Y3. Tensor Spherical Harmonics

- Since A and B are scalar fields (as long as we are not making any parity transformations, we can treat B as a scalar field; rotations of the spherical crd. system $\vartheta,\varphi \to \vartheta',\varphi'$ are ok), we can expand them in spherical harmonics in the usual manner:

$$A(\vartheta,\varphi) = \sum a_{lm} Y_l^m(\vartheta,\varphi)$$
$$B(\vartheta,\varphi) = \sum b_{lm} Y_l^m(\vartheta,\varphi) \qquad (1)$$

and the multipole coefficients transform as

$$a_{lm} = \sum_{m'} D^l_{mm'}(\alpha,\beta,\gamma)\, a'_{lm'}$$
$$b_{lm} = \sum_{m'} D^l_{mm'}(\alpha,\beta,\gamma)\, b'_{lm'} \qquad (2)$$

in a rotation of the crd. system. Here α,β,γ are the Euler angles of the $\vartheta,\varphi \to \vartheta',\varphi'$ rotation; (2) shows the inverse transformation of the a_{lm}, b_{lm}.

- Using (1) and (2.5) we can now write the polarization field P_{ab} as

$$\underline{P_{ab}(\vartheta,\varphi)} = \sum_{lm} a_{lm} \underbrace{\left(Y^m_{l\,;ab} - \tfrac{1}{2} g_{ab} Y_l^{m\,;c}{}_{;c}\right)}_{\equiv \frac{1}{N_l} Y^G_{(lm)ab}} + \sum_{lm} b_{lm} \cdot \underbrace{\tfrac{1}{2}\left(\varepsilon^c{}_b Y^m_{l\,;ac} + \varepsilon^c{}_a Y^m_{l\,;bc}\right)}_{\equiv \frac{1}{N_l} Y^C_{(lm)ab}}$$

$$= \underline{\sum_{lm} a^G_{lm} Y^G_{(lm)ab} + \sum_{lm} a^C_{lm} Y^C_{(lm)ab}} \qquad (3)$$

where we have defined the tensor spherical harmonics

$$\boxed{\begin{aligned} Y^G_{(lm)ab} &\equiv N_l\left(Y^m_{l\,;ab} - \tfrac{1}{2} g_{ab} Y_l^{m\,;c}{}_{;c}\right) \\ Y^C_{(lm)ab} &\equiv \tfrac{1}{2} N_l \left(\varepsilon^c{}_b Y^m_{l\,;ac} + \varepsilon^c{}_a Y^m_{l\,;bc}\right) \end{aligned}} \qquad (4)$$

and $\underline{N_l \equiv \sqrt{\frac{2(l-2)!}{(l+2)!}}}$ is a normalization factor. (5)

And we have just $a^G_{lm} = \frac{1}{N_l} a_{lm}$; $a^C_{lm} = \frac{1}{N_l} b_{lm}$. (6)

- Here the $Y^m_{l;ab}$ are second covariant derivatives of the ordinary (scalar) spherical harmonics Y^m_l.

$$Y^m_{l;ab} = (Y^m_{l,a})_{;b} = Y^m_{l,ab} - \Gamma^c_{ba} Y^m_{l,c} = Y^m_{l;ba} \qquad (7)$$

Using the Christoffel symbols from Eq. (2.2),

$$\begin{aligned} Y^m_{l;\vartheta\vartheta} &= Y^m_{l,\vartheta\vartheta} \\ Y^m_{l;\vartheta\varphi} &= Y^m_{l,\vartheta\varphi} - \cot\vartheta \cdot Y^m_{l,\varphi} \\ Y^m_{l;\varphi\varphi} &= Y^m_{l,\varphi\varphi} + \sin\vartheta\cos\vartheta\, Y^m_{l,\vartheta} \end{aligned} \qquad Y^{m;c}_{l\ \ ;c} = g^{cb} Y_{;bc} = Y_{;\vartheta\vartheta} + \frac{1}{\sin^2\vartheta} Y_{;\varphi\varphi} \qquad (8)$$

- What about $l=0$ and $l=1$? Eq. (5) gives $N_0 = N_1 = 0$. No problem.

$$Y^m_{l;ab} - \tfrac{1}{2} g_{ab} Y^{m;c}_{l\ \ ;c} = \varepsilon^c{}_b Y^m_{l;ac} + \varepsilon^c{}_a Y^m_{l;bc} = 0 \quad \text{for} \quad l=0,1 \qquad (9)$$

as a direct calculation (exercise) shows.

- The tensor spherical harmonics are orthonormal (we do not attempt to show this)

$$\boxed{\begin{aligned} \int d\Omega\, Y^{G*}_{(lm)ab}(\vartheta,\varphi)\, Y^{G\,ab}_{(l'm')}(\vartheta,\varphi) &= \int d\Omega\, Y^{C*}_{(lm)ab}(\vartheta,\varphi)\, Y^{C\,ab}_{(l'm')}(\vartheta,\varphi) = \delta_{ll'}\delta_{mm'} \\ \int d\Omega\, Y^{G*}_{(lm)ab}(\vartheta,\varphi)\, Y^{C\,ab}_{(l'm')}(\vartheta,\varphi) &= 0 \end{aligned}} \qquad (10)$$

so that

$$\boxed{\begin{aligned} a^G_{lm} &= \int d\Omega\, P_{ab}(\vartheta,\varphi)\, Y^{G\,ab*}_{(lm)}(\vartheta,\varphi) \\ a^C_{lm} &= \int d\Omega\, P_{ab}(\vartheta,\varphi)\, Y^{C\,ab*}_{(lm)}(\vartheta,\varphi) \end{aligned}} \qquad (11)$$

- This means that the monopole and dipole components of the scalar and pseudoscalar polarization potentials $A(\hat{n})$ and $B(\hat{n})$ make no contribution to the polarization $P_{ab}(\hat{n})$; by convention we set them to zero.

- Thus the a^G_{lm}, a^C_{lm}, $Y^G_{(lm)ab}$, $Y^C_{(lm)ab}$ <u>begin at $l=2$</u>, and

$$P_{ab}(\vartheta,\varphi) = \sum_{l=2}^{\infty}\sum_{m=-l}^{l} a^G_{lm} Y^G_{(lm)ab} + \sum_{l=2}^{\infty}\sum_{m=-l}^{l} a^C_{lm} Y^C_{(lm)ab} \qquad (12)$$

- From (2) and (6) follows that the multipole coefficients a^G_{lm}, a^C_{lm} transform the usual way

$$\boxed{a^{G/C}_{lm} = \sum_{m'} D^l_{mm'}(\alpha,\beta,\gamma)\, a^{G/C\,\prime}_{lm'}} \qquad (12\tfrac{1}{2})$$

in a rotation of the spherical coord. system.

- It may seem that we have multiplied the number of spherical harmonic components by a lot, with G, C, and $ab = \vartheta\vartheta, \vartheta\varphi, \varphi\vartheta, \varphi\varphi$. Actually, there are just two independent components.

First, the $Y^G_{(lm)ab}$, $Y^C_{(lm)ab}$ are traceless and symmetric by construction, so each of them has only two independent components. Moreover the components of $Y^G_{(lm)}$ and $Y^C_{(lm)}$ are related to each other, and we can write them as (exercise)

$$\boxed{Y^G_{(lm)ab} = \frac{N_l}{2}\begin{bmatrix} W_{lm} & X_{lm}\cdot\sin\vartheta \\ X_{lm}\cdot\sin\vartheta & -W_{lm}\cdot\sin^2\vartheta \end{bmatrix} \qquad Y^C_{(lm)ab} = \frac{N_l}{2}\begin{bmatrix} -X_{lm} & W_{lm}\cdot\sin\vartheta \\ W_{lm}\cdot\sin\vartheta & +X_{lm}\cdot\sin^2\vartheta \end{bmatrix}} \tag{13}$$

where the two independent functions are

$$\begin{aligned} \underline{W_{lm} \equiv Y_{lm;\vartheta\vartheta} - \frac{1}{\sin^2\vartheta}Y_{lm;\varphi\varphi}} &= \partial^2_\vartheta Y_{lm} - \frac{1}{\sin^2\vartheta}\partial^2_\varphi Y_{lm} - \frac{\cos\vartheta}{\sin\vartheta}\partial_\vartheta Y_{lm} \\ &= \left(\partial^2_\vartheta - \cot\vartheta\cdot\partial_\vartheta + \frac{m^2}{\sin^2\vartheta}\right)Y_{lm} = \underline{l(l+1)Y_{lm} + 2\partial^2_\vartheta Y_{lm}} \\ \underline{X_{lm} \equiv \frac{2}{\sin\vartheta}Y_{lm;\vartheta\varphi}} &= \frac{2}{\sin\vartheta}\partial_\vartheta\partial_\varphi Y_{lm} - \frac{2\cos\vartheta}{\sin^2\vartheta}\partial_\varphi Y_{lm} = \underline{\frac{2im}{\sin\vartheta}\left(\partial_\vartheta - \cot\vartheta\right)Y_{lm}} \end{aligned} \tag{14}$$

where we have used the properties of spherical harmonics

$$\begin{aligned} \hat{L}_z Y_{lm} &\equiv -i\partial_\varphi Y_{lm} = mY_{lm} \\ \hat{L}^2 Y_{lm} &\equiv -\left\{\frac{1}{\sin^2\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial}{\partial\vartheta}\right) + \frac{1}{\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2}\right\}Y_{lm} \\ &= -\cot\vartheta\,\partial_\vartheta Y_{lm} - \partial^2_\vartheta Y_{lm} - \frac{1}{\sin^2\vartheta}\partial^2_\varphi Y_{lm} = l(l+1)Y_{lm} \end{aligned} \tag{15}$$

where $\hat{L}_z$ and $\hat{L}^2$ are angular momentum operators.

- Note that in the orthonormal basis

$$P_{\hat{a}\hat{b}}(\vartheta,\varphi) = \frac{1}{2}\begin{bmatrix} Q & U \\ U & -Q \end{bmatrix} = \sum_{lm} a^G_{lm} Y^G_{(lm)\hat{a}\hat{b}} + \sum_{lm} a^C_{lm} Y^C_{(lm)\hat{a}\hat{b}}, \tag{16}$$

where

$$Y^G_{(lm)\hat{a}\hat{b}} \equiv \frac{N_l}{2}\begin{bmatrix} W_{lm} & X_{lm} \\ X_{lm} & -W_{lm} \end{bmatrix} \qquad Y^C_{(lm)\hat{a}\hat{b}} \equiv \frac{N_l}{2}\begin{bmatrix} -X_{lm} & W_{lm} \\ W_{lm} & +X_{lm} \end{bmatrix} \tag{17}$$

- For the Stokes parameters we have

$$\boxed{\begin{aligned} Q &= \sum_{lm} a^G_{lm} N_l W_{lm} - \sum_{lm} a^C_{lm} N_l X_{lm} \\ U &= \sum_{lm} a^G_{lm} N_l X_{lm} + \sum_{lm} a^C_{lm} N_l W_{lm} \end{aligned}} \tag{18}$$