P8. Including Different Momentum States; Observing

- Include now different photon momentum states $\vec{q}$. The Stokes operators need to be defined separately for each momentum $\vec{q}$:

$$
\begin{aligned}
\hat{I}(\vec{q}) &= a_x^+(\vec{q})a_x(\vec{q}) + a_y^+(\vec{q})a_y(\vec{q}) &&= \hat{n}_x(\vec{q}) + \hat{n}_y(\vec{q}) \equiv \hat{n}(\vec{q}) \\
\hat{Q}(\vec{q}) &= a_x^+(\vec{q})a_x(\vec{q}) - a_y^+(\vec{q})a_y(\vec{q}) &&= \hat{n}_x(\vec{q}) - \hat{n}_y(\vec{q}) \\
\hat{U}(\vec{q}) &= a_x^+(\vec{q})a_y(\vec{q}) + a_y^+(\vec{q})a_x(\vec{q}) \\
\hat{V}(\vec{q}) &= i a_y^+(\vec{q})a_x(\vec{q}) - i a_x^+(\vec{q})a_y(\vec{q})
\end{aligned}
\tag{1}
$$

where $a_x^+(\vec{q})$ creates one photon in the $|\vec{q}x\rangle$ state etc.

- The 1-particle reduced density operator $\hat{\rho}^{(1)}$ has matrix elements

$$\langle \vec{q}'s'|\hat{\rho}^{(1)}|\vec{q}s\rangle$$

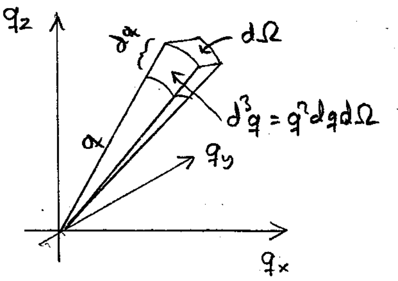
but we need only the 2x2 block diagonal

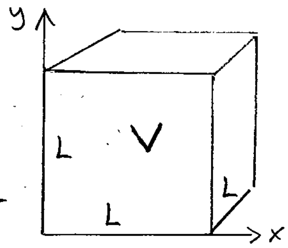
$$\langle \vec{q}s'|\hat{\rho}^{(1)}|\vec{q}s\rangle \equiv \underline{\rho^{(1)}_{s's}(\vec{q}) \equiv \langle a_s^+(\vec{q})a_{s'}(\vec{q})\rangle} = \underline{\frac{1}{2}\begin{bmatrix} I(\vec{q})+Q(\vec{q}) & U(\vec{q})-iV(\vec{q}) \\ U(\vec{q})+iV(\vec{q}) & I(\vec{q})-Q(\vec{q}) \end{bmatrix}} \tag{2}$$

to describe polarization.

- Thus we have different Stokes parameters $I(\vec{q}) \equiv \langle \hat{I}(\vec{q})\rangle$ etc for each $\vec{q} = q\hat{n}$.
In practice a range of nearby frequencies $\Delta q = \Delta\omega = 2\pi\Delta\nu$ and directions $\Delta\Omega$ can be grouped together and averaged over to get the momentum (frequency and direction) dependent Stokes parameters $I(\vec{q}), Q(\vec{q}), U(\vec{q}), V(\vec{q})$. The Stokes parameters (which have so far been dimensionless in this quantum discussion, referring to numbers of photons) can also be given in units related to the intensity of radiation. For this we need to know how many momentum states there are in a given range Δq and $\Delta\Omega$.

- So far we have used the pure momentum state description, so the photons are completely unlocalized. To get a handle on the density of momentum states, we need to consider states that are localized into some reference volume $V = L^3$.
The density of momentum states is then





$$\frac{V}{h^3} = \frac{V}{(2\pi)^3}$$

so there are

$$\frac{V}{(2\pi)^3}d^3q = \frac{V}{(2\pi)^3}q^2 dq\, d\Omega \tag{3}$$

momentum states with $q \in dq$, $\hat{n} \in d\Omega$.

Brightness in SI Units

- Since we shall make the connection to observing with instruments, we do the following in SI units.
- Density of states in momentum space is $\frac{V}{h^3} = \frac{V}{(2\pi\hbar)^3}$

$\Rightarrow$ there are $\frac{V}{h^3} d^3q = \frac{V}{h^3} q^2 \, dq \, d\Omega$ momentum states within dq and $d\Omega$.
$(d^3q = q^2 dq\, d\Omega)$

The total energy of these photons is

$$E = \langle \hat{H} \rangle = \sum_{\vec{q} \in d^3q} \varepsilon_{\vec{q}} \langle \hat{n}(\vec{q}) \rangle = \sum_{\vec{q} \in d^3q} qc \left(\langle \hat{n}_x(\vec{q}) \rangle + \langle \hat{n}_y(\vec{q}) \rangle \right) \tag{4}$$

$$= qc \sum_{\vec{q} \in d^3q} I(\vec{q}) \qquad \text{where} \quad \varepsilon_q = qc = h\nu = \hbar\omega$$

$$\Rightarrow \quad q = \frac{h\nu}{c} \quad \Rightarrow \quad dq = \frac{h}{c} d\nu$$

We now redefine $I(\vec{q})$ and the other Stokes parameters to be the average over the states $\vec{q} \in d^3q$

$$I(\vec{q}) \equiv \frac{\sum_{\vec{q} \in d^3q} I(\vec{q})}{\frac{V}{h^3} d^3q} \tag{5}$$

$\leftarrow$ number of terms in the sum $\sum_{\vec{q} \in d^3q}$

$$\therefore \quad E = qc \cdot I(\vec{q}) \cdot \frac{V}{h^3} q^2 \, dq \, d\Omega = \frac{q^3}{h^3} c \cdot I(\vec{q}) \cdot V dq \, d\Omega = \frac{h}{c^3} \nu^3 I(\vec{q}) \cdot V d\nu \, d\Omega \tag{6}$$

- The intensity $\mathcal{J}$ of this radiation is intensity $= \frac{\text{power}}{\text{area}} = \frac{\text{energy}}{\text{volume}} \cdot \text{velocity}$

$$\mathcal{J} = \frac{E}{V} \cdot c = \underbrace{\frac{h}{c^2} \nu^3 I(\vec{q})}_{B_\nu(\nu, \hat{n})} d\nu \, d\Omega \equiv d\mathcal{J} \tag{7}$$

(since this is really a differential intensity)

where we have defined the <u>brightness</u> (per unit frequency range) of the radiation

$$B_\nu(\nu, \hat{n}) \equiv \frac{d\mathcal{J}}{d\nu \, d\Omega} = \frac{h}{c^2} \nu^3 I(\vec{q}), \quad \text{where} \quad \vec{q} = \frac{h\nu}{c} \hat{n} \tag{8}$$

- The brightness is also called specific intensity. It gives the intensity (= power per area) per frequency range and solid angle at photon directions (or observing directions). For an observer it is natural to give the Stokes parameters in brightness terms, i.e. scaled by $\frac{h}{c^2} \nu^3$.

$$\boxed{\begin{aligned} I_B(\nu, \hat{n}) &\equiv B_\nu(\nu, \hat{n}) = \frac{h}{c^2} \nu^3 I(\vec{q}) & \qquad Q_B(\nu, \hat{n}) &\equiv \frac{h}{c^2} \nu^3 Q(\vec{q}) \\ U_B(\nu, \hat{n}) &\equiv \frac{h}{c^2} \nu^3 U(\vec{q}) & \qquad V_B(\nu, \hat{n}) &\equiv \frac{h}{c^2} \nu^3 V(\vec{q}) \end{aligned}} \tag{9}$$

- The dimension of brightness is $\dfrac{\text{power}}{\text{area} \cdot \text{frequency range} \cdot \text{solid angle}}$

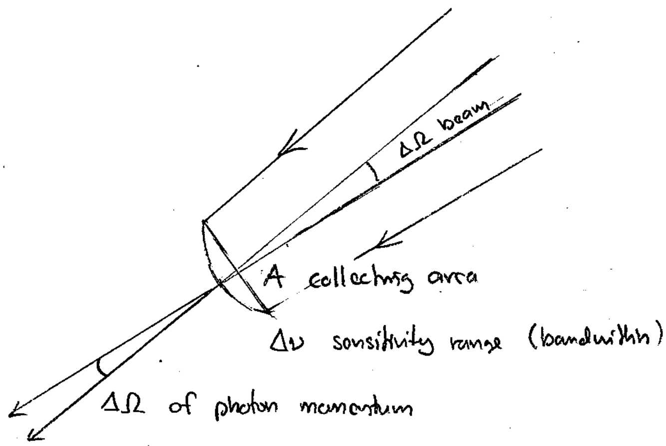
  so its SI unit is $\dfrac{W}{m^2 \cdot Hz \cdot sr}$ (sr = steradian)

- Since astronomical objects typically do not give very many W/m^2 here, radio astronomers use a smaller unit,

$$1 \text{ Jansky} \equiv 1 \text{ Jy} \equiv 10^{-26} \frac{W}{m^2\, Hz} \tag{10}$$

and the corresponding unit of brightness is Jy/sr.

- Consider now observing radiation e.g. with a telescope connected to a detector:



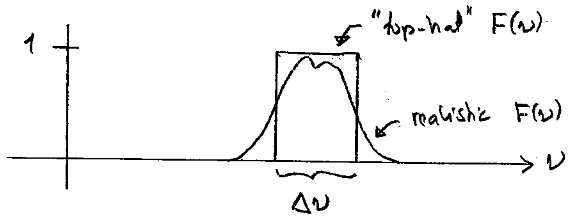
For a constant B_ν the power received by the detector is

$$P = B_\nu \cdot A \, \Delta\Omega \, \Delta\nu$$

Since the brightness $B_\nu(\nu, \hat{n})$ will depend on the frequency and direction, we have to integrate:

$$\boxed{P = A \int_{\Delta\Omega} d\Omega \int_{\Delta\nu} d\nu \; B_\nu(\nu, \hat{n})} \tag{11}$$

As an instrument description this is rather crude, assuming a "top-hat" frequency response and beam shape. In reality, the detector has some more complicated frequency response $F(\nu)$



and

$$P = A \int_{\Delta\Omega} d\Omega \int_0^\infty d\nu \; F(\nu)\, B_\nu(\nu, \hat{n}) \tag{12}$$

The (effective) bandwidth $\Delta\nu$ is defined then as $\Delta\nu \equiv \int_0^\infty d\nu \, F(\nu)$ (13)

Polarization-Sensitive Detector

- If the detector is sensitive to some particular linear polarization direction ψ only, then it will only detect photons with linear polarization ψ; and we have

$\hat{n}_\psi(\vec{q})$ instead of $\hat{n}_x(\vec{q}) + \hat{n}_y(\vec{q})$ in Eq. (4)

and
$$\langle \hat{n}_\psi(\vec{q}) \rangle = \frac{1}{2}\left[I(\vec{q}) + Q(\vec{q}) \cos 2\psi + U(\vec{q}) \sin 2\psi \right] \qquad (14)$$

(see Eq. 7.30) instead of $\langle \hat{n}_x(\vec{q}) \rangle + \langle \hat{n}_y(\vec{q}) \rangle = I(\vec{q})$.

Thus power measured by the polarization sensitive instrument is

$$\underline{P(\psi_{pol})} = A \int_{\Delta\Omega} d\Omega \underline{\int_0^\infty d\nu\, F(\nu) \cdot \frac{1}{2}\left[I_B(\nu,\hat{n}) + Q_B(\nu,\hat{n}) \cos 2\psi_{pol} + U_B(\nu,\hat{n}) \sin 2\psi_{pol} \right]} \qquad (15)$$

Collecting Area

- The collecting area A in the above need not be the same as the area of the telescope mirror (reflector) used to collect the photons and to focus them on the detectors. Instead it depends on the arrangement of the detectors (usually there is more than one detector for each frequency band in the instrument) in the focal plane.

The (effective) collecting area (also called the effective aperture) depends also on the direction the photons are coming from;

it is larger for the photons coming along the center of the beam, and falls off for directions further away from the center.

The part $A \int_{\Delta\Omega} d\Omega$ in Eqs. (12) and (15) needs to be replaced by something which describes in more detail how the effective aperture varies with photon direction; this is called the beam of the detector. The telescope geometry together with how the detector is arranged in the focal plane determine how effectively the detector detects photons coming from different directions. The treatment of the detector beam is discussed in the <u>Appendix: Detector Beam</u>