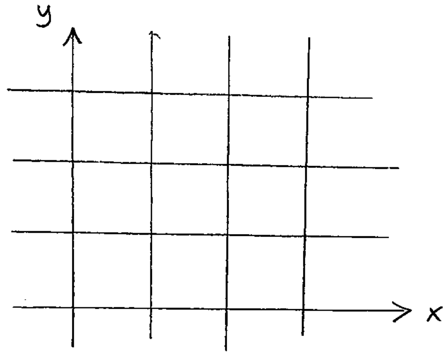
Y1. Flat-Sky Approximation



- Consider just a small part of the sphere (or of the sky, if we are thinking at the observer approach (2)), where we can ignore the curvature: We use Euclidean geometry and introduce Cartesian crd's x and y.

- Now
$$P_{ab}(x,y) = +\frac{1}{2}\begin{bmatrix} Q & U \\ U & -Q \end{bmatrix} = \begin{bmatrix} P_{xx} & P_{xy} \\ P_{yx} & P_{yy} \end{bmatrix} = \begin{bmatrix} P_{xx} & P_{xy} \\ P_{xy} & -P_{xx} \end{bmatrix} \quad (5)$$

We can introduce a vector field w^a related to the polarization field P_{ab}, by taking the divergence:
$$w^a \equiv p^{ab}{}_{,b} = p^{ba}{}_{,b} \quad (6)$$

(it doesn't matter whether an index is up or down, since we use Cartesian crds).
If we do a Fourier transformation on this 2-d plane, the divergence becomes

$$w^a = ik_b p^{ab}, \quad \text{or} \quad (7)$$

$$\begin{bmatrix} w^x \\ w^y \end{bmatrix} = \begin{bmatrix} ik_x p^{xx} + ik_y p^{xy} \\ ik_x p^{yx} + ik_y p^{yy} \end{bmatrix} = \begin{bmatrix} ik_x p^{xx} + ik_y p^{xy} \\ ik_x p^{xy} - ik_y p^{xx} \end{bmatrix} = i\begin{bmatrix} k_x & k_y \\ -k_y & k_x \end{bmatrix}\begin{bmatrix} p^{xx} \\ p^{xy} \end{bmatrix}$$

We can now invert this matrix relation:

$$\begin{bmatrix} p^{xx} \\ p^{xy} \end{bmatrix} = -i\begin{bmatrix} k_x & k_y \\ -k_y & k_x \end{bmatrix}^{-1}\begin{bmatrix} w^x \\ w^y \end{bmatrix} = \frac{-i}{k_x^2+k_y^2}\begin{bmatrix} k_x & -k_y \\ k_y & k_x \end{bmatrix}\begin{bmatrix} w^x \\ w^y \end{bmatrix} \quad (8)$$

Thus the polarization tensor field is determined by this vector field (up to a constant, corresponding to the $\vec{k} = (k_x, k_y) = 0$ Fourier mode). To express (8) as a derivative in $\vec{x}$-space, we need to get rid of the $k^2 \equiv k_x^2 + k_y^2$. Thus we define a new vector field $\vec{v}$ in Fourier space as

$$v^a = \frac{2}{k^2} w^a \quad (9)$$

In $\vec{x}$-space this corresponds to solving the $\vec{v}$ field from the differential equation

$$\nabla^2 \vec{v} = -2\,\vec{w} \qquad (10)$$

Besides $\vec{w}$, the solution depends on the boundary conditions. Using Fourier expansion corresponds to assuming periodic boundary conditions. (When we return to the sphere, there are no boundaries. Also, a constant nonzero linear polarization field on a sphere is not possible, since it unavoidably leads to a singularity at some point.)

Eq. (8) becomes

$$\begin{bmatrix} p^{xx} \\ p^{xy} \end{bmatrix} = -\frac{i}{2}\begin{bmatrix} k_x & -k_y \\ k_y & k_x \end{bmatrix}\begin{bmatrix} v^x \\ v^y \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -ik_x & +ik_y \\ -ik_y & -ik_x \end{bmatrix}\begin{bmatrix} v^x \\ v^y \end{bmatrix}. \qquad (11)$$

Returning to $\vec{x}$-space, we have

$$\begin{aligned} p^{xx} &= -p^{yy} = -\tfrac{1}{2}\partial_x v^x + \tfrac{1}{2}\partial_y v^y \\ p^{xy} &= p^{yx} = -\tfrac{1}{2}\partial_y v^x - \tfrac{1}{2}\partial_x v^y \end{aligned} \qquad (12)$$

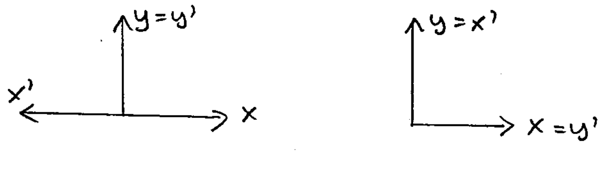
- A vector field can be divided into a curl-free and a divergence-free part. For a 3-dim vector field this can be written as

$$\vec{v} = \underbrace{-\nabla A}_{\substack{\text{scalar}\\ \text{potential}}} + \underbrace{\nabla\times\vec{B}}_{\substack{\text{vector}\\ \text{potential}}} \quad \text{or} \quad \begin{aligned} V_x &= -\partial_x A + \partial_y B_z - \partial_z B_y \\ V_y &= -\partial_y A + \partial_z B_x - \partial_x B_z \\ V_z &= -\partial_z A + \partial_x B_y - \partial_y B_x \end{aligned} \qquad (13)$$

We can move towards the 2-dim case by requiring that $V_z = 0$, and that the situation is homogeneous in the z-direction $\Rightarrow \partial_z = 0$. From this follows that

$$\begin{aligned} V_x &= -\partial_x A + \partial_y B_z \\ V_y &= -\partial_y A - \partial_x B_z \end{aligned} \qquad (14)$$

We can now call $B \equiv B_z$. A 2-dim vector field is thus defined by a scalar potential A and a pseudoscalar potential B. B is a pseudoscalar, since it must change sign in a parity transformation, e.g. $(x' = -x,\ y' = y)$ or $(x' = y,\ y' = x)$, since this requires $z' = -z \Rightarrow B_{z'} = -B_z$, for (x,y,z) to remain a right-handed crd. system



- Putting (12) and (14) together,

$$\begin{aligned} p^{xx} = -p^{yy} &= +\tfrac{1}{2}\left(\partial_x\partial_x A - \partial_x\partial_y B - \partial_y\partial_y A - \partial_y\partial_x B\right) \\ p^{xy} = p^{yx} &= +\tfrac{1}{2}\left(\partial_y\partial_x A - \partial_y\partial_y B + \partial_x\partial_y A + \partial_x\partial_x B\right) \end{aligned} \tag{15}$$

Using the Levi-Civita tensor ε_{ab}, whose components in a right-handed Cartesian crd. system are

$$\varepsilon_{ab} = \begin{bmatrix} & 1 \\ -1 & \end{bmatrix} \tag{16}$$

we can write (15) as a single tensor equation (Exercise)

$$P_{ab} = A_{,ab} - \tfrac{1}{2}\delta_{ab} A_{,cc} + \tfrac{1}{2}\left(\varepsilon_{cb} B_{,ac} + \varepsilon_{ca} B_{,bc}\right) \tag{17}$$

which divides the polarization tensor into two parts in a crd-independent way:

1) The part given by the scalar field A, that is called the E-mode, or the gradient mode, and
2) the part given by the pseudoscalar field B, that is called the B-mode, or the curl mode.