Y6. Correlation Functions and Angular Power Spectra

- The anisotropy and polarization of the CMB are described by the three sets of multipole coefficients

$$\boxed{\begin{aligned} a^T_{lm} &= \int d\Omega\, Y^{m*}_l(\vartheta,\varphi)\,\Theta(\vartheta,\varphi) \\ a^G_{lm} &= \int d\Omega\, Y^{Gab*}_{(lm)}(\vartheta,\varphi)\, P_{ab}(\vartheta,\varphi) \\ a^C_{lm} &= \int d\Omega\, Y^{Cab*}_{(lm)}(\vartheta,\varphi)\, P_{ab}(\vartheta,\varphi) \end{aligned}} \qquad (1)$$

We use X, Y to denote any of the T, G, C

Θ and P_{ab} are real $\Rightarrow$ $\underline{a^X_{l,-m} = (-1)^m a^{X*}_{lm}}$ (2)

- We assume the primordial perturbations were produced by a statistically isotropic Gaussian random process, and use $\langle\cdot\rangle$ to denote expectation values from that random process. We also assume that the observed a^X_{lm} were produced from primordial perturbations through linear physics (linear perturbation theory)

  $\Rightarrow$ they inherit the statistical properties of the primordial perturbations

  $\Rightarrow$ The expectation values $\langle a^X_{lm} a^{Y*}_{l'm'}\rangle$ vanish for $l \neq l'$, or $m \neq m'$ and are independent of m

$$\therefore \quad \langle a^X_{lm} a^{Y*}_{lm}\rangle = \langle a^X_{l,-m} a^{Y*}_{l,-m}\rangle = \underbrace{(-1)^{2m}}_{=1}\langle a^{X*}_{lm} a^Y_{lm}\rangle = \langle a^Y_{lm} a^{X*}_{lm}\rangle \qquad (3)$$

is <u>real and symmetric in XY.</u>

The statistical properties of the CMB are then fully described (all expectation values can be calculated from) by the six angular power spectra

$$\boxed{\begin{aligned} \langle a^T_{lm} a^{T*}_{l'm'}\rangle &= C^{TT}_l \delta_{ll'}\delta_{mm'} & \langle a^T_{lm} a^{G*}_{l'm'}\rangle &= C^{TG}_l \delta_{ll'}\delta_{mm'} \\ \langle a^G_{lm} a^{G*}_{l'm'}\rangle &= C^{GG}_l \delta_{ll'}\delta_{mm'} & \langle a^T_{lm} a^{C*}_{l'm'}\rangle &= C^{TC}_l \delta_{ll'}\delta_{mm'} \\ \langle a^C_{lm} a^{C*}_{l'm'}\rangle &= C^{CC}_l \delta_{ll'}\delta_{mm'} & \langle a^G_{lm} a^{C*}_{l'm'}\rangle &= C^{GC}_l \delta_{ll'}\delta_{mm'} \end{aligned}} \qquad (4)$$

- We also assume that the primordial random process was "parity conserving", i.e., it does not distinguish between the universe and its mirror image. Since the background universe is parity symmetric (equal to its mirror image); the perturbed universe is then statistically parity symmetric, i.e. the universe and its mirror image have the same statistical properties.

In the mirror universe the polarization pseudoscalar potential B has opposite sign, whereas the scalar potential A and the brightness function Θ are unchanged. We say, that in a parity transformation

$$\begin{aligned} \Theta \to \Theta \quad &\Rightarrow \quad a^T_{lm} \to a^T_{lm} \\ A \to A \quad &\Rightarrow \quad a^G_{lm} \to a^G_{lm} \\ B \to -B \quad &\Rightarrow \quad a^C_{lm} \to -a^C_{lm} \end{aligned} \tag{5}$$

From statistical parity symmetry

$$\begin{aligned} \Rightarrow \quad -\langle a^T_{lm} a^{C*}_{lm} \rangle &= \langle a^T_{lm} a^{C*}_{lm} \rangle = C^{TC} = 0 \\ -\langle a^G_{lm} a^{C*}_{lm} \rangle &= \langle a^G_{lm} a^{C*}_{lm} \rangle = C^{GC} = 0 \end{aligned} \tag{6}$$

$\therefore$ We have only four angular power spectra $\underline{C_l^{TT},\ C_l^{GG},\ C_l^{CC},\ C_l^{TG}}$

- In the spin-2 formalism conventions the three polarization-related power spectra are

$$\begin{aligned} C_l^{EE} &\equiv \langle a^E_{lm} a^{E*}_{lm} \rangle = 2C_l^{GG} \\ C_l^{BB} &\equiv \langle a^B_{lm} a^{B*}_{lm} \rangle = 2C_l^{CC} \\ C_l^{TE} &\equiv \langle a^T_{lm} a^{E*}_{lm} \rangle = -\sqrt{2}\, C_l^{TG} \end{aligned} \tag{7}$$

Correlation functions

- The temperature correlation function on the sphere (the sky) is defined

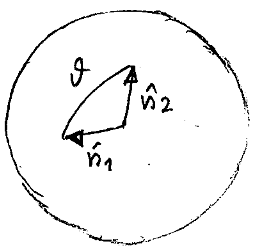
$$C^{TT}(\hat{n}_1, \hat{n}_2) \equiv \langle \Theta(\hat{n}_1)\Theta(\hat{n}_2)\rangle \qquad (8)$$

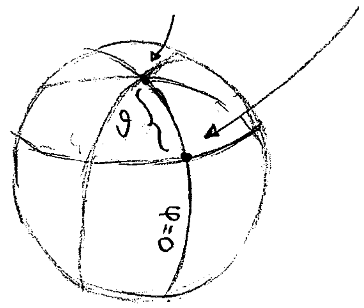
This is the predicted correlation from the random process, since $\langle\cdot\rangle$ denotes the expectation value. Due to statistical isotropy, this can only depend on the angle ϑ between the two directions

$$C^{TT}(\vartheta) \equiv \langle \Theta(\hat{n}_1)\Theta(\hat{n}_2)\rangle_{|\hat{n}_1\cdot\hat{n}_2 = \cos\vartheta} \qquad (9)$$

From the observed sky we get an estimate for this correlation function by taking an average over all pairs of directions separated by angle ϑ.

- Due to statistical isotropy we can thus choose $\hat{n}_1 = (0,0)$, $\hat{n}_2 = (\vartheta, 0)$





and we find $(\Theta = \Theta^*)$

$$\underline{C^{TT}(\vartheta)} = \langle \Theta(0,0)\Theta(\vartheta,0)\rangle = \sum_{lm}\sum_{l'm'} \langle a^{T*}_{lm} a^{T}_{l'm'}\rangle \underbrace{Y^{m*}_l(0,0)}_{\sqrt{\frac{2l+1}{4\pi}}\,\delta_{m0}} Y^{m'}_{l'}(\vartheta,0)$$

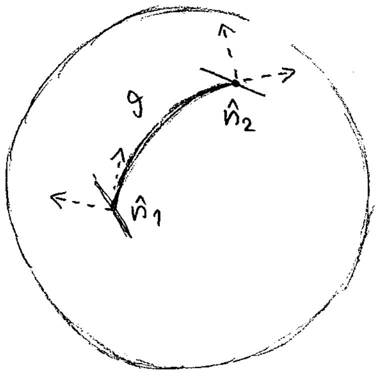
$$= \sum_{lm}\sum_{l'm'} C^{TT}_l \delta_{ll'}\delta_{mm'} \sqrt{\frac{2l+1}{4\pi}}\,\delta_{m0} \underbrace{Y^0_l(\vartheta,0)}_{\sqrt{\frac{2l+1}{4\pi}}\,P_l(\cos\vartheta)} = \underline{\sum_l \frac{2l+1}{4\pi} C^{TT}_l P_l(\cos\vartheta)} \qquad (10)$$

- Taking $\vartheta = 0$ $(\hat{n}_1 = \hat{n}_2)$ we get the temperature autocorrelation

$\equiv$ temperature variance

$$\underline{C^{TT}(0) = \langle \Theta(\hat{n})\Theta(\hat{n})\rangle = \left\langle \left(\frac{\delta T}{T}\right)^2 \right\rangle = \sum_l \frac{2l+1}{4\pi} C^{TT}_l} \qquad (11)$$

- For polarization, the definition of the correlation functions is more tricky, because of the crd. dependence of the Stokes parameters. Linear polarization has a direction on the sphere, given by Q and U. For correlating this direction between two points $\hat{n}_1$ and $\hat{n}_2$ on the sphere, the coordinate-independent way of comparing these directions is to relate them to the direction $\hat{n}_1$ and $\hat{n}_2$ are from each other. This direction is uniquely (up to a sign, which does not matter for polarization) determined by the great circle connecting $\hat{n}_1$ and $\hat{n}_2$.



Thus for the purpose of defining the polarization correlation between $\hat{n}_1$ and $\hat{n}_2$, we redefine the Stokes parameters with respect to a crd. system defined by this great circle, so that the "x" or "ϑ" direction is the direction along the great circle from $\hat{n}_1$ to $\hat{n}_2$ (the shorter way, so the separation is always $0 \le \vartheta \le \pi$), and the "y" or "φ" direction is orthogonal, so that $(\hat{x}, \hat{y}, \hat{n}_i)$ form a right-handed basis at both points. We call these redefined Stokes parameters Q_r and U_r. Positive Q_r corresponds to polarization along this great circle and negative Q_r orthogonal to it. U_r measures polarization along directions 45° from it. We can now define the correlation functions

$$
\begin{aligned}
C^{QQ}(\hat{n}_1,\hat{n}_2) &\equiv \langle Q_r(\hat{n}_1) Q_r(\hat{n}_2)\rangle & \qquad C^{TQ} &\equiv \langle \Theta_r(\hat{n}_1) Q_r(\hat{n}_2)\rangle \\
C^{UU}(\hat{n}_1,\hat{n}_2) &\equiv \langle U_r(\hat{n}_1) U_r(\hat{n}_2)\rangle & \qquad C^{TU} &\equiv \langle \Theta_r(\hat{n}_1) U_r(\hat{n}_2)\rangle \\
C^{QU}(\hat{n}_1,\hat{n}_2) &\equiv \langle Q_r(\hat{n}_1) U_r(\hat{n}_2)\rangle & &
\end{aligned}
\qquad (12)
$$

Statistical isotropy $\Rightarrow$ these functions depend only on the angle ϑ between $\hat{n}_1$ and $\hat{n}_2$,

$$\cos\vartheta = \hat{n}_1 \cdot \hat{n}_2, \qquad 0 \le \vartheta \le \pi$$

- Thus the polarization-related correlation functions are

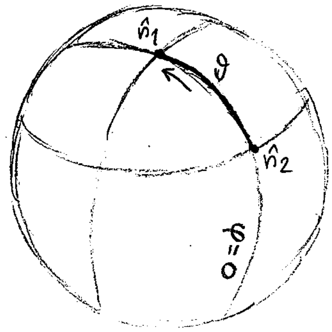
$$\boxed{\begin{array}{llll} C^{QQ}(\vartheta) \equiv \langle Q_r(\hat{n}_1) Q_r(\hat{n}_2)\rangle & & C^{TQ}(\vartheta) \equiv \langle \Theta_r(\hat{n}_1) Q_r(\hat{n}_2)\rangle \\ C^{UU}(\vartheta) \equiv \langle U_r(\hat{n}_1) U_r(\hat{n}_2)\rangle & & C^{TU}(\vartheta) \equiv \langle \Theta_r(\hat{n}_1) U_r(\hat{n}_2)\rangle \\ C^{QU}(\vartheta) \equiv \langle Q_r(\hat{n}_1) U_r(\hat{n}_2)\rangle & & \cos\vartheta \equiv \hat{n}_1\cdot\hat{n}_2 \end{array}} \qquad (13)$$

and for evaluating them, we can choose $\hat{n}_1 = (0,0)$, $\hat{n}_2 = (\vartheta, 0)$.

This makes the great circle connecting them equal to the ϑ coordinate with $\varphi = 0$, and the redefined Stokes parameters become thus the usual Stokes parameters defined wrt the ϑ,φ crd system: $\underline{Q_r = Q, \quad U_r = U}$ (this holds for $\underline{\vartheta_1 = \vartheta_2}$, even if $\varphi \neq 0$)

- Actually the usual Stokes parameters are ill-defined at the NP (also at the SP) where $\vartheta = 0$ but φ is multivalued. We can treat this by defining the Stokes parameters to be likewise <u>multivalued</u>:

$$\boxed{\begin{array}{lll} Q(0,\varphi) & \equiv & \lim\limits_{\vartheta\to 0} Q(\vartheta,\varphi) \\ U(0,\varphi) & \equiv & \lim U(\vartheta,\varphi) \end{array}} \qquad (14)$$



which gives the above $Q_r = Q$, $U_r = U$, with $\varphi = 0$.

- We now evaluate $C^{QQ}(\vartheta)$

From (3.18),
$$\underline{Q(\vartheta,\varphi) = \sum_{lm} N_l \left[a^G_{lm} W_{lm}(\vartheta,\varphi) - a^C_{lm} X_{lm}(\vartheta,\varphi)\right]} \qquad (15)$$

$$\Rightarrow \quad \underline{C^{QQ}(\vartheta)} \equiv \langle Q(0,0)\, Q(\vartheta,0)\rangle$$

$$= \sum_{lm}\sum_{l'm'} N_l N_{l'} \left\langle \left[a^G_{lm} W_{lm}(0,0) - a^C_{lm} X_{lm}(0,0)\right]\left[a^{G*}_{l'm'} W^*_{l'm'}(\vartheta,0) - a^{C*}_{l'm'} X^*_{l'm'}(\vartheta,0)\right]\right\rangle$$

$$= \sum_{lm}\sum_{l'm'} N_l N_{l'} \Big[W_{lm}(0,0) W^*_{l'm'}(\vartheta,0)\langle a^G_{lm} a^{G*}_{l'm'}\rangle - W_{lm}(0,0) X^*_{l'm'}(\vartheta,0) \underbrace{\langle a^G_{lm} a^{C*}_{l'm'}\rangle}_{=0}$$
$$- X_{lm}(0,0) W^*_{l'm'}(\vartheta,0) \underbrace{\langle a^C_{lm} a^{G*}_{l'm'}\rangle}_{=0} + X_{lm}(0,0) X^*_{l'm'}(\vartheta,0) \underbrace{\langle a^C_{lm} a^{C*}_{l'm'}\rangle}_{C^{CC}_l \delta_{ll'} \delta_{mm'}} \Big]$$

$$= \sum_{lm} N_l^2 \Big[\underbrace{W_{lm}(0,0)}_{\frac{1}{N_l}\sqrt{\frac{2l+1}{8\pi}}(\delta_{m2}+\delta_{m,-2})} W^*_{lm}(\vartheta,0)\, C^{GG}_l + \underbrace{X_{lm}(0,0)}_{\frac{i}{N_l}\sqrt{\frac{2l+1}{8\pi}}(\delta_{m2}-\delta_{m,-2})} X^*_{lm}(\vartheta,0)\, C^{CC}_l \Big]$$

(if do this for some other φ, the phases cancel here)

$$\underline{= \sum_l \sqrt{\frac{2l+1}{8\pi}}\, N_l \left\{ \left[W^*_{l2}(\vartheta,0) + W^*_{l,-2}(\vartheta,0)\right] C^{GG}_l + i\left[X^*_{l2}(\vartheta,0) - X^*_{l,-2}(\vartheta,0)\right] C^{CC}_l \right\}} \qquad (16)$$

- At $\varphi = 0$, the $W_{lm}(\vartheta,0)$ are real and the $X_{lm}(\vartheta,0)$ are imaginary, and from (6.4)

$$W_{l,-2} = W_{l2}^* \quad \text{and} \quad X_{l,-2} = X_{l2}^* \quad \text{so}$$

$$W_{l2}^*(\vartheta,0) + W_{l,-2}(\vartheta,0) = 2W_{l2}(\vartheta,0) \quad \text{and} \quad \underbrace{iX_{l2}^*(\vartheta,0)}_{-iX_{l2}(\vartheta,0)} - i\underbrace{X_{l,-2}^*(\vartheta,0)}_{X_{l2}(\vartheta,0)} = -2iX_{l2}(\vartheta,0)$$

and we get

$$\underline{C_l^{QQ}(\vartheta)} = \sum_l \sqrt{\frac{2l+1}{2\pi}}\, N_l \left[W_{l2}(\vartheta,0)\, C_l^{GG} - iX_{l2}(\vartheta,0)\, C_l^{CC}\right]$$

$$\underline{= \sum_l \frac{2l+1}{2\pi} N_l^2 \left[C_l^{GG} G_{l2}^+(\cos\vartheta) + C_l^{CC} G_{l2}^-(\cos\vartheta)\right]} \qquad (17)$$

- From (4.16), $$W_{l2}(0,0) = \frac{1}{N_l}\sqrt{\frac{2l+1}{8\pi}}, \quad X_{l2}(0,0) = \frac{i}{N_l}\sqrt{\frac{2l+1}{8\pi}} \qquad (18)$$

so the Q autocorrelation (variance) is

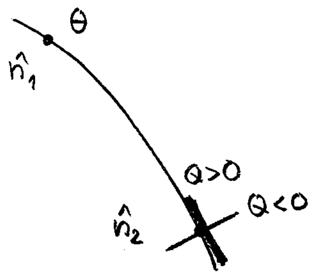
$$\boxed{\begin{aligned} C^{QQ}(0) = \langle Q(\hat{n})Q(\hat{n})\rangle = \langle Q^2\rangle &= \sum_l \frac{2l+1}{4\pi}\left(C_l^{GG} + C_l^{CC}\right) \\ &= \sum_l \frac{2l+1}{4\pi} \cdot \frac{C_l^{EE} + C_l^{BB}}{2} \end{aligned}} \qquad (19)$$

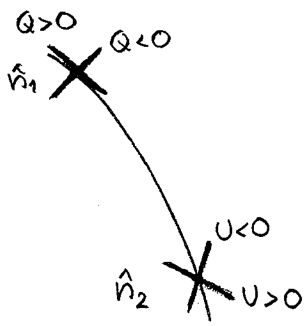
- Similarly we obtain (exercise)

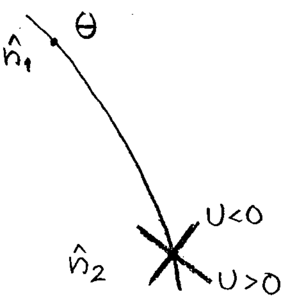
$$\boxed{\begin{aligned} C^{UU}(\vartheta) &= \sum_l \sqrt{\frac{2l+1}{2\pi}}\, N_l \left[W_{l2}(\vartheta,0)\, C_l^{CC} - iX_{l2}(\vartheta,0)\, C_l^{GG}\right] \\ &= \sum_l \frac{2l+1}{2\pi} N_l^2 \left[C_l^{GG} G_{l2}^-(\cos\vartheta) + C_l^{CC} G_{l2}^+(\cos\vartheta)\right] \\ C^{TQ}(\vartheta) &= \langle Q(0,0)\Theta(\vartheta,0)\rangle = \sum_l \frac{2l+1}{4\pi} N_l C_l^{TG} \overset{m=2,\text{ not square}}{P_l^2}(\cos\vartheta) \\ \text{and that}\quad C^{TU}(\vartheta) &= C^{QU}(\vartheta) = 0 \end{aligned}} \qquad (20)$$

- The vanishing of the TU and QU correlation is due to parity symmetry. (Remember, that they refer to Q_r and U_r !)

- Vanishing of TU and QU correlations:







The Stokes parameters $\Theta(\hat{n}_1), \Theta(\hat{n}_2), Q_r(\hat{n}_1), Q_r(\hat{n}_2)$ have the same value in a given universe and its mirror universe. But $U_r(\hat{n}_1), U_r(\hat{n}_2)$ have the opposite sign in the mirror universe. From parity conservation follows then

$$\langle Q_r(\hat{n}_1)U_r(\hat{n}_2)\rangle = -\langle Q_r(\hat{n}_1)U_r(\hat{n}_2)\rangle = 0 \quad \Rightarrow \quad C^{QU} = 0$$

$$\langle \Theta_r(\hat{n}_1)U_r(\hat{n}_2)\rangle = -\langle \Theta_r(\hat{n}_1)U_r(\hat{n}_2)\rangle = 0 \quad \Rightarrow \quad C^{TU} = 0$$

but for C^{UU} we get just $\langle U_r(\hat{n}_1)U_r(\hat{n}_2)\rangle = +\langle U_r(\hat{n}_1)U_r(\hat{n}_2)\rangle$, since both U_r change sign, and we get no condition.

- Evaluating $C^{UU}(\vartheta)$ and $C^{TQ}(\vartheta)$ at $\varphi = 0$, we get

$$\boxed{\begin{aligned} C^{UU}(0) &= \langle U^2\rangle = \sum_l \frac{2l+1}{4\pi}\left(C_l^{GG} + C_l^{CC}\right) = \sum_l \frac{2l+1}{4\pi}\frac{C_l^{EE}+C_l^{BB}}{2} = C^{QQ}(0) \\ C^{TQ}(0) &= \langle \Theta(\hat{n})Q(\hat{n})\rangle = \left\langle \frac{\delta T}{T}\cdot Q\right\rangle = \sum_l \frac{2l+1}{4\pi} N_l C_l^{TG} \underbrace{P_l^2(1)}_{=0} = 0 \end{aligned}} \qquad (21)$$

That the TQ autocorrelation (both perturbations evaluated at the same point) vanishes, is obvious from the statistical isotropy. We no longer have the connecting great circle to pick out a special direction, and therefore all polarization directions (both signs of Q (and U)) must be equally probable at the point where we are comparing T and Q.

- The Stokes parameter autocorrelation matrix $\langle (I,Q,U)\times(I,Q,U)\rangle$ is thus

$$\sum_l \frac{2l+1}{4\pi}\begin{bmatrix} C_l^{TT} & 0 & 0 \\ 0 & C_l^{GG}+C_l^{CC} & 0 \\ 0 & 0 & C_l^{GG}+C_l^{CC}\end{bmatrix} = \sum_l \frac{2l+1}{4\pi}\begin{bmatrix} C_l^{TT} & & \\ & \frac{C_l^{EE}+C_l^{BB}}{2} & \\ & & \frac{C_l^{EE}+C_l^{BB}}{2}\end{bmatrix} \qquad (22)$$

· In terms of the ${}_sY_L^m$, $\underline{C^{QQ}(\vartheta)} = \sum_L \sqrt{\frac{2L+1}{2\pi}} N_L \Big[\underbrace{W_{L2}(\vartheta,0)}_{\frac{1}{\sqrt{2}N_L}\left({}_2Y_L^2 + {}_{-2}Y_L^2\right)} C_L^{GG} \underbrace{- i X_{L2}(\vartheta,0)}_{\frac{-1}{\sqrt{2}N_L}\left({}_2Y_L^2 - {}_{-2}Y_L^2\right)} C_L^{CC} \Big]$

$$= \sum_L \sqrt{\frac{2L+1}{4\pi}} \left\{ \left[{}_2Y_L^2(\vartheta,0) + {}_{-2}Y_L^2(\vartheta,0) \right] C_L^{GG} - \left[{}_2Y_L^2(\vartheta,0) - {}_{-2}Y_L^2(\vartheta,0) \right] C_L^{CC} \right\}$$

$$= \sum_L \sqrt{\frac{2L+1}{4\pi}} \left\{ \frac{1}{2} \left[{}_2Y_L^2(\vartheta,0) + {}_{-2}Y_L^2(\vartheta,0) \right] C_L^{EE} - \frac{1}{2} \left[{}_2Y_L^2(\vartheta,0) - {}_{-2}Y_L^2(\vartheta,0) \right] C_L^{BB} \right\}$$

$$= \underline{\sum_L \sqrt{\frac{2L+1}{4\pi}} \left[\frac{1}{2}\left(C_L^{EE} - C_L^{BB}\right) {}_2Y_L^2(\vartheta,0) + \frac{1}{2}\left(C_L^{EE} + C_L^{BB}\right) {}_{-2}Y_L^2(\vartheta,0) \right]} \qquad (23)$$

and (from (17) and (20), just interchange E and B)

$$\boxed{\begin{aligned} C^{UU}(\vartheta) &= \sum_L \sqrt{\frac{2L+1}{2\pi}} \left[\frac{1}{2}\left(C_L^{BB} - C_L^{EE}\right) {}_2Y_L^2(\vartheta,0) + \frac{1}{2}\left(C_L^{EE} + C_L^{BB}\right) {}_{-2}Y_L^2(\vartheta,0) \right] \\ C^{TQ}(\vartheta) &= \sum_L \frac{2L+1}{4\pi} \underbrace{N_L \cdot \left(-\frac{1}{\sqrt{2}}\right)}_{-\sqrt{\frac{(L-2)!}{(L+2)!}}} C_L^{TE} P_L^2(\cos\vartheta) = - \sum_L \frac{2L+1}{4\pi} \sqrt{\frac{(L-2)!}{(L+2)!}} C_L^{TE} P_L^2(\cos\vartheta) \end{aligned}} \qquad (24)$$