Y2. Geometry of the Sphere

- We now return to the full sphere. The geometry is determined by the metric

$$ds^2 = d\vartheta^2 + \sin^2\vartheta \, d\varphi^2 \qquad g_{ab} = \begin{array}{cc} \vartheta & \varphi \end{array} \begin{bmatrix} 1 & 0 \\ 0 & \sin^2\vartheta \end{bmatrix} \tag{1}$$

The Christoffel symbols are

$$\boxed{\begin{aligned} \Gamma^\vartheta_{\varphi\varphi} &= -\sin\vartheta\cos\vartheta \\ \Gamma^\varphi_{\vartheta\varphi} &= \Gamma^\varphi_{\varphi\vartheta} = \cot\vartheta \end{aligned}} \tag{2}$$

$$g \equiv \det g = \sin^2\vartheta$$

$$\sqrt{g} = \sin\vartheta$$

Since the metric is diagonal (the spherical crd. system is orthogonal), it is easy to convert between the crd. basis and the corresponding orthonormal basis

$$P_{\hat a \hat b} = \frac{1}{2}\begin{bmatrix} Q & U \\ U & -Q \end{bmatrix}$$

$$P_{ab} = \sqrt{g_{aa}}\sqrt{g_{bb}}\, P_{\hat a \hat b} = \frac{1}{2}\begin{bmatrix} Q & \sin\vartheta . U \\ \sin\vartheta . U & -\sin^2\vartheta . Q \end{bmatrix} \tag{3}$$

The antisymmetric Levi-Civita tensor becomes in the crd. basis

$$\varepsilon_{ab} = \sqrt{g_{aa}}\sqrt{g_{bb}}\, \varepsilon_{\hat a \hat b} = \sqrt{g}\, \varepsilon_{\hat a \hat b} = \sin\vartheta \begin{bmatrix} & 1 \\ -1 & \end{bmatrix} \tag{4}$$

- We can now generalize the flat-sky result (1.17) to the curved manifold of the 2-sphere:

$$P_{ab} = A_{;ab} - \frac{1}{2} g_{ab} A^{;c}{}_{;c} + \frac{1}{2}\left(\varepsilon^c{}_b B_{;ac} + \varepsilon^c{}_a B_{;bc}\right) \tag{5}$$

Here

$$\varepsilon^c{}_b = g^{ca}\varepsilon_{ab} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\sin^2\vartheta} \end{bmatrix}\begin{bmatrix} 0 & \sin\vartheta \\ -\sin\vartheta & 0 \end{bmatrix} = \begin{bmatrix} 0 & \sin\vartheta \\ -\frac{1}{\sin\vartheta} & 0 \end{bmatrix} \tag{6}$$