P9. Photons in Thermal Equilibrium

- From Statistical Physics (grand canonical ensemble) we have that thermodynamical equilibrium corresponds to the density operator

$$\hat{\varrho} = \frac{1}{Z} e^{-\beta(\hat{H}-\mu\hat{N})} \tag{1}$$

where the partition function $Z = \mathrm{Tr}\, e^{-\beta(\hat{H}-\mu\hat{N})}$ so that $\mathrm{Tr}\,\hat{\varrho} = 1$.

- The 1-particle density matrix elements $\varrho^{(1)}_{s's}(\vec{q})$ are then

$$\begin{aligned} \varrho^{(1)}_{s's}(\vec{q}) &= \langle a^\dagger_s(\vec{q})\, a_{s'}(\vec{q})\rangle = \mathrm{Tr}\; a^\dagger_s(\vec{q})\, a_{s'}(\vec{q})\,\hat{\varrho} \\ &= \frac{1}{Z}\sum_{\{n_{\vec{q}s}\}} \langle \{n_{\vec{q}s}\} \,|\, a^\dagger_s(\vec{q})\, a_{s'}(\vec{q})\, e^{-\beta(\hat{H}-\mu\hat{N})} \,|\{n_{\vec{q}s}\}\rangle \end{aligned} \tag{2}$$

Assuming there are no particle interaction energies, the Fock basis states are energy $\hat{H}$ and number $\hat{N}$ eigenstates,

$$\begin{aligned} e^{-\beta(\hat{H}-\mu\hat{N})}\,|\{n_{\vec{q}s}\}\rangle &= e^{-\beta\left(\sum_{\vec{q}s}\varepsilon_{\vec{q}s} n_{\vec{q}s} - \mu \sum_{\vec{q}s} n_{\vec{q}s}\right)}\,|\{n_{\vec{q}s}\}\rangle \\ &= e^{-\beta E_{\{n_{\vec{q}s}\}} - \beta\mu N_{\{n_{\vec{q}s}\}}}\,|\{n_{\vec{q}s}\}\rangle \end{aligned}$$

so that the annihilation operator $a_{s'}(\vec{q})$ gets to act directly on $|\{n_{\vec{q}s}\}\rangle$ to produce a state that is orthogonal to $\langle\{n_{\vec{q}s}\}|\, a^\dagger_s$ unless $s = s'$. Thus

$$\varrho^{(1)}_{ss'}(\vec{q}) = \langle a^\dagger_s(\vec{q})\, a_s(\vec{q})\rangle\, \delta_{ss'} = \langle \hat{n}_s(\vec{q})\rangle\, \delta_{ss'} \tag{3}$$

is diagonal. (Actually the same argument shows that the full 1-particle reduced density matrix $\langle \vec{q}'s' | \hat{\varrho}^{(1)} | \vec{q}s\rangle$ is diagonal, i.e., it is zero also for $\vec{q}' \neq \vec{q}$.)

- From the above it can be calculated (Statistical Physics) that

$$\langle \hat{n}_s(\vec{q})\rangle = \frac{1}{e^{\beta(\varepsilon_{\vec{q}s}-\mu)} - 1} \tag{4}$$

for bosons (no upper limit on the occupation numbers $n_{\vec{q}s}$), and thus

$$\varrho^{(1)}_{ss'}(\vec{q}) = \frac{\delta_{ss'}}{e^{\beta(\varepsilon_{\vec{q}s}-\mu)} - 1} \tag{5}$$

The above discussion was for bosons in general, s referring to spin states, and $\vec{q}$ to momentum states. Apply it now to photons.

- The β and μ are Lagrangian multipliers related to constraints on the expectation values for energy $\langle \hat{H} \rangle$ and particle number $\langle \hat{N} \rangle$. For photons there is no photon number conservation law, so the photon number is not independently constrained but is determined by the energy $\langle \hat{H} \rangle$. Thus, for photons $\mu = 0$. The other Lagrange multiplier determines the temperature of the system $\beta = 1/k_B T$.

- For photons, $\varepsilon_{\vec{q}s} = qc$, and we have

$$\underline{g^{(1)}_{ss'}(\vec{q})} = \frac{\delta_{ss'}}{e^{qc/k_BT} - 1} = \underline{\frac{1}{e^{qc/k_BT} - 1} \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}} = \frac{1}{2} \begin{bmatrix} I(\vec{q}) + Q(\vec{q}) & U(\vec{q}) - iV(\vec{q}) \\ U(\vec{q}) + iV(\vec{q}) & I(\vec{q}) - Q(\vec{q}) \end{bmatrix} \quad (6)$$

$$\Rightarrow \quad I(\vec{q}) = \frac{2}{e^{qc/k_BT} - 1} = \frac{2}{e^{h\nu/k_BT} - 1} \quad (7)$$

and $Q(\vec{q}) = U(\vec{q}) = V(\vec{q}) = 0$ in thermal equilibrium.

- For the brightness (Eq. 8.8) we have

$$\underline{B_\nu(\nu, \hat{n}) = \frac{h}{c^2} \nu^3 I(\vec{q}) = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/k_BT} - 1}} \quad (8)$$

This is the Planck radiation law (black-body spectrum).

- Defining $\underline{x \equiv \frac{h\nu}{k_BT}} = \frac{qc}{k_BT}$ and approximating $\frac{1}{e^x - 1} \simeq \frac{1}{x}$ for $x \ll 1$

we get the Rayleigh-Jeans approximation

$$\underline{B_\nu(\nu, \hat{n}) \simeq 2\left(\frac{\nu^2}{c^2}\right) k_B T} \quad \text{for} \quad \underline{h\nu \ll k_B T} \quad (9)$$

This has led radio astronomers to define yet another way to express the radiation brightness: The antenna temperature T_A is defined (also for radiation not in equilibrium)

$$\underline{T_A(\nu, \hat{n}) \equiv \frac{1}{2k_B} \left(\frac{c^2}{\nu^2}\right) B_\nu(\nu, \hat{n}) = \frac{h\nu}{2k_B} I(\vec{q}) = \frac{qc}{2k_B} I(\vec{q})} \quad (10)$$

If the radiation is in thermal equilibrium, with some temperature T, then the antenna temperature is constant, $T_A(\nu, \hat{n}) = T$, for $h\nu \ll k_B T$, but falls below T at higher frequencies.

- All Stokes parameters can now be given also in antenna temperature units (antenna K);

$$\underline{Q_A(\nu, \hat{n}) \equiv \frac{h\nu}{2k_B} Q(\vec{q}) \equiv \frac{1}{2k_B} \left(\frac{c^2}{\nu^2}\right) Q_B(\nu, \hat{n}) \quad \text{etc}} \qquad (I_A = T_A) \quad (11)$$