# P6. Fock Space

- For a many-particle system we choose the Fock basis, where the basis state vectors are of the form

$$|\{n_k\}\rangle \equiv |n_1, n_2, \ldots, n_k, \ldots\rangle \tag{1}$$

where $n_k$ gives the occupation number of the 1-particle state $k$. For photons, the 1-particle state labels $k$ could represent, e.g., the pair $k = \vec{q}s$, where $s = x, y$, or $s = +, -$.

- In general, the number of the different orthogonal 1-particle states $k$ can be finite or infinite.

- We consider here only the case of bosons, where the occupation numbers are unlimited

$$0 \leq n_k < \infty \tag{2}$$

- For bosons, the annihilation and creation operators are defined

$$\begin{aligned} a_k |n_1, .., n_k, ..\rangle &= \sqrt{n_k}\, |n_1, .., n_k - 1, ..\rangle \\ a_k^+ |n_1, .., n_k, ..\rangle &= \sqrt{n_k + 1}\, |n_1, .., n_k + 1, ..\rangle \end{aligned} \tag{3}$$

If $n_k = 0$, $a_k$ produces $0$, the zero element of the vector space; not to be confused with the vacuum state $|0\rangle \equiv |0, 0, \ldots, 0, ..\rangle$.

These operators are not hermitian; but they are hermitian conjugates of each other. Acting onto the left (on bra-vectors)

$$\begin{aligned} \langle n_1, .., n_k, .. | a_k^+ &= \sqrt{n_k}\, \langle n_1, .., n_k - 1, .. | \\ \langle n_1, .., n_k, .. | a_k &= \sqrt{n_k + 1}\, \langle n_1, .., n_k + 1, .. | \end{aligned} \tag{4}$$

They satisfy the commutation relations

$$\begin{aligned} [a_k, a_{k'}] &= [a_k^+, a_{k'}^+] = 0 \\ [a_k, a_{k'}^+] &= \delta_{kk'} \end{aligned} \tag{5}$$

- The number operator corresponding to 1-particle state $k$ is

$$\hat{n}_k \equiv a_k^+ a_k \qquad \text{(Hermitean)} \tag{6}$$

$$\begin{aligned} \hat{n}_k |n_1, .., n_k, ..\rangle &= a_k^+ a_k |n_1, .., n_k, ..\rangle = n_k |n_1, .., n_k, ..\rangle \\ \langle n_1, .., n_k, .. | \hat{n}_k &= \langle n_1, .., n_k, .. | a_k^+ a_k = n_k \langle n_1, .., n_k, .. | \end{aligned} \tag{7}$$

Thus the Fock basis vectors are eigenstates of all the 1-particle state number operators $\hat{n}_k$.

- The total number of particles in a Fock basis state is $N = \sum n_k$.
  The space spanned by the Fock basis, the Fock space, does not have a fixed number of particles; rather the basis vectors include all values

$$0 \leq N < \infty .$$

- The vacuum state is $|0\rangle \equiv |0,0,\dots,0,\dots\rangle$ with $N=0$ particles.
  All vectors of the Fock basis can be created by repeated action of the creation operators $a_k^{\dagger}$ on the vacuum state (and normalization).

- If we restrict to the case where all photons have the same momentum $\vec{q}$, then there are only two independent 1-particle states, e.g., $|x\rangle$ and $|y\rangle$.
  The Fock basis is then

$$\{|n_x,n_y\rangle\} = \{\underbrace{|0,0\rangle}_{N=0}, \underbrace{|1,0\rangle, |0,1\rangle}_{N=1}, \underbrace{|2,0\rangle, |1,1\rangle, |0,2\rangle}_{N=2}, \underbrace{|3,0\rangle, |2,1\rangle, |1,2\rangle, |0,3\rangle}_{N=3}, \dots\}$$

  The two basis states with $N=1$ can be identified with the basis of the 1-photon system discussed earlier: $|1,0\rangle = |x\rangle$ and $|0,1\rangle = |y\rangle$.

- An arbitrary state $|\psi\rangle$ in Fock space is an arbitrary linear combination of the basis vectors, which in general correspond to different $N$. One can calculate the expectation number of particles as the expectation value of the total number operator $\hat{N} \equiv \sum_k \hat{n}_k$ as

$$\langle N\rangle \equiv \langle\hat{N}\rangle \equiv \langle\psi|\hat{N}|\psi\rangle \equiv \sum_k \langle\psi|\hat{n}_k|\psi\rangle \equiv \sum_k \langle\hat{n}_k\rangle$$