Y5. Spin-2 Spherical Harmonics

- There is another school of how to derive the spherical harmonic expansion of linear polarization. Instead of noting that the Stokes parameters form a tensor field and utilizing the familiar machinery of differential geometry on the curved manifold of the sphere; one focuses on the complex combinations of the Stokes parameters $Q+iU$ and $Q-iU$. There will be no reference to a coordinate basis; as the Stokes parameters are always given in terms of a local orthonormal basis. The key observation is that in a coordinate transformation $\vartheta, \varphi \to \vartheta', \varphi'$

$$Q' = Q\cos 2\psi + U\sin 2\psi$$
$$U' = -Q\sin 2\psi + U\cos 2\psi$$

$$\Rightarrow \quad \boxed{\begin{aligned} Q'+iU' &= e^{-i2\psi}\,.\,(Q+iU) \\ Q'-iU' &= e^{+i2\psi}\,.\,(Q+iU) \end{aligned}} \qquad (1)$$

- One then searches for a suitable way to do multipole expansion for functions on the sphere w such transformation properties, and finds differential operators which, acting on the ordinary (now called "spin-0") spherical harmonics produce so-called spin-2 (or spin 2 and spin -2) spherical harmonics ${}_2Y_\ell^m$ and ${}_{-2}Y_\ell^m$.

These differential operators are closely related to the covariant derivatives we used to obtain the tensor spherical harmonics, and the end result is that

$$\boxed{\begin{aligned} {}_2Y_\ell^m &= \frac{N_\ell}{\sqrt{2}}\left(W_{\ell m}+iX_{\ell m}\right) \\ {}_{-2}Y_\ell^m &= \frac{N_\ell}{\sqrt{2}}\left(W_{\ell m}-iX_{\ell m}\right) \end{aligned}} \qquad \begin{aligned} W_{\ell m} &= \frac{1}{\sqrt{2}N_\ell}\left({}_2Y_\ell^m + {}_{-2}Y_\ell^m\right) \\ X_{\ell m} &= \frac{-i}{\sqrt{2}N_\ell}\left({}_2Y_\ell^m - {}_{-2}Y_\ell^m\right) \end{aligned} \qquad (2)$$

Using (3.18) we then have that

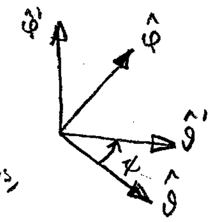
$$Q \pm iU = \sum_{\ell m} a_{\ell m}^G N_\ell \left(W_{\ell m} \pm iX_{\ell m}\right) + \sum_{\ell m} a_{\ell m}^C N_\ell \underbrace{\left(-X_{\ell m} \pm iW_{\ell m}\right)}_{\pm i\left(W_{\ell m} \pm iX_{\ell m}\right)} \qquad (3)$$

$$= \sum_{\ell m} \underbrace{\sqrt{2}\left(a_{\ell m}^G \pm i\, a_{\ell m}^C\right)}_{\equiv\, a_{\pm 2,\ell m}} {}_{\pm 2}Y_\ell^m$$

- The spin-2 spherical harmonics school of analyzing CMB polarization originates from Zaldarriaga & Seljak, PRD 55, 1830 (1997), while the tensor spherical harmonics school originates from Kamionkowski, Kosowsky, Stebbins, PRD 55, 7368 (1997). In the end they lead to the same; but the existence of two schools has led to different conventions regarding signs and factors of $\sqrt{2}$. The two conventions are distinguished by using the letters G and C for the two polarization modes in the tensor harmonic conventions; and the letters E and B in the spin-2 harmonic convention. We did the derivation in the tensor harmonic formalism, since it is physically more transparent; but the spin-2 harmonic formalism leads to a somewhat more compact math. The spin-2 harmonic conventions have become dominant in the field, and we shall later mostly use those.

- In general, quantities defined on the sphere, which transform as

$$f'(\hat{n}) = e^{-is\psi} f(\hat{n}) \qquad (4)$$



where ψ is the local (at $\hat{n}$) rotation angle of the $\hat{\vartheta}, \hat{\varphi}$ unit vectors, are called spin-s functions, and they are expanded in spin-s spherical harmonics ${}_sY_\ell^m(\vartheta,\varphi)$. The ordinary $Y_\ell^m(\vartheta,\varphi) \equiv {}_0Y_\ell^m(\vartheta,\varphi)$.

- The spin-s spherical harmonics (for a given s) form a complete set of orthonormal functions on the sphere,

$$\boxed{\int d\Omega \; {}_sY_\ell^{m}(\vartheta,\varphi)\, {}_sY_{\ell'}^{m'}(\vartheta,\varphi) = \delta_{\ell\ell'}\delta_{mm'}} \qquad (5)$$

- Thus we have

$$\boxed{\begin{aligned} a_{2,\ell m} &= \int d\Omega \; {}_2Y_\ell^{m*}(\vartheta,\varphi)\,(Q+iU)(\vartheta,\varphi) \\ a_{-2,\ell m} &= \int d\Omega \; {}_{-2}Y_\ell^{m*}(\vartheta,\varphi)\,(Q-iU)(\vartheta,\varphi) \end{aligned}} \qquad \boxed{\begin{aligned} Q+iU &= \sum_{\ell m} a_{2,\ell m}\, {}_2Y_\ell^m \\ Q-iU &= \sum_{\ell m} a_{-2,\ell m}\, {}_{-2}Y_\ell^m \end{aligned}} \qquad (6)$$

$$\boxed{\begin{aligned} Q &= \frac{1}{2}\sum_{\ell m}\left(a_{2,\ell m}\, {}_2Y_\ell^m + a_{-2,\ell m}\, {}_{-2}Y_\ell^m \right) \\ U &= -\frac{i}{2}\sum_{\ell m}\left(a_{2,\ell m}\, {}_2Y_\ell^m - a_{-2,\ell m}\, {}_{-2}Y_\ell^m \right) \end{aligned}} \qquad (7)$$

The E and B Modes

- ZS97: "Instead of $a_{2,lm}$, $a_{-2,lm}$ it is convenient to introduce their linear combinations"

$$\boxed{\begin{aligned} a^E_{lm} &\equiv -\tfrac{1}{2}(a_{2,lm} + a_{-2,lm}) = -\sqrt{2}\, a^G_{lm} \\ a^B_{lm} &\equiv \tfrac{i}{2}(a_{2,lm} - a_{-2,lm}) = -\sqrt{2}\, a^C_{lm} \end{aligned}} \quad (8)$$

$$\Rightarrow \begin{aligned} a_{2,lm} &= -a^E_{lm} - i a^B_{lm} \\ a_{-2,lm} &= -a^E_{lm} + i a^B_{lm} \end{aligned} \qquad \Rightarrow \begin{aligned} Q+iU &= -\sum (a^E_{lm} + i a^B_{lm})\, {}_2Y^m_l \\ Q-iU &= -\sum (a^E_{lm} - i a^B_{lm})\, {}_{-2}Y^m_l \end{aligned} \quad (9)$$

To close the loop, ZS define the spin-0 fields,

$$\begin{aligned} \tilde{E}(\hat{n}) &\equiv \sum_{lm} \sqrt{\frac{(l+2)!}{(l-2)!}}\, a^E_{lm} Y^m_l(\hat{n}) = -2 \sum_{lm} \frac{1}{N_l} a^G_{lm} Y^m_l = -2A(\hat{n}) \\ \tilde{B}(\hat{n}) &\equiv \sum_{lm} \sqrt{\frac{(l+2)!}{(l-2)!}}\, a^B_{lm} Y^m_l(\hat{n}) = -2 \sum_{lm} \frac{1}{N_l} a^C_{lm} Y^m_l = -2B(\hat{n}), \end{aligned} \quad (10)$$

the scalar and pseudoscalar polarization potentials we started from.

- The point in introducing the E and B combinations of the spin +2 and spin -2 multipoles is that a) we recombine the complex $Q \pm iU$ into real quantities E and B and b) they have definite parity: E is scalar and B is pseudoscalar.

- HEALPix (a widely used software package for operations on pixelized CMB temperature and polarisation maps and their angular power spectra) used the KKS (C and G) conventions for versions ≤ 1.1, and uses the ZS (E and B) conventions for versions ≥ 1.2.

Properties of Spin-2 Spherical Harmonics

- From the relation between the tensor and spin-2 harmonics we obtain the corresponding properties for the ${}_sY_L^m$.

$$\boxed{\begin{aligned} {}_{+2}Y_L^{-m} &= (-1)^m\,{}_{-2}Y_L^{m*} \\ {}_{-2}Y_L^{-m} &= (-1)^m\,{}_{+2}Y_L^{m*} \end{aligned}} \quad (11) \quad \Rightarrow \quad \boxed{\begin{aligned} a_{2,Lm}^* &= (-1)^m a_{-2,L-m} \\ a_{-2,Lm}^* &= (-1)^m a_{2,L-m} \end{aligned}} \quad (12)$$

(from the reality of Q and U)

- The antipode and North-South relations

$$\boxed{\begin{aligned} {}_2Y_L^m(-\hat{n}) &= (-1)^L\,{}_{-2}Y_L^m(\hat{n}) \\ {}_{-2}Y_L^m(-\hat{n}) &= (-1)^L\,{}_2Y_L^m(\hat{n}) \end{aligned}} \quad \boxed{\begin{aligned} {}_2Y_L^m(\pi-\vartheta,\varphi) &= (-1)^{L+m}\,{}_{-2}Y_L^m(\vartheta,\varphi) \\ {}_{-2}Y_L^m(\pi-\vartheta,\varphi) &= (-1)^{L+m}\,{}_2Y_L^m(\vartheta,\varphi) \end{aligned}} \quad (13)$$

Notice how all the above relations involve the spin-flip $+2 \Leftrightarrow -2$

- North and South Poles

$$\boxed{\begin{aligned} {}_2Y_L^m(0,\varphi) &= \sqrt{\tfrac{2L+1}{4\pi}}\,\delta_{m,-2}\,e^{-i2\varphi} & {}_2Y_L^m(\pi,\varphi) &= (-1)^L\sqrt{\tfrac{2L+1}{4\pi}}\,\delta_{m2}\,e^{i2\varphi} \\ {}_{-2}Y_L^m(0,\varphi) &= \sqrt{\tfrac{2L+1}{4\pi}}\,\delta_{m,2}\,e^{i2\varphi} & {}_{-2}Y_L^m(\pi,\varphi) &= (-1)^L\sqrt{\tfrac{2L+1}{4\pi}}\,\delta_{m,-2}\,e^{-i2\varphi} \end{aligned}} \quad (14)$$

So they are $\boxed{\text{multivalued at the NP, if } m=-s}$ and $\boxed{\text{multivalued at the SP, if } m=s}$ (for $s=\pm 2$; not for $s=0$)