# Y. SPHERICAL HARMONIC EXPANSION OF THE POLARIZATION FIELD

- The directional dependence of linear polarization is represented by a 2nd order tensor field on the sphere of the direction unit vectors,

$$\mathcal{P}_{\hat{a}\hat{b}}(\hat{n}) = \frac{1}{2}\begin{bmatrix} Q(\hat{n}) & U(\hat{n}) \\ U(\hat{n}) & -Q(\hat{n}) \end{bmatrix} \qquad (1)$$

The hats on the indices, $\hat{a}$ and $\hat{b}$, remind us that in Eq. (1) the components of the tensor are given in an orthonormal basis, the pair of unit vectors orthogonal to $\hat{n}$, with respect to whom the Stokes parameters are defined.

- The linear polarization tensor $\mathcal{P}_{\hat{a}\hat{b}}$ is real, symmetric, and traceless, and therefore (living in a 2-dim manifold, the unit 2-sphere) has only two independent components.

- The direction vector $\hat{n}$ can represent either
  1) the direction of photon momentum, $\hat{n} = \hat{n}_{photon}$, $\vec{q} = q\hat{n}$, or
  2) the direction of observation $\hat{n} = \hat{n}_{obs}$

This chapter (§Y) applies equally well to both cases. We apply approach 1) when we discuss the evolution of the CMB from the early universe and derive the Boltzmann equations for this. We apply approach 2) when we describe present-day observations and their analysis. When we obtain the relation between 1) and 2) we need to take account that

$$\hat{n}_{obs} = -\hat{n}_{photon}\,(\text{here and now}), \qquad (2)$$

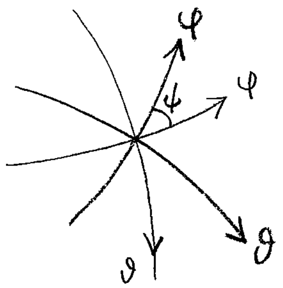
i.e, we need to "invert the sphere". However, this belongs to a later chapter. (§L)

- We shall use the usual spherical coordinates $\vartheta, \varphi$ on the sphere, so that

$$\hat{n} = (\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta) = (n_x, n_y, n_z) \qquad (3)$$

Note that there are many different spherical crd. systems for the sphere, corresponding to different orientations of the (orthogonal) x, y, and z axes. They can be obtained from each other by rotation. Note that $\hat{n}$ represents a direction vector, which is an object independent of crd. systems: a given direction vector has components $(n_x, n_y, n_z) = (\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\varphi)$ in one $\vartheta,\varphi$ crd. system, and other components $(n_{x'}, n_{y'}, n_{z'}) = (\sin\vartheta'\cos\varphi', \sin\vartheta'\sin\varphi', \cos\varphi')$ in another $\vartheta',\varphi'$ crd. system.

- The components $P_{\hat{a}\hat{b}}$, and likewise the Stokes parameters $Q, U$ for a given direction $\hat{n}$ change, when we change from one spherical crd. system to another. In a change of crd. systems $\vartheta,\varphi \to \vartheta',\varphi'$, the local crd. unit vectors $\hat{\vartheta},\hat{\varphi}$ at $\hat{n}$ are rotated by some angle $\psi$ (different for different $\hat{n}$)




$$\Rightarrow \quad \begin{aligned} Q' &= Q\cos 2\psi + U\sin 2\psi \\ U' &= -Q\sin 2\psi + U\cos 2\psi \end{aligned} \qquad (4)$$

- Thus the division of the polarization field into Q and U is not physically significant. However, it turns out that the linear polarization field on the sphere can be divided into two parts in a coordinate-independent way, and that the physics of those two parts is different. This division is analogous to the division of a vector field into a curl-free and a divergence-free part. The two parts are called E-mode (or G-mode, "gradient") and B-mode (or C-mode, "curl") polarization.

- We illustrate this division first using the "flat-sky approximation", i.e., ignoring the curvature of the sphere.