P5. Statistical One-Photon System

- Before going to discuss many-photon systems, consider a one-photon system: Suppose the photon is in a momentum eigenstate $\vec{q}$, but we have only statistical knowledge about its polarization. Then the remaining state space is 2-dimensional, and the density matrix ρ_{ij} is 2×2.

- In the $\{|x\rangle, |y\rangle\}$ basis the density matrix is

$$[\rho_{ij}] = \begin{bmatrix} \rho_{xx} & \rho_{xy} \\ \rho_{yx} & \rho_{yy} \end{bmatrix} \qquad \text{where } \rho_{xx}, \rho_{yy} \text{ are real}, \quad \rho_{yx} = \rho_{xy}^* \quad \text{(Hermitian)} \qquad (1)$$

$$\text{and} \quad \rho_{xx} + \rho_{yy} = 1 \quad \text{(normalization)}$$

$\therefore$ There are 3 degrees of freedom for ρ_{ij}: $\rho_{xx}-\rho_{yy}$, $\mathrm{Re}\,\rho_{xy}$, and $\mathrm{Im}\,\rho_{xy}$.

- From Eq. (3.2) we have the Stokes parameter operators in this basis:

$$\hat{I} = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} \quad \hat{Q} = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \quad \hat{U} = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} \quad \hat{V} = \begin{bmatrix} & -i \\ i & \end{bmatrix} \qquad (2)$$

Note that these are just the Pauli spin matrices (2.7).

- We can now calculate the expectation values for the Stokes parameters for a given density matrix

$$\underline{I \equiv \langle \hat{I} \rangle} = \mathrm{Tr}\,\hat{\rho}\hat{I} = \mathrm{Tr}\begin{bmatrix} \rho_{xx} & \rho_{xy} \\ \rho_{yx} & \rho_{yy} \end{bmatrix}\begin{bmatrix} 1 & \\ & 1 \end{bmatrix} = \underline{\rho_{xx} + \rho_{yy} = 1}$$

$$\underline{Q \equiv \langle \hat{Q} \rangle} = \mathrm{Tr}\,\hat{\rho}\hat{Q} = \mathrm{Tr}\begin{bmatrix} \rho_{xx} & \rho_{xy} \\ \rho_{yx} & \rho_{yy} \end{bmatrix}\begin{bmatrix} 1 & \\ & -1 \end{bmatrix} = \mathrm{Tr}\begin{bmatrix} \rho_{xx} & -\rho_{xy} \\ \rho_{yx} & -\rho_{yy} \end{bmatrix} = \underline{\rho_{xx} - \rho_{yy}} \qquad (3)$$

$$\underline{U \equiv \langle \hat{U} \rangle} = \mathrm{Tr}\,\hat{\rho}\hat{U} = \mathrm{Tr}\begin{bmatrix} \rho_{xx} & \rho_{xy} \\ \rho_{yx} & \rho_{yy} \end{bmatrix}\begin{bmatrix} & 1 \\ 1 & \end{bmatrix} = \mathrm{Tr}\begin{bmatrix} \rho_{xy} & \rho_{xx} \\ \rho_{yy} & \rho_{yx} \end{bmatrix} = \rho_{xy} + \rho_{yx} = \rho_{xy} + \rho_{xy}^* = \underline{2\,\mathrm{Re}\,\rho_{xy}}$$

$$\underline{V \equiv \langle \hat{V} \rangle} = \mathrm{Tr}\,\hat{\rho}\hat{V} = \mathrm{Tr}\begin{bmatrix} \rho_{xx} & \rho_{xy} \\ \rho_{yx} & \rho_{yy} \end{bmatrix}\begin{bmatrix} & -i \\ i & \end{bmatrix} = \mathrm{Tr}\begin{bmatrix} i\rho_{xy} & -i\rho_{xx} \\ i\rho_{yy} & -i\rho_{yx} \end{bmatrix} = i(\rho_{xy} - \rho_{yx}) = i(\rho_{xy} - \rho_{xy}^*) = \underline{-2\,\mathrm{Im}\,\rho_{xy}}$$

$$\therefore \left.\begin{aligned} \rho_{xx} &= \tfrac{1}{2}(I+Q) & \mathrm{Re}\,\rho_{xy} &= \tfrac{1}{2}U \\ \rho_{yy} &= \tfrac{1}{2}(I-Q) & \mathrm{Im}\,\rho_{xy} &= -\tfrac{1}{2}V \end{aligned}\right\} \Rightarrow \underline{[\rho_{ij}] = \frac{1}{2}\begin{bmatrix} I+Q & U-iV \\ U+iV & I-Q \end{bmatrix}} \quad \underline{\text{in the } \{|x\rangle,|y\rangle\} \text{ basis.}} \qquad (4)$$

I, Q, U, V are all real.

- Thus the Stokes parameters (i.e. their expectation values) contain exactly the same information as the density operator,

$$\underline{\hat{\rho} = \tfrac{1}{2}(I\hat{I} + Q\hat{Q} + U\hat{U} + V\hat{V})} \quad \text{in general} \qquad (5)$$

and

$$\underline{\rho = \tfrac{1}{2}(I\mathbb{1} + Q\sigma_3 + U\sigma_1 + V\sigma_2)} \quad \text{in the } \underline{\{|x\rangle,|y\rangle\}} \text{ basis} \qquad (6)$$

$\uparrow$ density matrix

Pure and Mixed States

- From the density matrix $\boxed{\rho = \frac{1}{2}\begin{bmatrix} I+Q & U-iV \\ U+iV & I-Q \end{bmatrix}}$ we find (exercise)

$$\underline{\mathrm{Tr}\,\rho^2 = \dots = \tfrac{1}{2}\left(I^2+Q^2+U^2+V^2\right)} = \tfrac{1}{2}\left(1+Q^2+U^2+V^2\right) \qquad (7)$$

For a pure state, $\mathrm{Tr}\,\rho^2 = 1 \Rightarrow \underline{Q^2+U^2+V^2 = I^2}$ $(=1)$ (8)

For a mixed state, $0 \le \mathrm{Tr}\,\rho^2 < 1 \Rightarrow \underline{0 \le Q^2+U^2+V^2 < I^2}$ (9)

- The normalization of the density operator sets $I=1$. This can be understood to represent the number N of photons; here $N=1$.