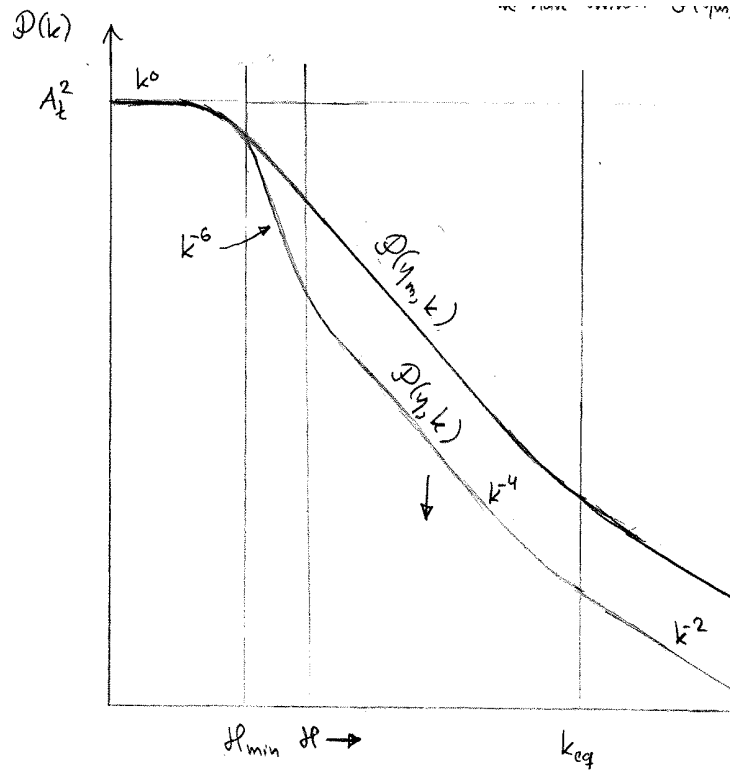


Figure 14: Tensor power spectrum for scale-invariant primordial perturbations during the vacuum-dominated epoch. For $n_t \neq 0$, just add n_t to slopes. Note that this figure shows the situation at some time in the future; today \mathcal{H} is still close to \mathcal{H}_{\min} , so that we do not yet have a clear k^{-6} part between the two bends.

30 Modified Gravity

A major question in cosmology is the cause of the acceleration of the expansion of the universe. The two main classes of proposed explanations are *dark energy* and *modified gravity*.⁶⁹ The dividing line between these two classes of explanations is not sharp; there are suggested models that could be interpreted to belong into either one or as a mixture of both; but in simple terms a dark energy model is one where GR is valid but there is a new energy component with an equation of state with negative pressure (a total $p/\rho < -1/3$ is needed for acceleration) at least since $z \sim 1$; whereas a modified gravity model can not be expressed this way.

Both a modified gravity explanation and a dark energy explanation should modify the expansion law $a(t)$ to get this acceleration ($\ddot{a} > 0$ since $z \sim 1$). At the background level modified gravity and dark energy cannot be distinguished from each other observationally, since we have only one function $a(t)$ to explain, and in principle any expansion history $a(t)$ can be explained with a suitable behavior of the dark energy equation of state $w_{\text{de}}(t)$. The two classes of explanations are distinguished by their effect on perturbations; since in a dark energy model $w_{\text{de}}(t)$ determines also the perturbations according to the GR-based perturbation equations we have discussed; whereas in a modified gravity model the perturbations will behave differently.

If there is no effect on perturbations then the two explanations cannot be distinguished: if gravity is modified by adding the cosmological constant Λ to the Einstein equation, the effect is the same as introducing dark energy with $w_{\text{de}} \equiv -1$, i.e., vacuum energy.

A conservative approach to modifying gravity (general relativity) is to assume that the nature of gravity is still curvature of spacetime, which can be expressed with a metric; and that the *equivalence principle* is still valid, i.e., freely falling test particles follow geodesics and energy-momentum is locally “covariantly conserved”, i.e., the equation

$$T^{\mu\nu}_{;\nu} = 0 \quad (30.1)$$

is still valid. The same arguments as in standard GR lead then to the assumption that at large scales and early times the metric is the RW metric with perturbations. From (30.1) then follows that our fluid perturbation equations remain valid.

With these assumptions the modification of gravity is a modification of the Einstein equation

$$G^{\mu\nu} = 8\pi G T^{\mu\nu}. \quad (30.2)$$

At the background level this modification will lead to a modification of the Friedmann equations, which should lead to the observed acceleration.

At the linear perturbation level, we got from the Einstein equation four equations for scalar perturbations; but these are not independent of the fluid equations, so that if we use the fluid equations we need to supplement them with just two of the Einstein equations for the metric perturbations. We pick the two constraint equations

$$\begin{aligned} k^2 \Psi &= -4\pi G a^2 \rho \delta^C \\ \Psi - \Phi &= 8\pi G p \Pi. \end{aligned} \quad (30.3)$$

Based on observations the acceleration issue becomes relevant only when the universe is already matter dominated. Thus we can ignore the effects of radiation and consider the cosmic fluid to be matter, so that $p = 0$. The second constraint equation then becomes $\Psi = \Phi$, or $\Psi/\Phi = 1$.

⁶⁹It has also been suggested that this could be a backreaction effect, i.e., due to nonlinear effects of inhomogeneities in standard GR; or an effect of inhomogeneities on observations that has caused them to be misinterpreted. Both of these explanations rely on effects that are difficult to calculate but are usually expected to be small. For them to explain the (real or apparent) acceleration, the effect would have to be large enough to explain away $\Omega_\Lambda \sim 0.7$. As observations become more accurate while remaining reasonably consistent with Λ CDM, it becomes less plausible that such effects could conspire to mimic Λ CDM to such accuracy; whereas one can think of modified gravity or dark energy models that are close to, but not exactly, Λ CDM.

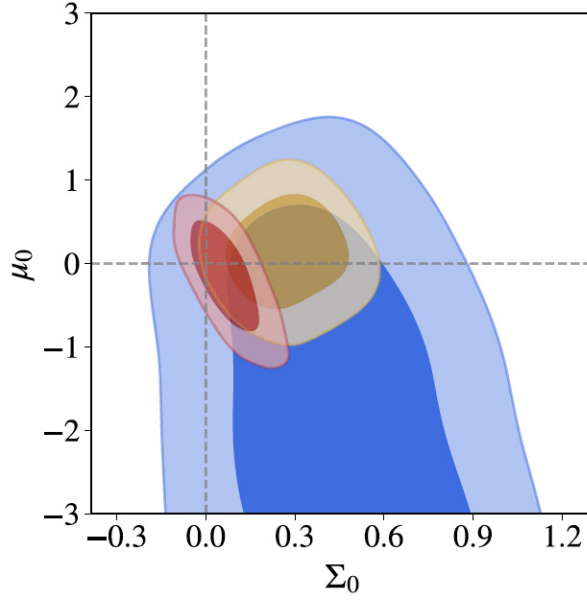


Figure 15: The 68% and 95% confidence constraints on modified gravity parameters Σ_0 and μ_0 from DES first-year data (blue), other cosmological data (brown), and their combination (red), assuming a Λ CDM background. GR corresponds to $\Sigma_0 = \mu_0 = 0$ (the intersection of the dashed lines). Note that the two datasets constrain the Λ CDM parameters differently and therefore the red region differs significantly from the intersection of the blue and brown regions. From [14].

There are many suggested models for modified gravity and the power of observational data to distinguish between them will be limited, especially in the near term. It makes sense to approach the application of observations to theory in a phenomenological way: to search if there is evidence for modification of (30.3) and how should they be modified. One can accomplish this by introducing modified constraint equations [13, 10]

$$\begin{aligned} k^2\Psi &= -4\pi GQ(a, k)a^2\rho\delta^C \\ \Psi &= \eta(a, k)\Phi, \end{aligned} \quad (30.4)$$

where $Q(a, k)$ and $\eta(a, k)$ are two time- and scale-dependent functions to be determined from observations. These functions can be derived from different modified gravity theories and models and compared to observations. The simplest approach is to use Λ CDM as the background model and just modify the perturbations this way. In GR, $Q \equiv \eta \equiv 1$.

Different kinds of observations are more sensitive to Φ (growth of structure and peculiar velocities) or $\Phi + \Psi$ (ISW and weak lensing), i.e., to the combinations Q/η or $Q/\eta + Q$, so we define

$$\begin{aligned} \mu &\equiv \frac{Q}{\eta} \\ \Sigma &\equiv \frac{1}{2}(Q + Q/\eta) = \frac{1}{2}(\mu\eta + \mu) \end{aligned} \quad (30.5)$$

so that we can write (30.4) as

$$\begin{aligned} k^2\Phi &= -4\pi Ga^2\mu(a, k)\rho\delta^C \\ k^2(\Phi + \Psi) &= -8\pi Ga^2\Sigma(a, k)\rho\delta^C, \end{aligned} \quad (30.6)$$

and one can use the pair of functions (μ, Σ) instead of (Q, η) .

Of course the data will never be sufficient to determine these two functions exactly, so one must parametrize them with a small number of parameters to be fit to observations. The simplest

parametrization is to assume them to be constants over both the observationally relevant times and scales. The Dark Energy Survey (DES) collaboration used the parametrization

$$\mu(z) = 1 + \mu_0 \frac{\Omega_\Lambda(z)}{\Omega_\Lambda}, \quad \Sigma(z) = 1 + \Sigma_0 \frac{\Omega_\Lambda(z)}{\Omega_\Lambda} \quad (30.7)$$

(with Λ CDM as the background universe, giving the $\Omega_\Lambda(z)$). The motivation was to let the modification become larger closer to present times when a modification is needed to account for the acceleration. Using the DES first-year data combined with other cosmological data they obtained the observational constraints[14]

$$\begin{aligned} \Sigma_0 &= 0.06 \pm 0.08 \\ \mu_0 &= -0.1 \pm 0.5 \end{aligned} \quad (30.8)$$

(see Fig. 15), consistent with GR.

A General Perturbation

From Eq. (3.8) we have that the general perturbed metric (around the flat Friedmann model) is

$$ds^2 = a^2(\eta) \left\{ -(1 + 2A)d\eta^2 - 2B_i d\eta dx^i + [(1 - 2D)\delta_{ij} + 2E_{ij}] dx^i dx^j \right\}. \quad (\text{A.1})$$

The Christoffel symbols are

$$\begin{aligned} \Gamma_{00}^0 &= \mathcal{H} + A' \\ \Gamma_{0i}^0 &= -\mathcal{H}B_i + A_{,i} \\ \Gamma_{ij}^0 &= \mathcal{H}[(1 - 2A - 2D)\delta_{ij} + 2E_{ij}] + \frac{1}{2}(B_{i,j} + B_{j,i}) - \delta_{ij}D' + E'_{ij} \\ \Gamma_{00}^i &= -\mathcal{H}B_i - B'_i + A_{,i} \\ \Gamma_{0j}^i &= \mathcal{H}\delta_{ij} + \frac{1}{2}(B_{j,i} - B_{i,j}) - D'\delta_{ij} + E'_{ij} \\ \Gamma_{jk}^i &= \mathcal{H}\delta_{jk}B_i - \delta_{jk}D_{,k} - \delta_k^i D_{,j} + \delta_{jk}D_{,i} + E_{ij,k} + E_{ik,j} - E_{jk,i}. \end{aligned} \quad (\text{A.2})$$

and we have the Christoffel sums

$$\begin{aligned} \Gamma_{0\mu}^\mu &= 4\mathcal{H} + A' - 3D' \\ \Gamma_{i\mu}^\mu &= A_{,i} - 3D_{,i}. \end{aligned} \quad (\text{A.3})$$

Note that the Christoffel sums contain only scalar perturbations. Thus for vector and tensor perturbations, these sums contain only the background value $\Gamma_{0\mu}^\mu = 4\mathcal{H}$.

The Einstein tensor is

$$\begin{aligned} G_0^0 &= -3a^{-2}\mathcal{H}^2 + a^{-2}[-2\nabla^2 D + 6\mathcal{H}D' + 6\mathcal{H}^2 A - 2\mathcal{H}B_{i,i} - E_{ik,ik}] \\ G_i^0 &= a^{-2}[-2D'_{,i} - 2\mathcal{H}A_{,i} - \frac{1}{2}(B_{i,kk} - B_{k,ik}) - E'_{ik,k}] \\ G_0^i &= a^{-2}[2D'_{,i} + 2\mathcal{H}A_{,i} + \frac{1}{2}(B_{i,kk} - B_{k,ik}) + 2\mathcal{H}'B_i - 2\mathcal{H}^2 B_i + E'_{ik,k}] \\ G_j^i &= a^{-2}(-2\mathcal{H}' - \mathcal{H}^2)\delta_{ij} \\ &\quad + a^{-2}[2D'' - \nabla^2(D - A) + \mathcal{H}(2A' + 4D') + (4\mathcal{H}' + 2\mathcal{H}^2)A - B'_{k,k} - 2\mathcal{H}B_{k,k} - E_{kl,kl}]\delta_j^i \\ &\quad + a^{-2}[(D - A)_{,ij} + \frac{1}{2}(B'_{i,j} + B'_{j,i}) + \mathcal{H}(B_{i,j} + B_{j,i}) + E'_{ij} - \nabla^2 E_{ij} + E_{ik,jk} + E_{jk,ik} + 2\mathcal{H}E'_{ij}] \end{aligned} \quad (\text{A.4})$$

For scalar perturbations $B_i = -B_{,i}$, $E_{ij} = E_{,ij} - \frac{1}{3}\delta_{ij}E_{,kk}$, and $\psi = D + \frac{1}{3}E_{,kk}$, so that the Christoffel symbols become

$$\begin{aligned} \Gamma_{00}^0 &= \mathcal{H} + A' \\ \Gamma_{0i}^0 &= \mathcal{H}B_{,i} + A_{,i} \\ \Gamma_{ij}^0 &= \mathcal{H}[(1 - 2A - 2\psi)\delta_{ij} + 2E_{,ij}] - \delta_{ij}\psi' + E'_{,ij} \\ \Gamma_{00}^i &= \mathcal{H}B_{,i} + B'_{,i} + A_{,i} \\ \Gamma_{0j}^i &= \mathcal{H}\delta_{ij} - \psi'\delta_{ij} + E'_{,ij} \\ \Gamma_{jk}^i &= -\mathcal{H}\delta_{jk}B_{,i} - \delta_j^i\psi_{,k} - \delta_k^i\psi_{,j} + \delta_{jk}\psi_{,i} + E_{,ijk}. \end{aligned} \quad (\text{A.5})$$

For scalar perturbations, the perturbation in the Einstein tensor becomes

$$\delta G_0^0 = a^{-2} [-2\nabla^2 \psi + 6\mathcal{H}D' + 6\mathcal{H}^2 A + 2\mathcal{H}\nabla^2 B] \quad (\text{A.6})$$

$$\begin{aligned} \delta G_i^0 &= a^{-2} [-2\psi'_{,i} - 2\mathcal{H}A_{,i}] \\ \delta G_0^i &= a^{-2} [2\psi'_{,i} + 2\mathcal{H}A_{,i} - 2\mathcal{H}'B_{,i} + 2\mathcal{H}^2 B_{,i}] \\ \delta G_j^i &= a^{-2} [2D'' - \nabla^2(D - A) + \mathcal{H}(2A' + 4D') + (4\mathcal{H}' + 2\mathcal{H}^2)A + \nabla^2 B' + 2\mathcal{H}\nabla^2 B] \delta_j^i \\ &\quad + a^{-2} \left[-\frac{1}{3}\nabla^2(\nabla^2 E) - \frac{1}{3}\nabla^2 E'' - \frac{2}{3}\mathcal{H}\nabla^2 E' \right] \delta_j^i \\ &\quad + a^{-2} (D - A - B' - 2\mathcal{H}B + E'' + \frac{1}{3}\nabla^2 E + 2\mathcal{H}E')_{,ij} , \\ &= a^{-2} [(4\mathcal{H}' + 2\mathcal{H}^2)A + 2\mathcal{H}A' + 2\psi'' + 4\mathcal{H}\psi' + \nabla^2 \mathcal{D}] \delta_j^i - a^{-2} \mathcal{D}_{,ij} \end{aligned} \quad (\text{A.7})$$

where $\psi = D + \frac{1}{3}\nabla^2 E$ and

$$\mathcal{D} \equiv A - \psi + 2\mathcal{H}(B - E') + (B - E')'. \quad (\text{A.8})$$

The trace of the space part is

$$\begin{aligned} \delta G_i^i &= a^{-2} [6D'' - 2\nabla^2 \psi + 2\nabla^2 A + 3\mathcal{H}(2A' + 4D') + (12\mathcal{H}' + 6\mathcal{H}^2)A + 2\nabla^2 B' + 4\mathcal{H}\nabla^2 B] \\ &= 3a^{-2} [(4\mathcal{H}' + 2\mathcal{H}^2)A + 2\mathcal{H}A' + 2\psi'' + 4\mathcal{H}\psi'] + 2a^{-2}\nabla^2 \mathcal{D}. \end{aligned} \quad (\text{A.9})$$

In Fourier space these read as

$$\begin{aligned} \delta G_0^0 &= a^{-2} [2k^2 \psi + 6\mathcal{H}D' + 6\mathcal{H}^2 A - 2\mathcal{H}kB] \quad (\text{A.10}) \\ \delta G_i^0 &= a^{-2} [-2ik_i \psi' - 2i\mathcal{H}k_i A] \\ \delta G_0^i &= a^{-2} [2ik_i \psi' + 2i\mathcal{H}k_i A - 2i\mathcal{H}'k_i B + 2i\mathcal{H}^2 k_i B] \\ \delta G_j^i &= a^{-2} [2D'' + k^2(D - A) + \mathcal{H}(2A' + 4D') + (4\mathcal{H}' + 2\mathcal{H}^2)A - kB' - 2\mathcal{H}kB] \delta_j^i \\ &\quad + a^{-2} \left[-\frac{1}{3}k^2 E + \frac{1}{3}E'' + \frac{2}{3}\mathcal{H}E' \right] \delta_j^i \\ &\quad - k_i k_j a^{-2} \left[D - A - \frac{1}{k}B' - 2\frac{\mathcal{H}}{k}B - \frac{1}{3}E + \frac{1}{k^2}(E'' + 2\mathcal{H}E') \right] \\ \delta G_i^i &= a^{-2} [6D'' + 2k^2 \psi - 2k^2 A + \mathcal{H}(6A' + 12D') + (12\mathcal{H}' + 6\mathcal{H}^2)A - 2kB' - 4\mathcal{H}kB]. \end{aligned}$$

(The last two could also be simplified with ψ and \mathcal{D} .)

The general perturbed energy tensor is

$$\begin{aligned} T_0^0 &= -\bar{\rho} - \delta\rho \\ T_i^0 &= (\bar{\rho} + \bar{p})(v_i - B_i) \\ T_0^i &= -(\bar{\rho} + \bar{p})v_i \\ T_j^i &= \bar{p}\delta_j^i + \delta p\delta_j^i + \bar{p}\Pi_{ij} \end{aligned} \quad (\text{A.11})$$

The energy continuity equations

$$T_{\nu;\mu}^\mu \equiv T_{\nu,\mu}^\mu + \Gamma_{\alpha\nu}^\mu T_\nu^\alpha - \Gamma_{\nu\mu}^\alpha T_\alpha^\mu = 0 \quad (\text{A.12})$$

become the background equation

$$\bar{\rho}' + 3\mathcal{H}(\bar{\rho} + \bar{p}) = 0 \quad (\text{A.13})$$

and the fluid perturbation equations

$$\begin{aligned} \delta\rho' &= -3\mathcal{H}(\delta\rho + \delta p) + (\bar{\rho} + \bar{p})(3D' - \nabla \cdot \vec{v}) \\ (\bar{\rho} + \bar{p})(v_i - B_i)' &= -(\bar{\rho} + \bar{p})'(v_i - B_i) - 4\mathcal{H}(\bar{\rho} + \bar{p})(v_i - B_i) \\ &\quad - \delta p_{,i} - \bar{p}\Pi_{ij,j} - (\bar{\rho} + \bar{p})A_{,i} \end{aligned} \quad (\text{A.14})$$

For scalar perturbations, the fluid perturbation equations become

$$\begin{aligned} \delta\rho' &= -3\mathcal{H}(\delta\rho + \delta p) + (\bar{\rho} + \bar{p})(3D' + \nabla^2 v) \\ (\bar{\rho} + \bar{p})(v - B)' &= -(\bar{\rho} + \bar{p})'(v - B) - 4\mathcal{H}(\bar{\rho} + \bar{p})(v - B) \\ &\quad + \delta p + \frac{2}{3}\bar{p}\nabla^2 \Pi + (\bar{\rho} + \bar{p})A. \end{aligned} \quad (\text{A.15})$$

Using $\delta \equiv \delta\rho/\bar{\rho}$ and background relations, these can be written

$$\begin{aligned} \delta' &= (1 + w)(\nabla^2 v + 3D') + 3\mathcal{H}\left(w\rho - \frac{\delta p}{\bar{\rho}}\right) \\ (v - B)' &= -\mathcal{H}(1 - 3c_s^2)(v - B) + \frac{\delta p}{\bar{\rho} + \bar{p}} + \frac{2}{3}\frac{w}{1 + w}\nabla^2 \Pi + A. \end{aligned} \quad (\text{A.16})$$

The factor $\mathcal{H}(1 - 3c_s^2)$ can also be written as $(1 - 3w)\mathcal{H} + w'/(1 + w)$.

In Fourier space these are

$$\begin{aligned} \delta\rho' &= -3\mathcal{H}(\delta\rho + \delta p) + (\bar{\rho} + \bar{p})(3D' - kv) \\ (\bar{\rho} + \bar{p})(v - B)' &= -(\bar{\rho} + \bar{p})'(v - B) - 4\mathcal{H}(\bar{\rho} + \bar{p})(v - B) \\ &\quad + k\delta p - \frac{2}{3}k\bar{p}\Pi + k(\bar{\rho} + \bar{p})A \\ \delta' &= (1 + w)(-kv + 3D') + 3\mathcal{H}\left(w\delta - \frac{\delta p}{\bar{\rho}}\right) \\ (v - B)' &= -\mathcal{H}(1 - 3w)(v - B) - \frac{w'}{1 + w}(v - B) + \frac{k\delta p}{\bar{\rho} + \bar{p}} - \frac{2}{3}\frac{w}{1 + w}k\Pi + kA. \end{aligned} \quad (\text{A.17})$$

B Rules to Convert between Different Time Coordinates

The cosmic time t and the conformal time η are related by

$$dt = a d\eta \Rightarrow \frac{d}{dt} = \frac{1}{a} \frac{d}{d\eta} \Rightarrow ()^\cdot = \frac{1}{a} ()' \quad \text{or} \quad ()' = a ()^\cdot \quad (\text{B.1})$$

For any two functions of time, f and g , we have

$$\frac{df}{dg} = \frac{\dot{f}}{\dot{g}} = \frac{f'}{g'} \quad (\text{B.2})$$

The ordinary Hubble parameter H and the conformal (or comoving) Hubble parameter are related by

$$\mathcal{H} = aH = \dot{a} = \frac{a'}{a} \quad (\text{B.3})$$

Sometimes it's convenient to use the scale factor a or its logarithm $\ln a$ as the time coordinate. We have the following relations between the derivatives wrt these time coordinates:

$$\mathcal{H}^{-1} f' = H^{-1} \dot{f} = a \frac{df}{da} = \frac{df}{d \ln a} \quad (\text{B.4})$$

$$\mathcal{H}^{-2} f'' = H^{-2} \ddot{f} + H^{-1} \dot{f} = a^2 \frac{d^2 f}{da^2} + \frac{1-3w}{2} a \frac{df}{da} = \left(\frac{d}{d \ln a} \right)^2 f - \frac{1+3w}{2} \frac{df}{d \ln a} \quad (\text{B.5})$$

In many equations the combination

$$\mathcal{H}^{-2} f'' + 2\mathcal{H}^{-1} f' = H^{-2} \ddot{f} + 3H^{-1} \dot{f} \quad (\text{B.6})$$

appears. We also have that

$$\mathcal{H}' = a\ddot{a} = a^2 \left(\dot{H} + H^2 \right). \quad (\text{B.7})$$

C Equations

All fluid quantities below are in the Newtonian gauge. All equations are in Fourier space.

The Einstein equations, from Sec. 10, are

$$\mathcal{H}^{-1}\Psi' + \Phi + \frac{1}{3}\left(\frac{k}{\mathcal{H}}\right)^2\Psi = -\frac{1}{2}\delta \quad (\text{C.1})$$

$$\mathcal{H}^{-1}\Psi' + \Phi = \frac{3}{2}(1+w)\frac{\mathcal{H}}{k}v \quad (\text{C.2})$$

$$\mathcal{H}^{-2}\Psi'' + \mathcal{H}^{-1}(\Phi' + 2\Psi') - 3w\Phi - \frac{1}{3}\left(\frac{k}{\mathcal{H}}\right)^2(\Phi - \Psi) = \frac{3}{2}\frac{\delta p}{\rho} \quad (\text{C.3})$$

$$\left(\frac{k}{\mathcal{H}}\right)^2(\Psi - \Phi) = 3w\Pi, \quad (\text{C.4})$$

The fluid equations for the “real universe”, from Secs. 21 and 22, are

$$\begin{aligned} \mathcal{H}^{-1}\delta'_c + \left(\frac{k}{\mathcal{H}}\right)v_c - 3\mathcal{H}^{-1}\Psi' &= 0 \\ \mathcal{H}^{-1}\delta'_b + \left(\frac{k}{\mathcal{H}}\right)v_b - 3\mathcal{H}^{-1}\Psi' &= 0 \\ \mathcal{H}^{-1}\delta'_\gamma + \frac{4}{3}\left(\frac{k}{\mathcal{H}}\right)v_\gamma - 4\mathcal{H}^{-1}\Psi' &= 0 \\ \mathcal{H}^{-1}\delta'_\nu + \frac{4}{3}\left(\frac{k}{\mathcal{H}}\right)v_\nu - 4\mathcal{H}^{-1}\Psi' &= 0 \\ \mathcal{H}^{-1}v'_c + v_c - \left(\frac{k}{\mathcal{H}}\right)\Phi &= 0 \\ \mathcal{H}^{-1}v'_b + v_b - \left(\frac{k}{\mathcal{H}}\right)\Phi &= an_e\sigma_T\frac{4\rho_\gamma}{3\rho_b}(v_\gamma - v_b) \\ \mathcal{H}^{-1}v'_\gamma - \frac{1}{4}\left(\frac{k}{\mathcal{H}}\right)\delta_\gamma - \left(\frac{k}{\mathcal{H}}\right)\Phi &= -\frac{1}{6}\left(\frac{k}{\mathcal{H}}\right)\Pi_\gamma + an_e\sigma_T(v_b - v_\gamma) \\ \mathcal{H}^{-1}v'_\nu - \frac{1}{4}\left(\frac{k}{\mathcal{H}}\right)\delta_\nu - \left(\frac{k}{\mathcal{H}}\right)\Phi &= -\frac{1}{6}\left(\frac{k}{\mathcal{H}}\right)\Pi_\nu, \end{aligned} \quad (\text{C.5})$$

and, from Sec. 18,

$$\mathcal{H}^{-1}S'_{ij} = -\frac{k}{\mathcal{H}}(v_i - v_j) \quad \text{where} \quad S_{ij} = \frac{\delta_i}{1+w_i} - \frac{\delta_j}{1+w_j}. \quad (\text{C.6})$$

From Sec. 16 we have the relation between the Bardeen potentials and the comoving curvature perturbation,

$$\frac{2}{3}\mathcal{H}^{-1}\Psi' + \frac{5+3w}{3}\Psi = -(1+w)\mathcal{R} + \frac{2}{3}(\Psi - \Phi), \quad (\text{C.7})$$

and the evolution equation

$$\mathcal{H}^{-1}\mathcal{R}' = \frac{2}{3(1+w)}\left(\frac{k}{\mathcal{H}}\right)^2[c_s^2\Psi + \frac{1}{3}(\Psi - \Phi)] + 3c_s^2\mathcal{S}. \quad (\text{C.8})$$

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