Y4. Properties of Tensor Spherical Harmonics

- The scalar spherical harmonics are $\quad Y_l^m(\vartheta,\varphi) = (-1)^m \sqrt{\frac{2l+1}{4\pi}} \sqrt{\frac{(l-m)!}{(l+m)!}}\, P_l^m(\cos\vartheta)\, e^{im\varphi}$ (1)

where the associated Legendre functions $P_l^m(\cos\vartheta)$ are real and

$$P_l^{-m} = (-1)^m \frac{(l-m)!}{(l+m)!} P_l^m \qquad (2)$$

$$\Rightarrow \underline{Y_l^{-m}} = (-1)^{-m} \sqrt{\frac{2l+1}{4\pi}} \sqrt{\frac{(l+m)!}{(l-m)!}}\, P_l^{-m} e^{-im\varphi} = \sqrt{\frac{2l+1}{4\pi}} \sqrt{\frac{(l-m)!}{(l+m)!}}\, P_l^m e^{-im\varphi} = \underline{(-1)^m Y_l^{m\,*}} \qquad (3)$$

- From this follows that also (using Eqs. (3.4) and (3.13))

$$\boxed{\begin{aligned} Y^{G/C}_{(l,-m)ab} &= (-1)^m Y^{G/C\,*}_{(lm)ab} \\ W_{l,-m} &= (-1)^m W^*_{lm} \\ X_{l,-m} &= (-1)^m X^*_{lm} \end{aligned}} \qquad (4)$$

- From (3.14) we can calculate the W_{lm} and X_{lm} in terms of the $P_l^m(\cos\vartheta)$: (exercise)

$$\boxed{\begin{aligned} W_{lm}(\vartheta,\varphi) &= l(l+1)Y_l^m + 2\partial_\vartheta^2 Y_l^m = .. = (-1)^m \cdot 2\sqrt{\frac{2l+1}{4\pi}}\sqrt{\frac{(l-m)!}{(l+m)!}}\, G^+_{lm}(\cos\vartheta)\, e^{im\varphi} \\ -iX_{lm}(\vartheta,\varphi) &= \frac{2m}{\sin\vartheta}(\partial_\vartheta - \cot\vartheta)Y_l^m = .. = (-1)^m \cdot 2\sqrt{\frac{2l+1}{4\pi}}\sqrt{\frac{(l-m)!}{(l+m)!}}\, G^-_{lm}(\cos\vartheta)\, e^{im\varphi} \end{aligned}} \qquad (5)$$

where

$$\boxed{\begin{aligned} G^+_{lm}(\cos\vartheta) &\equiv -\left[\frac{l-m^2}{\sin^2\vartheta} + \frac{1}{2}l(l-1)\right]P_l^m(\cos\vartheta) + (l+m)\frac{\cos\vartheta}{\sin^2\vartheta} P^m_{l-1}(\cos\vartheta) \\ G^-_{lm}(\cos\vartheta) &\equiv \frac{m}{\sin^2\vartheta}\left[(l-1)\cos\vartheta \cdot P_l^m(\cos\vartheta) - (l+m)P^m_{l-1}(\cos\vartheta)\right] \end{aligned}} \qquad (6)$$

are real,

where we have used the associated Legendre equation

$$\frac{1}{\sin\vartheta}\frac{d}{d\vartheta}\left(\sin\vartheta \cdot \frac{d}{d\vartheta}P_l^m(\cos\vartheta)\right) + \left[l(l+1) - \frac{m^2}{\sin^2\vartheta}\right]P_l^m(\cos\vartheta) = 0$$

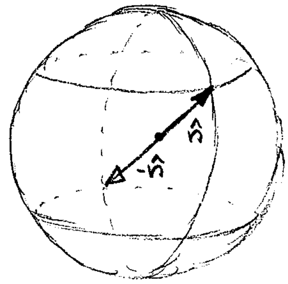
$$\Rightarrow \frac{d^2}{d\vartheta^2}P_l^m(\cos\vartheta) = x\frac{dP_l^m}{dx} + \frac{m^2}{1-x^2}P_l^m - l(l+1)P_l^m \qquad \text{where } x=\cos\vartheta \qquad (7)$$

and the recursion relation (Abramowicz & Stegun 8.5.4)

$$(x^2-1)\frac{d}{dx}P_l^m = lxP_l^m - (l+m)P^m_{l-1} \qquad (8)$$

- We see that the φ-dependence of $W_{lm}, X_{lm}, Y^G_{lm}, Y^C_{lm}$ is trivial, $e^{im\varphi}$, and that for $m=0$: W_{l0} is real and X_{l0} is imaginary

Antipodes



- The two directions opposite to each other, $\hat{n}$ and $-\hat{n}$, have ends

$$\hat{n} = (\vartheta, \varphi) \quad \text{and} \quad -\hat{n} = (\vartheta^-, \varphi^-) = (\pi - \vartheta, \varphi + \pi)$$

$$\Rightarrow \quad x^- = \cos\vartheta^- = -\cos\vartheta = -x$$

$$\underline{e^{im\varphi^-}} = e^{im\pi} e^{im\varphi} = \underline{(-1)^m e^{im\varphi}}$$

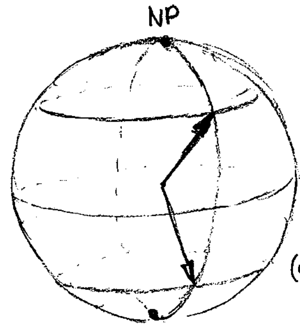
- The parity of the associated Legendre functions is $\underline{P_\ell^m(-x) = (-1)^{\ell+m} P_\ell^m(x)}$ (9)

$$\therefore \quad \underline{Y_\ell^m(-\hat{n})} = (-1)^m \cdot (-1)^{\ell+m} Y_\ell^m(\hat{n}) = \underline{(-1)^\ell Y_\ell^m(\hat{n})} \qquad (10)$$

- From the definitions of the $G^\pm_{\ell m}$, Eq. (6), we get easily (exercise)

$$\underline{G^\pm_{\ell m}(-x) = \pm(-1)^{\ell+m} G^\pm_{\ell m}(x)} \quad \Rightarrow \quad \boxed{\begin{aligned} W_\ell^m(-\hat{n}) &= (-1)^\ell W_\ell^m(\hat{n}) \\ X_\ell^m(-\hat{n}) &= -(-1)^\ell X_\ell^m(\hat{n}) \end{aligned}} \qquad (11)$$

North-South reflection



- If we just make a reflection wrt the equator, i.e.,

$$\vartheta \to \pi - \vartheta \quad \Rightarrow \quad \cos\vartheta \to -\cos\vartheta$$

but keep φ the same, we do not get the $e^{im\pi} = (-1)^m$ factor, and we have the results

$$\boxed{\begin{aligned} Y_\ell^m(\pi-\vartheta, \varphi) &= (-1)^{\ell+m} Y_\ell^m(\vartheta, \varphi) \\ W_\ell^m(\pi-\vartheta, \varphi) &= (-1)^{\ell+m} W_\ell^m(\vartheta, \varphi) \\ X_\ell^m(\pi-\vartheta, \varphi) &= -(-1)^{\ell+m} X_\ell^m(\vartheta, \varphi) \end{aligned}} \qquad (12)$$

North and South Poles

- The north and south poles (NP and SP) of the ϑ,φ crd. system correspond to $\vartheta = 0, \pi$.

$$x = \cos\vartheta = +1, -1$$

The φ coordinate is multivalued at those points (all values of φ correspond to the same point).

The $P_\ell^m(x)$ have the values
$$P_\ell^m(\pm 1) = 0 \quad \text{for } m \neq 0$$
$$P_\ell(1) = 1 \quad \text{and} \quad P_\ell(-1) = (-1)^\ell \tag{13}$$

From this we get
$$\boxed{\begin{aligned} Y_\ell^m(NP) &= \sqrt{\frac{2\ell+1}{4\pi}}\,\delta_{m0} \\ Y_\ell^m(SP) &= (-1)^\ell\sqrt{\frac{2\ell+1}{4\pi}}\,\delta_{m0}\end{aligned}} \tag{14}$$

- The $G^\pm_{\ell m}(\pm 1)$ are tricky to derive, since from (6) & (13) we just get $\frac{0}{0}$ or $\infty - \infty$, as $\sin\vartheta = 0$ at the poles. To evaluate them, we need to take the limits $\vartheta \to 0, \pi$.

From Cabella & Kamionkowski astro-ph/0403392 Eqs. (75) and (76),

$$P_\ell^m(\cos\vartheta) \sim \frac{(-1)^m}{2^m\, m!}\frac{(\ell+m)!}{(\ell-m)!}\vartheta^m \quad \text{for } m>0 \tag{15}$$
$$\text{as } \vartheta \to 0$$
$$P_\ell(\cos\vartheta) \sim 1 - \tfrac{1}{4}\ell(\ell+1)\vartheta^2$$

Using this in (6) we get (exercise) <u>for $m \geq 0$</u>, that

$$G^\pm_{\ell 2}(1) = \frac{1}{4}\frac{(\ell+2)!}{(\ell-2)!} \quad \Rightarrow \quad W_{\ell 2}(0,\varphi) = \frac{1}{N_\ell}\sqrt{\frac{2\ell+1}{8\pi}}\,e^{i2\varphi}$$
$$X_{\ell 2}(0,\varphi) = \frac{i}{N_\ell}\sqrt{\frac{2\ell+1}{8\pi}}\,e^{i2\varphi}$$

but other $G^\pm_{\ell m}(1) = W_{\ell m}(0,\varphi) = X_{\ell m}(0,\varphi) = 0$.

Using (4) to get the negative m, we have for the North Pole

$$\boxed{\begin{aligned} W_\ell^m(0,\varphi) &= \frac{1}{N_\ell}\sqrt{\frac{2\ell+1}{8\pi}}\left(\delta_{m2}\,e^{i2\varphi} + \delta_{m,-2}\,e^{-i2\varphi}\right) \\ X_\ell^m(0,\varphi) &= \frac{i}{N_\ell}\sqrt{\frac{2\ell+1}{8\pi}}\left(\delta_{m2}\,e^{i2\varphi} - \delta_{m,-2}\,e^{-i2\varphi}\right)\end{aligned}} \tag{16}$$

We get the South Pole using (12):

$$\boxed{\begin{aligned} W_\ell^m(\pi,\varphi) &= \frac{(-1)^\ell}{N_\ell}\sqrt{\frac{2\ell+1}{8\pi}}\left(\delta_{m2}\,e^{i2\varphi} + \delta_{m,-2}\,e^{-i2\varphi}\right) \\ X_\ell^m(\pi,\varphi) &= \frac{-i(-1)^\ell}{N_\ell}\sqrt{\frac{2\ell+1}{8\pi}}\left(\delta_{m2}\,e^{i2\varphi} - \delta_{m,-2}\,e^{-i2\varphi}\right)\end{aligned}} \tag{17}$$

Thus the $W_\ell^m(\vartheta,\varphi)$ and $X_\ell^m(\vartheta,\varphi)$ are <u>multivalued</u> at the poles for $m = \pm 2$!